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# On the smallest non-Hamiltonian locally Hamiltonian graph

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## ABSTRACT

We show that the smallest non-Hamiltonian, connected locally Hamiltonian graph is of order 11 and size 27.

## 1. INTRODUCTION

Let $G=(V,E)$ be a finite undirected graph of order $n$ which contains no loops and no parallel edges. The subgraph of $G$ induced by the set $N(x,G)=\{y:xy\in E\}$ of neighbours of a vertex $x$ in $G$ is denoted by $G/x$. The degree of $x$ in $G$ is denoted by $d(x,G)$.

Following Skupień (1965, 1966), $G$ is called an LH (locally Hamiltonian) graph if, for every $x$ in $V$, the graph $G/x$ is non-empty and Hamiltonian.

The investigations in this paper were prompted by the statement without proof in Skupień (1965, p. 616) which can be stated as follows:

*1.1. Theorem.* The graph $F_0$ shown in Fig. 1 is the smallest (i.e. of minimum order and minimum size) connected LH graph which is not Hamiltonian.

*1.2. Remark.* It is known that $F_0$ is the unique non-Hamiltonian maximal planar graph of order 11. Hence owing to Skupień's result (Theorem 2.5 below) $F_0$ is the only smallest connected LH non-Hamiltonian graph.

The main result of this paper is that the statement is correct. The proof is by contradiction. We suppose that $G_0$ of order $n\leqslant 10$ is a smallest connected LH and non-Hamiltonian graph. Next we show that $n>7$ and $G_0$ contains a circuit $C$ of order $n-1$ which avoids a vertex $x_0$ of degree 3 in $G_0$. We consider all distributions of vertices of $G_0$ along $C$ and show that $G_0$ does not exist.

Note that the graph $F_0$ was found by Reynolds (1932) (see also Whitney (1931), and Skupień (1965, Fig. 1)) as an example of the smallest non-Hamiltonian simplicial 3-polytopal (i.e. maximal planar) graph, but the proof of this property of $F_0$ appears only in Barnette & Jucovič (1970). Hence a connected LH graph need not be

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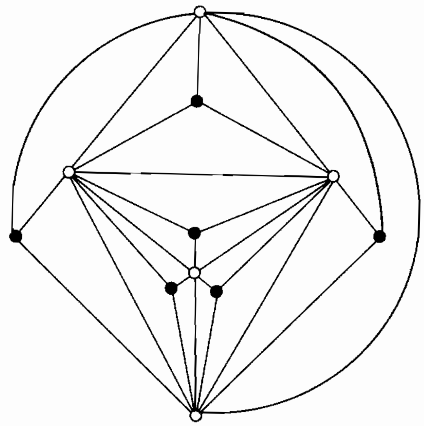

**Fig. 1.** The graph $F_0$.

Hamiltonian because maximal planar graphs of order greater than 3 are clearly connected LH. This was observed in Skupień (1965), but was overlooked by Oberly & Sumner (1979). Though infinite families of non-Hamiltonian maximal planar graphs were known as early as 1963 (Grünbaum & Walther 1973), the paper by Oberly & Sumner (1979) has stimulated new interest in connected LH non-Hamiltonian graphs.

In order to shorten the presentation of results we either omit or outline some proofs.

## 2. GENERAL RESULTS

In this section we report some general results related to LH graphs.

*2.1. Lemma.* A connected LH graph is 3-connected.

For the proof see Theorem 1 in Skupień (1968, pp. 255 & 258).

The following results (Corollary 2.2–2.5) are borrowed from Skupień (1966).

*2.2. Corollary.* If $G$ is an LH graph then $\delta(G) \geqslant 3$.

*2.3. Corollary.* Four is the smallest order of an LH graph.

*2.4. Lemma.* If $G$ is connected LH and $x$ is a vertex of degree 3, then each neighbour of $x$ in $G$ has a degree $\geqslant 4$ provided $G \neq K_4$.

*2.5. Theorem.* A connected and LH graph of order $n$ ($\geqslant 4$) contains at least $3n-6$ edges and it is a maximal planar graph iff it contains exactly $3n-6$ edges.

*2.6.* Observe that the graph $F_0$ presented in Fig. 1 is an LH graph which is non-Hamiltonian and has degree sequence $(6 \times 3, 2 \times 6, 3 \times 8)$ with maximum degree $\Delta(F_0) = 8 = n-3$ $(n = 11)$.

