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On universally maximal and minimal ideals

ANWAR T. AL-LAHHAM

Department of Mathematics, Faculty of Science,
University of Damascus, Syria

ABSTRACT

In this paper we prove that a left ideal L of a semigroup S is the universally minimal left ideal of S if and only if $LH = HL = L$ for all left ideals of S.

A semigroup S has both the universally minimal left ideal L and the universally minimal right ideal R if and only if S has the kernel T_* which is a group. In this case $L_* = R_* = T_*$. A two-sided ideal A of a monoid S, with no increasing elements, is the universally maximal left, right, and two-sided ideal of S if and only if $S - A$ is a subgroup of S.

If a semigroup S with L_*, R_*, L^* and R^* has the property $L^* = R^* = L_*$ then either S is the union of two disjoint groups or L_* is a singleton having a non-idempotent e.

INTRODUCTION

We shall follow the notation and terminology of Clifford & Preston (1961, 1967) for all concepts not defined in this paper. Throughout this paper all semigroups under consideration are without zero. We shall state theorems with respect to 'left'. The dual case (i.e. 'right') is similar.

Definition. A two-sided (left, right) ideal of a semigroup S is called minimal if it does not properly contain any two-sided (left, right) ideal of S.

A two-sided (left, right) ideal of a semigroup S is called universally minimal if it is contained in every two-sided (left, right) ideal of S.

A two-sided (left, right) ideal of a semigroup S is called universally maximal if it is proper and contains every proper two-sided (left, right) ideal of S.

An element x of a semigroup S is called right (left) increasing element of S if there exists a proper subset D of S such that $Dx = S$ ($xD = S$).

Notation. In this paper we shall denote the ensemble of all left (right, two-sided) ideals of an arbitrary semigroup S by $\mathscr{A}$ ($\mathscr{B}$, $\mathscr{D}$).
$\mathscr{A}$ is a semigroup with respect to the opertion defined by:

$$L_1, L_2 \in \mathscr{A} \qquad L_1 L_2 = \{xy\colon x \in L_1,\ y \in L_2\}$$

Similarly $\mathscr{B}$ and $\mathscr{D}$ are semigroups.

L_* (R_*, T_*) denotes the universally minimal left (right, two-sided) ideal of a semigroup S, and L^* (R^*, T^*) denotes the universally maximal left (right, two-sided) ideal of S.

The term 'universally minimal' is the synonym of 'least' or 'smallest', that is

$$L_* = \bigcap_{H \in \mathscr{A}} H,\ R_* = \bigcap_{H \in \mathscr{B}} H,\ T_* = \bigcap_{H \in \mathscr{D}} H.$$

Of course L_*, R_*, $T_* \neq \emptyset$ are required.

If L (R) is a minimal left (right) ideal of a semigroup S, then L (R) is a right (left) zero element of $\mathscr{A}$ ($\mathscr{B}$) but the converse is not true in general, since the union of any collection of minimal left (right) ideals of S is a right (left) zero of $\mathscr{A}$ ($\mathscr{B}$) but not a minimal left (right) ideal of S.

It is obvious that a universally minimal left (right, two-sided) ideal is a minimal left (right, two-sided) ideal. We note that a minimal two-sided ideal is a universally minimal two-sided ideal, equivalently, the smallest. In fact, let T be a minimal ideal of a semigroup S and D be any ideal of S. Then $T \cap D$ is a non-empty ideal of S since $TD \subseteq T \cap D$; $T \cap D \subseteq T$ implies $T \cap D = T$ by the minimality of T. Hence $T \subseteq D$. Thus T is least. The least ideal is also called the kernel. Unlike minimal left (right) ideals, K is the kernel of S if and only if K is the zero element of $\mathscr{D}$.

Theorem 1. Let L be a left ideal of a semigroup S. Then $L = L_*$ if an only if $LH = HL = L$ for all left ideals H of S.

Proof. (i) Necessity. Assume $L = L_*$. Then $L \subseteq HL$ follows from the fact that HL is a left ideal of S. On the other hand $HL \subseteq L$ since L is a left ideal of S. Hence $HL = L$. Next, since $LH \subseteq H$ for all left ideals H of S, LH $\subseteq L$. But $LH = L$ since $L = L_*$.

(ii) Sufficiency. Since $L = LH \subseteq H$ for all left ideals H of S, we have $L = L_*$.

Assume a semigroup S has a minimal left ideal and let $\{Li\colon i \in I\}$ be the set of all minimal left ideals of S.

Lemma 1. A minimal left ideal of a semigroup S is contained in any two-sided ideal of S.

