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On the compactness theorem of Luxemburg and Zaanen

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ABSTRACT

The compactness criterion for integral operators (Luxemburg & Zaanen 1963) is stated in a generalized form and several known facts about compactness of integral operators are discussed from the point of view of this result. It is also shown that every integral operator from L^p to L^q, $p > q \geqslant 1$ with positive kernel is compact.

1. INTRODUCTION

This note deals with integral operators. We formulate an extension of the compactness theorem of Luxemburg & Zaanen (1963) based on a theorem of Korotkov (1977) and we indicate some examples of its applications, which except for the last one seem to be of interest because of their simplicity rather than novelty.

2. NOTATION

X, Y are complete σ-finite measure spaces with measures denoted by dx and dy. $M(\cdot)$ denotes the space of all scalar valued, finite a.e., measurable functions on the space indicated in parentheses, equipped with the metrizable complete vector topology of convergence in measure on subsets of finite measure. For $k \in M(X \times Y)$ the integral operator with the kernel k is the linear operator from $M(Y)$ to $M(X)$ with the domain

$$\mathscr{D}_K = \{u \in M(Y);\ \int_Y |k(x,y)||u(y)|dy < \infty \text{ a.e.}\}$$

and given by the formula

$$Ku(x) = \int_Y k(x,y)u(y)dy, \quad u \in \mathscr{D}_K .$$

We also write

$$|K|u(x) = \int_Y |k(x,y)|u(y)dy \quad \text{with } \mathscr{D}_{|K|} = \mathscr{D}_K .$$

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3. CONTINUITY

We recall the following result.

Theorem. If K is an integral operator, if $A \subset M(Y)$, $B \subset M(X)$ are complete metrizable vector spaces with both inclusions continuous, if $A \subset \mathscr{D}_K$ and if $KA \subset B$ then $K|_A : A \to B$ is continuous.

A prototype of this theorem was proved by Banach (1922) and various versions of it were proved subsequently by Gribanov (1967), Halmos & Sunder (1978), Schachermayer (1979) and Schep (1980a). The version quoted above is taken from Aronszajn & Szeptycki (1966).

We note that the definitions in Section 2 and the above theorem remain valid without the assumption of σ-finiteness of X and Y (Szeptycki 1979). We do not know if this is true of what follows.

4. COMPACTNESS

We consider the situation as in Section 3 with the additional hypothesis that A, B are Banach spaces. We recall that a Banach space $A \subset M$ is solid if the conditions $v \in A$, $u \in M$, $|u| \leqslant |v|$ a.e., imply that $u \in A$ and $||u||_A \leqslant ||v||_A$.

The subspace of A of functions with absolutely continuous norms is defined by

$$A^c = \{u \in A; \quad ||\chi_{E_n} u||_A \to 0 \text{ for every } E_n \downarrow \varnothing\}.$$

In the above, χ_E denotes the characteristic function of a measurable set E and for a sequence of such sets $\{E_n\}$ we write $E_n \downarrow \varnothing$ to indicate that the sequence is decreasing and that $\cap E_n$ is of measure 0.

We also denote by A' the associated space of A

$$A' = \{v \in M : \int |uv| dx < \infty \text{ for all } u \in A\};$$

with the norm $|| \quad ||_{A'}$ given by

$$||v||_{A'} = \sup\{|\int uv\, dx|; \quad u \in A, \quad ||u||_A \leqslant 1\}, \quad v \in A',$$

A' is a solid Banach space.

Any solid Banach space A can be enlarged to a solid Banach space $A^\#$ given by

$$A^\# = \{u \in M; \quad \{v \in A; \quad |v| \leqslant |u|\} \text{ is bounded in } A\}.$$

It is easily shown that if $A \subset M(Y)$, $B \subset M(X)$ are solid Banach spaces, if B has the weak Fatou property and if $A \subset \mathscr{D}_K$, $KA \subset B$ for some kernel k then $A^\# \subset \mathscr{D}_K$ and $KA^\# \subset B$. We assume in what follows that all spaces under consideration coincide with their extensions by $^\#$.

Theorem 1 (Korotkov 1977). Suppose that K is an integral transformation and $A \subset M(Y)$ is a solid Banach space such that $(A')^c = A'$ and $A \subset \mathscr{D}_K$. Then $K|_A : A \to M(X)$ is compact.