*2.7. Lemma.* If $G$ is an LH graph of order $n$ with $\Delta(G) \geqslant n-2$, then $G$ is Hamiltonian.

In fact, a Hamiltonian circuit of $G$ can be obtained from that of $G/x_0$ if $d(x_0, G) = \Delta(G)$.

*Remark*. The condition $\Delta(G) \geqslant n-2$ in Lemma 2.7 can not be weakened (by Lemma 2.6).

*2.8. Lemma*. If $G$ is a connected LH graph with $\Delta(G) \leqslant 4$, then $G$ is Hamiltonian. Otherwise a circuit of maximum length could be extended.

*2.9. Lemma*. If $G$ is a connected LH graph with order $n \leqslant 7$, then $G$ is Hamiltonian.

*2.10. Construction*. (cf. Skupień 1966, pp. 196–7). Let $G_0$ be a connected LH graph of order at least 5 and let $x_0$ be a vertex of $G_0$ such that $3 \leqslant d(x_0, G_0) \leqslant 4$. Define

$$G' = \begin{cases} G_0 - x_0 \text{ if } G_0/x_0 \text{ is a complete graph (so in particular if } d(x_0, G_0) = 3); \\ (G_0 - x_0) \cup e \text{ otherwise,} \end{cases}$$

where $e$ is a new edge whose both end-vertices are non-adjacent neighbours of $x_0$.

*2.11. Claim*. $G'$ is a connected LH graph (Skupień 1966).

*2.12. Claim*. If $G'$ has a Hamiltonian circuit, say $C$, and $G_0$ is not Hamiltonian, then in $G_0$ no two neighbours of $x_0$ are neighbours on $C$.

*2.13. Corollary*. If $G_0$ is non-Hamiltonian and $G'$ is Hamiltonian, then the edge $e$ introduced in Construction 2.10 does not belong to any Hamiltonian circuit of $G'$.

*2.14. Claim*. No two adjacent vertices of $G_0$ are of degree 3.

## 3. THE STRUCTURE OF NON-HAMILTONIAN GRAPH $G_0$

In this section, we assume that $G_0$ is not Hamiltonian and $G'$ is Hamiltonian where $G'$ and $G_0$ are as described in Construction 2.10. Before we can give some more properties of $G_0$, we shall need some conventions.

*3.1.* Denote by $C$ the Hamiltonian circuit of $G'$. Let $x_i$ ($i = 1, 2, \ldots, d(x_0, G_0)$) be neighbours of $x_0$ in $G_0$ taken in an order of their appearance on $C$. Let $x_i$ & $x_j$ be successive neighbours of $x_0$ on the circuit $C$, where for instance $\{i,j\} \in \{\{1,2\},\{2,3\},\{3,1\}\}$ if $d(x_0, G_0) = 3$. By Claim 2.12, vertices $x_i$ and $x_j$ are separated on $C$ by vertices non-adjacent to $x_0$. Denote by $x_{ij}$ and $x_{ji}$ (possibly $x_{ij} = x_{ji}$) vertices adjacent to $x_i$ and $x_j$, respectively, which belong to the same component of $C - N(x_0, G_0)$. Because $C$ is a circuit of maximum length in $G_0$ we have the following four observations.

*3.2. Claim*. $x_{ij}x_{jk} \notin E(G_0)$ for any admissible, mutually different subscripts $i, j$ and $k$.

*3.3. Claim*. If $d(x_0, G_0) = 4$ and $x_{ij} = x_{ji}$, then $x_{ij}x_{k\ell} \notin E(G_0)$ for any admissible subscripts such that $\{i, j, k, \ell\} = \{1,2,3,4\}$.

*Proof.* In fact, otherwise if the notation is chosen so that $x_i$ and $x_k$ are separated on $C$ by $x_j$ and $x_\ell$, then the circuit $[x_i, x_0, x_k] \cup C\,[x_k, \ldots, x_j, \ldots, x_{ij}] \cup [x_{ij}, x_{k\ell}] \cup C\,(x_{k\ell}, \ldots, x_\ell, \ldots, x_i]$ would be a Hamiltonian circuit of $G_0$, a contradiction.

