J. Univ. Kuwait (Sci.) **16**, 1989

A note on groups acting on connected graphs

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ABSTRACT

Serre (1977) gave the definition of a graph under the condition that no edge is equal to its inverse. He also defined the action of a group on a graph under the condition that no element of the group transforms any edge to its inverse. In this paper we drop these conditions to give a new definition of a graph and then establish a result obtained under the action of a group acting on a connected graph.

INTRODUCTION

We begin by giving preliminary definitions. By a graph X we understand a pair of disjoint sets $V(X)$ and $E(X)$, with $V(X)$ non-empty, together with a mapping $E(X) \to V(X) \times V(X)$, $y \to (o(y), t(y))$, and a mapping $E(X) \to E(X)$, $y \to \bar{y}$ satisfying $\bar{\bar{y}} = y$ and $o(\bar{y}) = t(y)$, for all $y \in E(X)$. The case $\bar{y} = y$ is possible for some $y \in E(X)$.

A path in a graph X is defined to be either a single vertex $v \in V(X)$ (a trivial path), or a finite sequence of edges $y_1, y_2, \ldots, y_n$, $n \geqslant 1$ such that $t(y_i) = o(y_{i+1})$ for $i = 1, 2, \ldots, n-1$.

A path $y_1, y_2, \ldots, y_n$ is reduced if $y_{i+1} \neq \bar{y}_i$, for $i = 1, 2, \ldots, n-1$. A graph X is connected, if for every pair of vertices u and v of $V(X)$ there is a path $y_1, y_2, \ldots, y_n$ in X such that $o(y_1) = u$ and $t(y_n) = v$.

A graph X is called a tree if for every pair of vertices of $V(X)$ there is a unique reduced path in X joining them.

A subgraph Y of a graph X consists of sets $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$ such that if $y \in E(Y)$, then $\bar{y} \in E(Y)$, $o(y)$ and $t(y)$ are in $V(Y)$. We write $Y \subseteq X$. We take any vertex to be a subtree without edges.

A reduced path $y_1, y_2, \ldots, y_n$ is called a circuit if $o(y_1) = t(y_n)$, and $o(y_i) \neq o(y_j)$ when $i \neq j$. It is clear that a graph X is a tree if X is connected and contains no circuits.

If X_1 and X_2 are two graphs, then the map $f\colon X_1 \to X_2$ is called a morphism, if f takes vertices to vertices and edges to edges such that

$$f(\bar{y}) = \overline{f(y)}$$

$$f(o(y)) = o(f(y))$$

and
$$f(t(y)) = t(f(y)), \qquad \text{for all } y \in E(X_1);$$

f is called an isomorphism if it is one-to-one and onto, and is called an automorphism if it is an isomorphism and $X_1 = X_2$. The automorphisms of X form a group under composition of maps, denoted by Aut(X).

We say that a group G acts on a graph X, if there is a group homomorphism $\phi : G \to \text{Aut}(X)$. If $x \in X$ is a vertex or an edge, we write $g(x)$ for $\phi(g)(x)$. If $y \in E(X)$, then $g(\bar{y}) = \overline{g(y)}$, $g(o(y)) = o(g(y))$, and $g(t(y)) = t(g(y))$. The case $g(y) = \bar{y}$ for some $y \in E(X)$ and $g \in G$ may occur. If $y \in X$, (vertex or edge), we define $G(y) = \{g(y) \mid g \in G\}$ and this set is called an orbit. If $x, y \in X$, we define $G(x, y) = \{g \in G \mid g(y) = x\}$.

DEFINITION AND THE MAIN THEOREM

Definition. Let G be a group acting on a connected graph X. A subtree T of X is called a tree of representatives for the action of G on X if T contains exactly one vertex from each G-vertex orbit. A subgraph Y of X containing a tree of representatives, T(say), is called a fundamental domain for the action of G on X if each edge in Y has at least one end in T and Y contains exactly one edge, y(say), from each G-edge orbit such that $G(\bar{y}, y) = \phi$ and exactly one pair y and $\bar{y}$ from each G-edge orbit such that $G(\bar{y}, y) \neq \phi$.

The main result of this paper is the following theorem.

Theorem. Let G be a group acting on a connected graph X. Then there exists a tree of representatives and a fundamental domain for the action of G on X.

The theorem will follow from the following propositions.

Proposition 1. Let S be the set of all T, where T is a subtree of X containing at most one vertex from each G-vertex orbit and at most one edge from each G-edge orbit. Then S has a maximal element.

