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Fixed points of mappings and set-valued mappings

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ABSTRACT

Let (X,d) be a complete metric space and let $B(X)$ be the set of all non-empty, bounded subsets of X. The function $\delta(A,B)$ with A, B in $B(X)$ is defined by

$$\delta(A,B) = \sup\{d(a,b): a \in A,\ b \in B\}.$$

If F is a mapping of X into $B(X)$ then a point z is said to be a fixed point of F if z is in Fz.

Now let F be a mapping of X into $B(X)$ and let I be a continuous mapping of X into itself. Further let F and I commute and let the range of I contain the range of F. It is proved that if

$$\delta(Fx,Fy) \le c \cdot \max\{d(Ix,Iy),\ \delta(Ix,Fx),\ \delta(Iy,Fy),\ \delta(Ix,Fy),\ \delta(Iy,Fx)\}$$

for all x,y in X, where $0 \le c < 1$, then F and I have a unique common fixed point z and further $Fz = \{z\}$.

Two corollaries are deduced. The first where F is replaced by a mapping T of X into itself and the second where I is the identity mapping on X.

INTRODUCTION

Let (X,d) be a complete metric space and let $B(X)$ be the set of all non-empty, bounded subsets of X. The function $\delta(A,B)$ with A and B in $B(X)$ is defined by

$$\delta(A,B) = \sup\{d(a,b): a \in A,\ b \in B\}.$$

If A consists of a single point a we write

$$\delta(A,B) = \delta(a,B)$$

and if B also consists of a single point b we write

$$\delta(A,B) = \delta(a,b) = d(a,b).$$

It follows easily from the definition that

$$\delta(A,B) = \delta(B,A) \ge 0$$

for all A and B in $B(X)$.

If F is a mapping of X into $B(X)$ we say that a point z is a fixed point of F if z is in Fz, see Kaulgud & Pai (1975).

We now prove the following theorem.

Theorem. Let F be a mapping of a complete metric space (X,d) into $B(X)$ and I a continuous mapping of X into itself satisfying the inequality

$$\delta(Fx,Fy) \leq c\cdot\max\{d(Ix,Iy),\ \delta(Ix,Fx),\ \delta(Iy,Fy),\ \delta(Ix,Fy),\ \delta(Iy,Fx)\} \tag{1}$$

for all x, y in X, where $0 \leq c < 1$. If F and I commute and the range of I contains the range of F, then F and I have a unique common fixed point z and further, $Fz = \{z\}$.

Proof. Let x_0 be an arbitrary point in X and choose a point y_1 in the set $Fx_0 = Y_1$. Since the range of I contains the range of F there exists a point x_1 with $Ix_1 = y_1$. In general, having chosen the point x_{n-1}, choose a point y_n in the set $Fx_{n-1} = Y_n$ and then a point x_n with $Ix_n = y_n$.

Let us now suppose that the sequence of real numbers $\{\delta(Fx_n,Fx_1)\}$ is unbounded. Then there exists an integer n such that

$$(1-c)\delta(Fx_n,Fx_1) > c\delta(Fx_1,Fx_0)$$

and

$$\delta(Fx_n,Fx_1) > \max\{\delta(Fx_r,Fx_1):\ 0 \leq r < n\}. \tag{2}$$

These inequalities imply that for $r = 0, 1, \ldots, n$

$$c\delta(Fx_r,Fx_0) \leq c[\delta(Fx_r,Fx_1) + \delta(Fx_1,Fx_0)] < \delta(Fx_n,Fx_1)$$

and so

$$\delta(Fx_n,Fx_1) > c\cdot\max\{\delta(Fx_r,Fx_0):\ 0 \leq r \leq n\}. \tag{3}$$

We now prove by induction that

$$\delta(Fx_n,Fx_1) \leq c^k\cdot\max\{\delta(Fx_r,Fx_s):\ 1 \leq r,s \leq n\} \tag{4}$$

for $k = 1, 2, \ldots$. Using inequality (1) we have

$$\begin{aligned}\delta(Fx_n,Fx_1) &\leq c\cdot\max\{d(Ix_n,Ix_1),\ \delta(Ix_n,Fx_n),\ \delta(Ix_1,Fx_1),\ \delta(Ix_n,Fx_1),\ \delta(Ix_1,Fx_n)\}\\ &\leq c\cdot\max\{\delta(Fx_{n-1},Fx_0),\ \delta(Fx_{n-1},Fx_n),\ \delta(Fx_0,Fx_1),\ \delta(Fx_{n-1},Fx_1),\\ &\qquad \delta(Fx_0,Fx_n)\}.\end{aligned}$$

Using inequalities (2) and (3), this inequality reduces to

$$\delta(Fx_n,Fx_1) \leq c\delta(Fx_{n-1},Fx_n).$$

Inequality (4) therefore holds when $k = 1$.