Theorem 2 (Szeptycki 1980). Assume that K and A are as in Theorem 1 and that $B \subset M(X)$ is a solid Banach space such that $KA \subset B$. If for every sequence $E_n \subset X$ such that $E_n \downarrow \varnothing$ the norms $||\chi_{E_n}K||_{A\to B}\to 0$ then $K|_A : A\to B$ is compact. If $KA\subset B^c$ then the above condition is also necessary for the compactness of $K|_A : A\to B$.

Theorem 2 is an immediate consequence of Theorem 1 and the following easily proved proposition.

A subset $C\subset B^c$ is compact if and only if: (i) C is compact in M; (ii) for every sequence $E_n\downarrow\varnothing$ the limit

$$\lim_{n\to\infty} \chi_{E_n}u = 0$$

(in B) is uniform with respect to $u\in C$.

Theorem 2 extends partly the result of Luxemburg & Zaanen (1963) in as much that it does not impose conditions on $|K|$ other than the inclusion $A\subset \mathcal{D}_K$.

It is well known that without the hypothesis that $(A')^c = A'$ the conclusion of Theorem 1 is not valid. A characterization of compactness of integral operators in absence of this hypothesis seems to be an open problem.

5. EXAMPLES

We give now some examples of situations where Theorem 2 can be applied directly:

(i) With A, K as in Theorem 2 suppose that $B_1\subset B_2\subset M(X)$ are solid Banach spaces such that $||\chi_E u||_{B_2}\leqslant C(E)||u||_{B_1}$ for every $u\in B_1$ and for every measurable $E\subset X$, where $C(E)$ is independent of u and has the property that $C(E_n)\to 0$ whenever $E_n\downarrow\varnothing$. Assume also that $KA\subset B_1$. Then $K|_A$ considered as an operator from A to B_2 is compact. This follows readily from the estimate

$$||\chi_E K||_{A\to B_2} \leqslant C(E)||K||_{A\to B_1}\,.$$

A special case of the above situation occurs when the space of multipliers $m(B_1,B_2)$ satisfies the condition $m(B_1,B_2)^c = m(B_1,B_2)$; we have then $C(E) = ||\chi_E||_{m(B_1,B_2)}$. This case was considered by Schep (1980b) who has shown that it accounts for the results in Halmos & Sunder (1978) and Dodds (1977), with $A = L^p$, $B_i = L^{p_i}$, $p_1 > p_2 \geqslant 1$.

(ii) With K,A,B as in Theorem 2 K is called U-bounded from A to B provided that $|Ku(x)|\leqslant f(x)||u||_A$ for some fixed $f\in B$. Clearly in this case we have $||\chi_E K||\leqslant||\chi_E f||_B$ and $K|_A : A\to B$ is compact if $f\in B^c$. This condition is clearly fulfilled if $B = B^c$—in this case the result is stated with references in Krein (1972, ch. III, §5).

(iii) We consider now the case when A,B are Orlicz spaces; for notations and terminology we use Krasnoselskii & Rutickii (1961) where the following result can be found (§15).

Let M_1,M_2,Φ be N-functions with complementary functions N_1,N_2,Ψ and suppose that Φ satisfies the following condition:

$u\in L^*_{M_1}(Y)$, $v\in L^*_{N_2}(X)$ implies that $w(x,y) = u(x)v(y)$ is in $L^*_\Phi\ (X\times Y)$ with $||w||_\Phi\leqslant \text{const}\ ||u||_{M_1}\ ||v||_{N_2}$.

Then the condition $k\in L^*_\Psi(X\times Y)$ implies that $L^*_{M_1}\subset\mathcal{D}_K$ and $KL^*_{M_1}\subset L^*_{M_2}$ with $||K||_{L^*_{M_1}\to L^*_{M_2}}\leqslant\text{const}\ ||k||_\Psi$.

The point we want to make in this example is that if $k\in E_\Psi = (L^*_\Psi)^c$ and if $L^*_{N_1} = E_{N_1}$ then $K: L^*_{M_1}\to L^*_{M_2}$ is compact. These conditions are satisfied for example if N_1 and Ψ have the Δ_2-property.

The above statement is checked immediately by observing that $||\chi_E K||$ $L^*_{M_1}\to L^*_{M_2}\leqslant\text{const}\ ||\chi_E K||_\Psi$.

One can also check that various compactness criteria in Krasnoselskii & Rutickii (1961, §16) can be derived directly from Theorem 2.

The same is true about the compactness criteria for operators in L^p spaces given in Krasnoselskii *et al.* (1966).