Proof. S is not empty, since every vertex of X is a subtree, and hence is in S. For T_1 and T_2 in S, we define $T_1 \leqslant T_2$ if T_1 is a subtree of T_2. Hence S becomes a partially ordered set. Let $\{T_i/i \in I\}$ be a linearly ordered subset of S. Define $T^* = \bigcup_{i \in I} T_i$. We need to show that T^* is a subtree of X. It is connected, for if $u, v \in V(T^*)$, then $u \in V(T_l)$ and $v \in V(T_j)$ for some $l, j \in I$. We can suppose by symmetry that $T_l \leqslant T_j$, so $u, v \in V(T_j)$, and there is a path in T^* from u to v. The path has no circuits, for if $y_1, \ldots, y_n$ is a circuit, then $y_1 \in E(T_{i_1}), \ldots, y_n \in E(T_{i_n})$, so $y_1, \ldots, y_n$ will all be edges of T_i, where $T_i = \max\{T_{i_1}, \ldots, T_{i_n}\}$, contradicting the fact that T_i is a tree.

It is clear that T^* has at most one vertex and one edge from each orbit under G. Hence $T^* \in S$ and T^* is an upper bound for $\{T_i/i \in I\}$. By Zorn's Lemma, S has a maximal element, T_0 say.

Proposition 2. T_0 is a tree of representatives for the action of G on X.

Proof. Claim: T_0 contains exactly one vertex from each G-vertex orbit. Suppose that it does not. Suppose $v \in V(X)$ is such that $V(T_0) \cap G(v) = \phi$, where $G(v)$ is the orbit containing v. Since X is connected, there is a shortest path $y_1, \ldots, y_n$ joining a vertex of T_0 to v. Let y be the first edge of this path such that $V(T_0) \cap G(o(y_i)) \neq \phi$ and

$V(T_0) \cap G(t(y_i)) = \phi$. Then there exists $g \in G$ such that $o(g(y_i)) \in V(T_0)$, $t(g(v_i)) \notin V(T_0)$ and so $g(y_i) \notin E(T_0)$. Let T' be the subgraph with $V(T') = V(T_0) \cup \{t(g(y_i))\}$ and $E(T') = E(T_0) \cup \{g(y_i), g(\bar{y}_i)\}$. It is clear that T' is a subtree of X that properly contains T_0 and at most one vertex from each G-vertex orbit. This contradicts the maximality of T_0 in S. Thus T_0 is a tree of representatives for the action of G on X.

Now we need to prove the existence of a fundamental domain for the action of G on X.

Proposition 3. There exists a fundamental domain for the action of G on X.

Proof. Let Λ be the set of all Y, where Y is a subgraph of X containing the chosen tree of representatives T_0 such that each edge in Y has at least one end in T_0 and contains at most one edge from $G(y)$, unless $G(\bar{y}, y) \neq \phi$, in which case it contains at most one pair y, $\bar{y}$ from $G(y)$, for all $y \in E(X)$. Since T_0 contains at most one edge from each G-edge orbit, $T_0 \in \Lambda$. For Y_1 and Y_2 in Λ, we define $Y_1 \leqslant Y_2$ if Y_1 is a subgraph of Y_2, so Λ becomes a partially ordered set. As in the proof of the existence of a tree of representatives, we can show that Λ contains a maximal element Y_0, (say). Let $y \in E(X)$. We need to show that Y_0 contains exactly one edge from $G(y)$, if $G(\bar{y}, y) = \phi$ and exactly one pair y, $\bar{y}$ from $G(y)$, if $G(\bar{y}, y) \neq \phi$. Suppose there exists $y \in E(X)$ such that $E(Y_0) \cap G(y) = \phi$. Since X is connected, there is a shortest path $y_1, \ldots, y_n = y$ joining a vertex in Y_0 to $t(y)$.

Let y_i be the first edge of this path such that $E(Y_0) \cap G(y_i) = \phi$. Since $V(T_0) \cap G(o(y_i)) \neq \phi$ and T_0 is the tree of representatives in Y_0, there exists $g \in G$ such that $o(g(y_i)) \in V(T_0)$.

Let Y be the subgraph with $V(Y) = V(Y_0) \cup \{t(g(y_i))\}$ and $E(Y) = E(Y_0) \cup \{g(y_i), g(\bar{y}_i)\}$. It is clear that each edge of Y has at least one end in T_0. Moreover, it is clear that $E(Y)$ satisfies the conditions on edges for elements of Λ and properly contains Y_0. This contradicts the maximality of Y_0 in Λ.

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(Received 29 August 1987, revised 3 September 1988)

ملاحظة على الزمر المؤثرة على البيانات المتصلة

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خلاصة

في هذا البحث نبرهن أنه أذا أثرت الزمرة G على البيان المتصل X فانه توجد شجرة جزئية T من X وبيان جزئي Y من X بحيث ان T تحتوي بالضبط على رأس واحدة من كل مدار من مدارات رؤوس X ، وان Y تحتوي على T بالاضافة إلى ان كل حرف في Y له على الأقل نهاية واحدة في T ، وان Y تحتوي بالضبط على حرف واحد من كل مدار من مدارات أحرف X إذا كان ذلك الحرف لا يقع في نفس المدار الذي يحتوي على عكسه . ويحتوي Y على حرف واحد مع عكسه إذا وقع ذلك الحرف وعكسه في مدار واحد من مدارات أحرف X .