Now assume that inequality (4) holds for some k. Then

$$\begin{aligned}\delta(Fx_n,Fx_1) &\leq c^k\cdot\max\{\delta(Fx_r,Fx_s):\ 1 \leq r,s \leq n\}\\ &\leq c^{k+1}\cdot\max\{\delta(Fx_{r-1},Fx_{s-1}),\ \delta(Fx_{r-1},Fx_r),\ \delta(Fx_{s-1},Fx_s),\\ &\qquad \delta(Fx_{r-1},Fx_s),\ \delta(Fx_{s-1},Fx_r):\ 1 \leq r,s \leq n\}\\ &\leq c^{k+1}\cdot\max\{\delta(Fx_r,Fx_s):\ 0 \leq r,s \leq n\}.\end{aligned}$$

Using inequality (3), this inequality reduces to

$$\delta(Fx_n,Fx_1) \leq c^{k+1}\cdot\max\{\delta(Fx_r,Fx_s):\ 1 \leq r,s \leq n\}$$

and inequality (4) now follows by induction.

Letting k tend to infinity in inequality (4) it now follows that

$$\delta(Fx_n,Fx_1) = 0,$$

contradicting inequality (2). Our assumption that the sequence $\{\delta(Fx_n,Fx_1)\}$ is unbounded is therefore false. It follows that

$$M = \sup\{\delta(Fx_r,Fx_s): r,s = 0,1,2,\ldots\}$$
$$\leq \sup\{\delta(Fx_r,Fx_1)+\delta(Fx_1,Fx_s): r,s = 0,1,2,\ldots\}$$

is finite.

Now for abritrary $\varepsilon > 0$ choose an integer N such that

$$c^N M < \varepsilon$$

Using inequality (1) for $m,n > N$ we have

$$\begin{aligned}\delta(Fx_m,Fx_n) &\leq c\cdot\max\{\delta(Fx_{m-1},Fx_{n-1}),\ \delta(Fx_{m-1},Fx_m),\ \delta(Fx_{n-1},Fx_n),\\ &\qquad \delta(Fx_{m-1},Fx_n),\ \delta(Fx_{n-1},Fx_m)\}\\ &\leq c\cdot\max\{\delta(Fx_r,Fx_s),\ \delta(Fx_r,Fx_{r'}),\ \delta(Fx_s,Fx_{s'}):\\ &\qquad m-1 \leq r,r' \leq m;\ n-1 \leq s,s' \leq n\}\\ &\leq c^2\cdot\max\{\delta(Fx_r,Fx_s),\ \delta(Fx_r,Fx_{r'}),\ \delta(Fx_s,Fx_{s'}):\\ &\qquad m-2 \leq r,r' \leq m;\ n-2 \leq s,s' \leq n\}\\ &\leq c^N\cdot\max\{\delta(Fx_r,Fx_s),\ \delta(Fx_r,Fx_{r'}),\ \delta(Fx_s,Fx_{s'}):\\ &\qquad m-N \leq r,r' \leq m;\ n-N \leq s,s' \leq n\}\\ &\leq c^N M < \varepsilon.\end{aligned}$$

Thus if z_n is an arbitrary point in Fx_n for $n=1,2,\ldots$ we have

$$d(z_m,z_n) \leq \delta(z_m,Fx_n) \leq \delta(Fx_m,Fx_n) < \varepsilon$$

for $m,n > N$. It follows that the sequence $\{z_n\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X. Since I is continuous, the sequence $\{Iz_n\}$ converges to Iz. Further, there exists $\eta > 0$ such that $d(Iz_m,Iz_n) < \varepsilon$ whenever $d(z_m,z_n) < \eta$, since I is continuous, however each point z_n is chosen in Fx_n. But there exists an integer N' such that $d(z_m,z_n) < \eta$ if $m,n \geq N'$ and so $d(Iz_m,Iz_n) < \varepsilon$ if $m,n \geq N'$. Since z_n is an arbitrary point in Fx_n, it follows that $\delta(Iz_m,IFx_n) \leq \varepsilon$ for $m,n \geq N'$. In particular the point Ix_{n+1} is in Fx_n and so the sequence $\{Ix_n\}$ converges to z, the sequence $\{I^2x_n\}$ converges to Iz and

$$\delta(Ix_m,Fx_n) < \varepsilon,\ \delta(I^2x_m,IFx_n) < \varepsilon$$

for $m,n > \max\{N,N'\}$.