Proof. Let L be a minimal left ideal of S and T a two-sided ideal of S. Then $L = TL \subseteq TS \subseteq T$.

Lemma 2. Let L be a minimal left ideal of S and R a minimal right ideal of S. Then

$$\bigcup_{i \in I} L_i = LS = LR = T_*.$$

Proof. Let x be an element of S and y be an element of Lx. There exists $z \in L$ such that $y = zx$. Then

$$Sy \bigcup \{y\} = Szx \bigcup \{zx\} = (Sz \bigcup \{z\})x = Lx.$$

It follows that Lx is a minimal left ideal of S. So we have

$$L_iS = \bigcup_{x \in S} L_ix \subseteq \bigcup_{i \in I} L_i \qquad \text{for every } i \in I$$

and

$$(\bigcup_{i \in I} L_i)S \subseteq \bigcup_{i \in I} (L_iS) \subseteq \bigcup_{i \in I} L_i$$

We have shown that

$$\bigcup_{i \in I} L_i$$

is a two-sided ideal of S. By Lemma 1, it is contained in every ideal of S. Therefore

$$\bigcup_{i \in I} L_i = T_*.$$

By a similar argument

$$LR \subseteq LS = \bigcup_{x \in S} Lx = \bigcup_{i \in I} L_i = T_*.$$

Since LR is a two-sided ideal of S, $LR = LS = T_*$.

In consequence of Lemma 2, if S has a minimal left ideal then S has the kernel.

Proposition 1. If a semigroup S has the universally minimal left ideal L_* then $L_* = T_*$.

Proof. By Theorem 1, $L_*H = HL_* = L_*$ for all left ideals H of S. In particular $L_*S = L_*$, but Lemma 2 yields $L_* = T_*$.

Remark 1. The converse of Proposition 1 does not hold. The following example illustrates that. Let S be the semigroup defined by the table

	a	b	c	d	e	f
a	b	b	e	e	e	b
b	b	b	e	e	e	b
c	f	f	d	d	d	f
d	f	f	d	d	d	f
e	b	b	e	e	e	b
f	f	f	d	d	d	f

Then S has the kernel $K = \{b, d, e, f\}$. But S has neither L_* nor R_*. The semigroup S is an inflation of rectangular band K.

Remark 2. The existence of one of L_* and R_* in a semigroup S does not imply the existence of the other.

Example. Let $\mathscr{F}$ be the semigroup of all transformations of the non-empty set Q, and H be the set of all constant transformations of Q. Then

(1) Every element $\alpha \in H$ constitutes a minimal left ideal of $\mathscr{F}$, and $\mathscr{F}$ has no other minimal left ideals.

To prove that, let α be an element of H, then for every $\psi \in \mathscr{F}$ and for every $x \in Q$, $x\psi\alpha = y\alpha = x\alpha$, that is, $\psi\alpha = \alpha$. So $\mathscr{F}\alpha = \{\alpha\}$ and $\{\alpha\}$ is a minimal left ideal of $\mathscr{F}$.

Now let L be a minimal left ideal of $\mathscr{F}$, and θ be any element of L. Then $x\alpha\theta = \lambda\theta = \lambda' = x\beta$; ($\beta \in H$), which implies that $\alpha\theta = \beta \in H$. But $\alpha\theta \in L$, because L is a left ideal, which implies that $L = \{\beta\}$ by the minimality of L.

(2) $\mathscr{F}$ has the kernel $T_* = H$ by Lemma 2.
So H is a right ideal of $\mathscr{F}$.

(3) H is the universally minimal right ideal of $\mathscr{F}$.

To prove this, let R be any right ideal of $\mathscr{F}$. Then for every $\psi \in R$ and for every $\alpha \in H$, we have $\psi\alpha = \alpha$. But $\psi\alpha \in R$, so $\alpha \in R$, which implies that $H \subseteq R$. So $\mathscr{F}$ has R_* but it has no L_*.

Theorem 2. A semigroup S has both L_* and R_* if and only if S has the kernel T_* which is a group. In this case $L_* = R_* = T_*$.

Proof. (i) Necessity. By Proposition 1, $L_* = T_*$. Similarly $R_* = T_*$. Let x be any element of S, R be any right ideal of S, L be any left ideal of S. Then $T_* x \subseteq Rx \subseteq RS \subseteq R$, $xT_* \subseteq xL \subseteq SL \subseteq L$.
So for every $a \in T_*$, we have $aT_* = T_* a = T_*$.
Therefore T_* is a group.