*3.4. Claim.* If $x_{ij}=x_{ji}$, then no neighbours of $x_{ij}$ are neighbours on $C$.

*Proof.* Suppose $x$ and $y$ are neighbours of $x_{ij}$ such that $xy \in E(C)$. Then the circuit $(C-xy-x_{ij}) \cup [x,x_{ij},y] \cup [x_i,x_0,x_j]$ would be a Hamiltonian circuit of $G_0$, a contradiction.

*3.5. Claim.* If $x_{ij}x$ is a diagonal of $C$ in $G_0$, then $x_{ji}$ is not adjacent to any neighbour of $x$ on $C$.

## 4. $G_0$ IS A SMALLEST NON-HAMILTONIAN GRAPH

From 2.6 and Theorem 2.5, it follows that in order to prove the main result (Theorem 1.1) it is sufficient to show that the following supposition leads to a contradition.

Suppose that $G_0$ is a smallest connected LH non-Hamiltonian graph of order $n \leqslant 10$.

*4.1. Claim.* $3 \leqslant \delta(G_0) \leqslant 4$.

*Proof.* In fact, $\delta(G_0) \geqslant 3$ by Corollary 2.2; and $\delta(G_0) \leqslant 4$ since $n \geqslant 2\delta(G_0)+1$, by a well-known theorem of Dirac (1952).

*4.2. Claim.* $G_0$ contains no vertex of degree 4.

*Proof.* Let $x_0$ be a vertex of $G_0$ such that $d(x_0, G_0)=4$. Construct $G'$ as in Construction 2.10. Then, by minimality of the order of $G_0$ and by Claim 2.11 and Corollary 2.13, $G'$ has a Hamiltonian circuit by edges of $G_0$. Now, by Claim 2.12, $G_0$ must be of order 9 or 10. Consequently, by 3.1, $x_{ij}=x_{ji}$ with one exception, say of $\{i,j\}=\{1,2\}$, iff $n=10$. Now, by Corollary 2.2 and Claims 3.2 and 3.3, we have that $x_{34}$ $(=x_{43})$ is adjacent to a vertex say $x_1$, from among $x_1$ and $x_2$. Now we can see that no two neighbours $x_0$, $x_{12}$, $x_{34}$, and $x_{14}$ of $x_1$ belong to the same component of $(G_0/x_1)-\{x_2,x_3,x_4\}$. This means that deleting common neighbours of $x_0$ and $x_1$ from $G_0/x_1$ results in a subgraph with the number of components greater than that of deleted vertices, contrary to Hamiltonicity of $G_0/x_1$, which proves Claim 4.2.

*4.3. Corollary.* $\delta(G_0)=3$ and $8 \leqslant n \leqslant 10$.

In order to obtain our main result it is enough to show that $G_0$ with $n \leqslant 10$ does not exist. According to the above results, in what follows we may assume

*4.4.* $d(x_0, G_0)=3$ and neighbours $x_1, x_2, x_3$ of $x_0$ induce a triangle in $G_0$, which consists of diagonals of $C$, a fixed Hamiltonian circuit of $G_0-x_0$. So the structure of $G_0$ depends on distribution of and connection between the remaining vertices including vertices of the form $x_{ij}$.

*4.5. Claim.* Each subgraph $(G_0/x_j)-\{x_0,x_i,x_k\}$ where $\{i,j,k\}=\{1,2,3\}$ has a Hamiltonian path.

*Proof.* In fact, because $G_0$ is LH the graph $(G_0/x_j)-x_0$ has a Hamiltonian path with end-vertices $x_i$ and $x_k$ which are the only neighbours of $x_0$ in $G/x_j$. Hence the result (Claim 4.5) follows.