Now

$$\begin{aligned}d(I^2x_{n+1},Ix_{n+1}) &\leq \delta(IFx_n,Fx_n)\\ &= \delta(FIx_n,Fx_n)\\ &\leq c\cdot\max\{d(I^2x_n,Ix_n),\ \delta(I^2x_n,FIx_n),\ \delta(IX_n,Fx_n),\ \delta(I^2x_n,Fx_n),\ \delta(Ix_n,FIx_n)\}\\ &\leq c\cdot\max\{d(I^2x_n,Ix_n),\ \varepsilon,\ \varepsilon, d(I^2x_n,Ix_n)+\varepsilon,\ d(IX_n,I^2x_n)+\varepsilon\}\\ &= c[d(I^2x_n,Ix_n)+\varepsilon]\end{aligned}$$

for $n > \max\{N,N'\}$. Letting n tend to infinity we have

$$d(Iz,z) \leq c[d(Iz,z)+\varepsilon]$$

and it follows that

$$Iz = z.$$

Further, since z_n is in Fx_n

$$\begin{aligned}\delta(Fz,z_n) &\leq \delta(Fz,Fx_n)\\ &\leq c\cdot\max\{d(Iz,Ix_n), \delta(Iz,Fz), \delta(Ix_n,Fx_n), \delta(Iz,Fx_n), \delta(Ix_n,Fz)\}\\ &= c\cdot\max\{d(z,Ix_n), \delta(z,Fz), \delta(Ix_n,Fx_n), \delta(z,Fx_n), \delta(Ix_n,Fz)\}\\ &\leq c\cdot\max\{\varepsilon, \delta(z,Fz), \varepsilon, \varepsilon, \delta(Ix_n,Fz)\}\end{aligned}$$

for $n > \max\{N,N'\}$. Letting n tend to infinity we have

$$\delta(Fz,z) \leq c\cdot\max\{\varepsilon, \delta(Fz,z)\}$$

and it follows that

$$\delta(Fz,z) = 0.$$

Thus, z is in Fz and $Fz=\{z\}$. The point z is therefore a common fixed point of F and I.
Now suppose that F and I have a second common fixed point w. Then

$$\begin{aligned}\delta(w,Fw) &\leq \delta(Fw,Fw)\\ &\leq c\cdot\max\{d(Iw,Iw), \delta(Iw,Fw)\}\\ &= c\cdot\delta(w,Fw)\end{aligned}$$

so that

$$\delta(w,Fw) = 0.$$

It follows that $Fw=\{w\}$ and so

$$d(z,w) = \delta(Fz,Fw) \leq cd(z,w).$$

The uniqueness of z follows. This completes the proof of the theorem.
The following corollaries follow easily.

Corollary 1. Let T be a mapping and let I be a continuous mapping of a complete metric space (X,d) into itself satisfying the inequality

$$d(Tx,Ty) \leq c\cdot\max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), d(Ix,Ty), d(Iy,Tx)\}$$

for all x, y in X, where $0 \leq c < 1$. If T and I commute and the range of I contains the range of T, then T and I have a unique common fixed point z.
The result of this corollary was given by Das & Vishwanatha Naik (1979).

Corollary 2. Let F be a mapping of a complete metric space (X,d) into $B(X)$ satisfying the inequality

$$\delta(Fx,Fy) \leq c\cdot\max\{d(x,y), \delta(x,Fx), \delta(y,Fy), \delta(x,Fy), \delta(y,Fx)\}$$

for all x, y in X, where $0 \leq c < 1$. Then F has a unique fixed point z and further, $Fz=\{z\}$.

This corollary generalises a result given in Fisher (1981) where the extra condition that F maps a bounded set into a bounded set was given.

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النقط الثابتة للتطبيقات والتطبيقات مجموعية القيمة

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خلاصة

ليكن (X,d) فضاء مقاسيا تاما ولتكن $B(X)$ مجموعة كل المجموعات الجزئية المحدودة وغير الخالية من المجموعة X . لنعرف الدالة $\delta\,(A,B)$ بالشكل

$$\delta(A,B) = \sup\{d(a,b)\colon a \in A,\ b \in B\}.$$

حيث A و B في $B(X)$.
إذا كان F تطبيقا للفضاء X في $B(X)$ عندئذ نقول بأن z هي نقطة ثابتة لـ F إذا كانت z في Fz .
لنفرض أن F هو تطبيق للفضاء X في $B(X)$ وليكن I تطبيقا مستمرا للفضاء X في نفسه . لنفرض أيضا أن F و I متبادلان فيما بينهما وأن مدى I يحتوي مدى F .
نبرهن هنا أنه إذا تحققت المتباينة

$$\delta(Fx,Fy) \le c\cdot\max\{d(Ix,Iy),\ \delta(Ix,Fx),\ \delta(Iy,Fy),\ \delta(Ix,Fy),\ \delta(Iy,Fx)\}$$

من أجل جميع y,x في X حيث $0 \le c < 1$ ، عندئذ فإن لـ I,F نقطة ثابتة مشتركة وحيدة Z ، كما يكون $Fz = \{z\}$

وقد تم الحصول على نتيجتين : الأولى باستبدال F بتطبيق T للفضاء X في نفسه والثانية بجعل I التطبيق المحايد على X .