We note that in this example it suffices to use the classical version of Theorem 2 with the conditions imposed on $|K|$ rather than on K.

(iv) In this example as in (iii) only the modulus of K version of Theorem 2 is used; however, the verification of the condition in the theorem is more difficult.

Let $1 \leqslant q < p < \infty$ and let k be a kernel such that $L^p \subset \mathscr{D}_K$ and that $|K| L^p \subset L^q$. Then $K: L^p \to L^q$ is compact.

Proof. We use the following result of Gagliardo (1965). If $k \geqslant 0$ and if $K: L^p \to L^q$ for some p,q, $1 \leqslant q < p < \infty$ then for every $\varepsilon > 0$ there exist $\phi > 0$, $\psi > 0$ such that $\phi \in L^p(Y)$, $\psi \in L^{q'}(X)$, $1/q' + 1/q = 1$ and such that with $C = ||K||_{L^p \to L^q}$ we have a.e.

$$K\phi(x) = \int k(x,y)\phi(y)dy \leqslant (C+\varepsilon)\psi(x)^{q'/q},\ {}^tK\psi(y) = \int k(x,y)\psi(x)dx \leqslant (C+\varepsilon)\phi(y)^{p'/p}$$

and

$$\int\int k(x,y)\phi(y)\psi(x)dy\,dx \leqslant (C+\varepsilon)\,,$$

where $1/p + 1/p' = 1$.

To estimate $||\chi_E K||_{L^p \to L^q}$ we note that

$$||\chi_E K||_{L^p \to L^q} \leqslant \sup\{\int_E\int_Y |k(x,y)u(y)v(x)|dy\,dx;\quad ||u||_{L^p} \leqslant 1,\quad ||v||_{L^{q'}} \leqslant 1\}$$

and with k replaced by $|k|$ we choose ϕ and ψ as in the theorem above.

We can write with $|k(x,y)| = k$, $|u(y)| = u$, $|v(x)| = v$, $\phi(y) = \phi$, $\psi(x) = \psi$;

$$\int_E\int_Y kuv\,dy\,dx = \int_E\int_Y k^{1/p}\frac{\psi^{1/p}}{\phi^{1/p'}}\,\mathrm{u}\,\mathrm{k}^{1/q'}\frac{\phi^{1/q'}}{\psi^{1/q}}\,v\,(k\phi\psi)^{1/q-1/p}dy\,dx$$

$$\leqslant \left(\int_Y \frac{{}^tK\,\psi}{\phi^{p/p'}}\,u^p dy\right)^{1/p}\left(\int_X \frac{K\,\phi}{\psi^{q^1/q}}\,v^{q'}\,dx\right)^{1/q'}\left(\int_D\int_Y k\phi\psi dy\,dx\right)^{1/q-1/p}$$

$$\leqslant (C+\varepsilon)^{1/p+1/q'}\,||u||_p\,||v||_{q'}\,(\int_E\int_Y k\phi\psi dy\,dx)^{1/q-1p},$$

where in the middle step we applied the Hölder's inequality.

Clearly the condition of Theorem 2 is verified and the result follows.

The result mentioned at the end of (i) states that if $K: L^p \to L^{p_1} \subset L^{p_2}$, $p_1 \geqslant p_2 \geqslant 1$, $p > 1$, (i.e. the measure space X is finite) then K considered as an operator from L^p to L^{p_2} is compact.

By the result above this can be amended: if $p > p_2$ and $k \geqslant 0$ then there is no need for the intermediate space L^{p_1}. It would be of some interest to know if this amendment remains valid if the condition $k \geqslant 0$ is relaxed.

Another question is that of possibility of an extension of the Gagliardo's theorem to Orlicz spaces other than the L^p-s.

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حول نظرية لوكسمبورغ وزانن في التراص

باڤل شبتسكي
قسم الرياضيات بجامعة كانساس ، لورانس ،
كانساس ٦٦٠٤٥ ، الولايات المتحدة الامريكية

خلاصة

ينص هنا على خاصة التراص في المؤثرات التكاملية بشكل معمم حسب مفهوم لوكسمبورغ وزانن ، ونناقش انطلاقا من هذه النتيجة عدة حقائق معروفة بالنسبة لتراص المؤثرات التكاملية .

ونبرهن أيضا على أن كل مؤثر تكاملي من L^p إلى L^q ، حيث $p > q \geqq 1$ وذي نواة موجبة هو متراص .