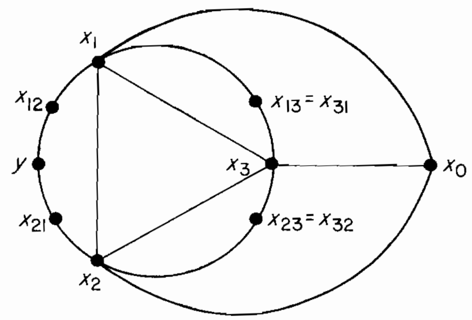


**Fig. 2.**

Now notice that if all vertices of $G_0$ are of the form either $x_\alpha$ with $\alpha=0,1,2,3$ or $x_{\alpha\beta}$ with $\{\alpha,\beta\}\in\{\{1,2\},\{2,3\},\{3,1\}\}$ and, for certain admissible integers $i$ and $j, x_{ij}=x_{ji}$ then, by Claim 3.2, the vertex $x_{ji}$ is adjacent to neighbours of $x_0$ only. Consequently, the vertices $x_{ji}$ and $x_{jk}$ with $\{i,j,k\}=\{1,2,3\}$ belong to different components of $(G/x_j)-\{x_0,x_i,x_k\}$, contrary to Claim 4.5. Hence, by Corollary 4.3 and 4.4, we have

*4.6. Corollary.* $9\leqslant n\leqslant 10$ and if $n=9$ then $G_0$ contains a subgraph isomorphic to that shown in Fig. 2.

Moreover, according to Claims 3.2 and 4.5, we have

*4.7. Corollary.* If $\{i,j,k\}=\{1, 2, 3\}$ and $x_{ij}=x_{ji}$ then each of the neighbours $x_{ji}$ and $x_{jk}$ of $x_j$ is adjacent to another neighbour of $x_j$ different from $x_0,x_i,x_k,x_{ji}$ and $x_{jk}$.

Now consider the case $n=9$. According to Corollary 4.6 we may assume that the notation is chosen as in Fig. 2. From Claim 3.2 we deduce that among vertices not included in $N(x_0,G_0)$ only $y$ can be adjacent to either $x_{13}$ or $x_{23}$. Consequently, by Corollary 4.7 (with $j=1,2,3$), $y$ is of degree $7=n-2$ in $G_0$. Thus, by Lemma 2.7, $G_0$ is Hamiltonian, a contradiction. So, by Corollary 4.6, we have proved the following:

*4.8. Claim.* $n=10$.

Here it will suffice to consider all cases of distribution of six vertices different from $x_i$ $(i=0,1,2,3)$ on three paths in $C$ with end-vertices among $x_1,x_2,x_3$. By Claim 2.12, each of those paths contains an inner vertex and therefore each admissible distribution in question will have three distributions: $1+1+4$, $1+2+3$ and $2+2+2$.

Now we may assume that the notation is chosen so that $G_0$ contains a factor coinciding with one of the following three graphs depicted in Fig. 3.

Note that in Figs 3a and 3b we introduce symbols $y_1,y_2$ and $y$ for corresponding vertices different from $x_i$ and $x_{ij}$ with any admissible $i,j$ and $k$.

*Case of Fig. 3a.* Now by Claim 3.2, $N(x_{13}=x_{31}, G_0)\subseteq\{y_1,y_2,x_2,x_1,x_3\}$ but because of Claim 3.4 only $y_1$ or $y_2$ (not both) can be a neighbour of $x_{13}$. Therefore, by Corollary 2.2 and Claim 4.2, $d(x_{13}, G_0)=3$. Hence, by Claim 4.5 with $j=2$, $x_2x_{13}\notin E(G_o)$.

Similarly, $d(x_{32},G_0)=3$, $x_1x_{32}\notin E(G_0)$ and one of the following three cases occurs.

$(a_1)$ $x_{13}$ and $x_{32}$ have a common neighbour, say $y_1$, among $y_1$ and $y_2$.
$(a_2)$ $x_{13}$ and $x_{32}$ are adjacent to $y_1$ and $y_2$, respectively.
$(a_3)$ $x_{13}$ and $x_{32}$ are adjacent to $y_2$ and $y_1$, respectively.

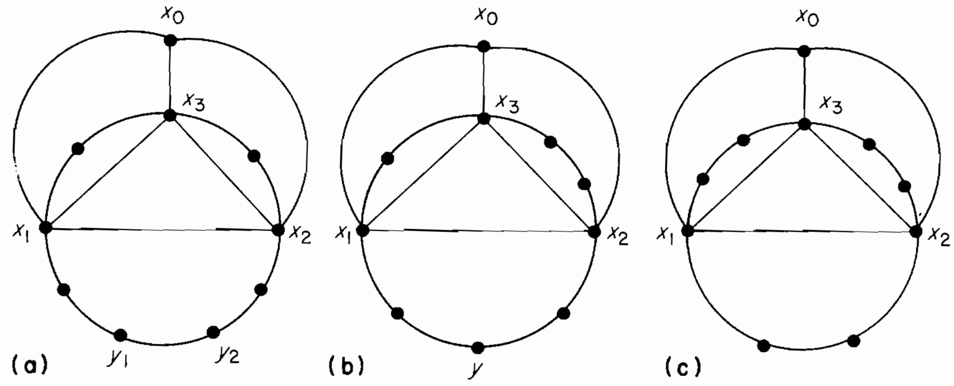


**Fig. 3.**

In case $(a_1)$, by Hamiltonicity of $G_0/x_{13}$ and $G_0/x_{32}$, $y_1$ is adjacent to $x_1$, $x_3$, and $x_2$. Now, because $G_0$ is not Hamiltonian, $x_{12}$ is not adjacent to both $x_{21}$ and $y_2$. Therefore $x_{12}$ is of degree 3 and is adjacent to $x_2$ or $x_3$. Analogously, $x_{21}$ is of degree 3 and is adjacent to $x_1$ or $x_3$. Hence, by Lemma 2.4 and Claim 4.2, $y_2$ is adjacent to $x_1$, $x_2$ and $x_3$. Therefore, one of vertices $x_1$, $x_2$, and $x_3$ is of degree bigger than 7. Hence, by Lemma 2.7, $G_0$ is Hamiltonian, a contradiction.

In case $(a_2)$, because $G_0$ is not Hamiltonian, $x_{12}$ is not adjacent to $x_{21}$, $y_2$, and $x_2$ as well as to $x_{31}$ and $x_{32}$ (Claim 3.2). Thus $x_{12}$ is adjacent to $x_3$, $x_1$, and $y_1$ only. Similarly, $x_{21}$ is adjacent to $x_3$, $x_2$, and $y_2$ only. Now, by Hamiltonicity of $G_0/x_{12}$ and $G_0/x_{21}$, $x_3$ is adjacent to $y_1$ and $y_2$. Hence $d(x_3,G_0)=9$, contrary to Lemma 2.7 and non-Hamiltonicity of $G_0$.

Case $(a_3)$ can be dealt with similarly.

*Case of Fig. 3b.* By Claims 3.2 and 4.2, $d(x_{13},G_0)=3$ and $x_{13}$ is adjacent to exactly one of $x_2$ and $y$. But, because of Claim 4.5 with $j=2$, $x_{13}$ has to be adjacent to $y$. Consequently, $y$ is adjacent to $x_1$ and $x_3$ because $G/x_{13}$ has to be Hamiltonian.

Now, because $G_0$ is not Hamiltonian, $x_{12}$ is not adjacent to both $x_{21}$ and $x_{32}$. Hence, by Corollary 2.2 and Claims 3.2 and 4.2, $d(x_{12},G_0)=3$ and $x_{12}$ is adjacent to exactly one of the vertices $x_2$ and $x_3$. Now, by Claim 4.5 with $j=3$, $x_{12}$ must be adjacent to $x_2$. Hence, $yx_2 \in E(G_0)$ because $G_0/x_{12}$ is Hamiltonian.

Similarly, $x_{21}x_{23} \notin E(G_0)$, $d(x_{21},G_0)=3$ and $x_{21}$ is adjacent to exactly one of the vertices $x_3$ and $x_1$.

Further, by Claims 2.14 and 4.2, one of the vertices $x_{23}$ and $x_{32}$ is of degree at least 5. However, $d(x_{32},G_0)=3$ because otherwise $x_2x_{32} \in E(G_0)$ and $G_0$ would be Hamiltonian by Lemma 2.7. Therefore $N(x_{23},G_0)=\{x_{32},x_3,x_2,x_1,y\}$. Hence $\Delta(G_0) \geqslant 8$, contrary to Lemma 2.7 and non-Hamiltonicity of $G_0$.

*Case of Fig. 3c.* For all admissible values of $i$ and $j$, one of the adjacent vertices $x_{ij}$ and $x_{ji}$ is of degree not less than 5 because of Lemma 2.4 and Claim 4.2. Let $x_{12}$ be a vertex of maximum degree among vertices of the form $x_{ij}$. We claim that $d(x_{12},G_0)<6$. To prove this, note that $x_{12}$ can not be adjacent to $x_0$, $x_{23}$ and $x_{31}$, by Claim 3.2. Hence $d(x_{12}, G_0) \leqslant 6$. Now, if $d(x_{12},G_0)=6$, then by Claims 3.2 and 3.5, $d(x_{21},G_0)=2$, a contradic-

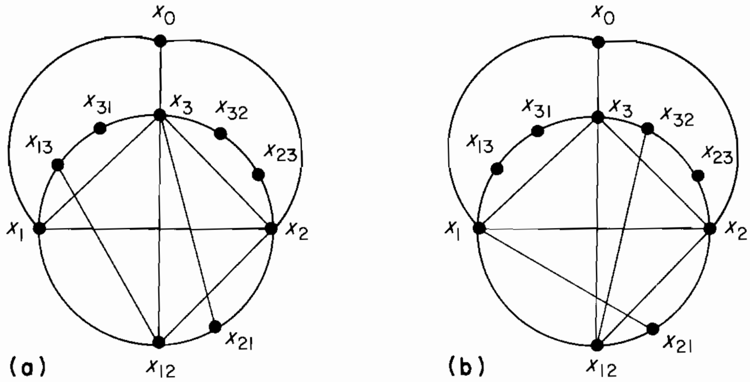


**Fig. 4.**

tion. Hence the degree of each vertex denoted by $x_{ij}$ is either 3 or 5 and at least one of every two vertices $x_{ij}$ and $x_{ji}$ is of degree 5.

Given a vertex $x_{ij}$, let

$$\tilde{N}(x_{ij}) = \{x \in V(G_0) \mid xx_{ij} \text{ is a diagonal of } C\} .$$

Now consider the case where $d(x_{12}, G_0) = 5$. There are four possible sets $\tilde{N}(x_{12})$. For each possible set $\tilde{N}(x_{12})$ there must exist a (non-empty) set $\tilde{N}(x_{21})$ such that Corollary 4.3 and Claims 3.2 and 3.5 are satisfied. Therefore, $\tilde{N}(x_{12})$ can be either $\{x_2, x_3, x_{13}\}$ or $\{x_2, x_{32}, x_3\}$ only. Then the corresponding sets $\tilde{N}(_{21})$ are $\{x_3\}$ and $\{x_1\}$, respectively, which we represent in Figs 4a and 4b. Notice now that $\{3,5\}$ is the degree set of each pair of vertices $x_{ij}$ and $x_{ij}$.

Now, by Claims 3.2 and 3.5, $G_0$ can contain at most one edge $e$ which connects two vertices from the set $A = \{x_{13}, x_{31}, x_{32}, x_{23}\}$ and does not appear in Fig. 4. Moreover, $e \in \{x_{13}x_{23}, x_{31}x_{32}\}$. But we know that among the vertices in $A$ in each case, there are two vertices of degree 5 and the other two of degree 3. Hence $G_0$ contains at least five more edges which are incident to three neighbours of $x_0$ and do not appear in Fig. 4. So, by the pigeonhole principle, one of the neighbours of $x_0$ is of degree at least 8 which, by Lemma 2.7, implies that $G_0$ is Hamiltonian, a contradiction.

## PROBLEMS

1. Generalize Goodey's result (Goodey 1972) on non-traceable maximal planar graphs by showing that 14 is the smallest order of a connected non-traceable locally Hamiltonian graph.
2. Is a regular locally Hamiltonian graph different from $K_4$, 4-connected?
3. Does there exist a locally Hamiltonian regular graph which is not Hamiltonian?

## REMARK

There is a variety of problems similar to that answered in our paper as well as to Problem 1. For instance, is 9 the minimum order of a connected non-traceable locally traceable graph? For other possible pairs of global and local properties see Skupień (1976). Some other problems are stated in Skupień (1969).

## ACKNOWLEDGMENT

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# حول البيان غير الهاملتوني الأصغر الهاملتوني محليا

شاندرا موهان باريك وچسواف سكوبين
قسم الرياضيات بجامعة الكويت

## خلاصة

نبين ان البيان غير الهاملتوني الأصغر المتصل ، الهاملتوني محليا ذو ترتيب ١١ وحجم ٢٧ .