(ii) Sufficiency. Let L be a left ideal of S. Since $T_* L \subseteq T_* \cap L$, $T_* \cap L$ is a non-empty left ideal of T_*. But T_* is a group; we have $T_* \cap L = T_*$, that is, $T_* \subseteq L$. Hence $T_* = L_*$.
Similarly $T_* = R_*$.

Proposition 2. A semigroup S contains the kernel T_* if and only if

$$\bigcap_{a \in S} SaS \neq \varnothing.$$

In this case, necessarily

$$T_* = \bigcap_{a \in S} SaS$$

Proof. Let

$$D = \bigcap_{a \in S} SaS.$$

(i) Necessity. Since SaS is an ideal of S, $T_* \subseteq SaS$ for all $a \in S$, hence $T_* \subseteq D$, $D \neq \varnothing$.
Let $t \in T_*$. Then $D \subseteq StS \subseteq T_*$. As D is an ideal of S, we have $D = T_*$.

(ii) Sufficiency. Assume $D \neq \varnothing$, D is certainly an ideal of S. Let I be an ideal of S.
For $x \in I$, $D \subseteq SxS \subseteq I$.
Thus D is contained in all ideals I, hence $D = T_*$.

Theorem 3. A two-sided ideal A of a monoid S, with no increasing elements, is the universally maximal left, right, and two-sided ideal of S if and only if $S - A$ is a subgroup of S.

Proof. (i) Necessity. Assume A is the universally maximal left, right and two-sided ideal of S. Then

$$\bar{A} = \{a \in S\colon Sa = aS = S\} \qquad \text{(see Al-Lahham (1974))}$$

Obviously the identity e of the monoid S belongs to $\bar{A}$. If $a \in \bar{A}$ then there exists $a', a'' \in \bar{A}$ such that $aa' = a''a = e$. It follows that $a' = a''$ and $Sa' = a'S = S$. So $a' \in \bar{A}$.
Therefore $\bar{A}$ is a subgroup of S.

(ii) Sufficiency. Assume $\bar{A}$ is a subgroup of S. Let B be any left ideal of S not properly contained in A. Then $b \in B - A$ implies $\bar{A}b \subseteq SB \subseteq B$. But $\bar{A}b = \bar{A}$, since $\bar{A}$ is a subgroup of S and $b \in \bar{A}$. So $\bar{A} \subseteq B$, which implies that e belongs to B. Hence $B \supseteq SB \supseteq S$ and $B = S$.

Therefore A is the universally maximal left ideal of S. Consequently A is the universally maximal right and two-sided ideal of S. (see Al-Lahham (1981)).

Theorem 4. If a semigroup S with L_*, R_*, L^* and R^* has the property $L^* = R^* = L_*$. Then either S is the union of two disjoint groups or L_* is a singleton having a non-idempotent element e.

Proof. Since S has L_* and R_* then S has T_* and moreover $T_* = L_* = R_*$ is a subgroup of S.

Since S has $L^* = R^*$ then S has no increasing elements and moreover it has $T^* = L^* = R^*$ where

$$T^* = \{a \in S\colon a \cup Sa = a \cup aS = S\} \quad \text{(see Al-Lahham (1974))}$$

If $|\bar{T}^*| > 1$, then $\bar{T}^*$ is a subgroup of S and S is the union of two disjoint groups.

If $|\bar{T}^*| = 1$, then $\bar{T}^*$ has only one element e, where either $e \in eS \cap Se$ and e is the identity of S, so $e^2 = e$ and S is the union of two disjoint groups, or $e \in Se \cap eS$ then $e^2 \neq e$.

Example. Let $S = G \cup \{e\}$; $e \notin G$, where G is a group and $e^2 = 1$ (1 is the identity of G). $eg = ge = g$ for every $g \in G$. Then S is a semigroup with

$$T^* = L^* = R^* = T_* = L_* = R_* = G.$$

Corollary. If a monoid S, with no increasing elements, has the property $L_* = R_* = T^*$, then S is the union of two disjoint groups.

Proof. S has $L^* = R^* = T^*$ (see Al-Lahham (1981)) and T^* is a subgroup of S, also S has $T_* = R_* = L_*$, by Proposition 1, which is a group. So S is the disjoint union of two groups.

Example. Let $S = G_1 \cup G_2$; $G_1 \cap G_2 = \varnothing$. Where G_1 and G_2 are two groups, and

$$xy = yx = x \text{ for every } x \in G_1,\ y \in G_2$$

Then S is a semigroup with the property

$$T^* = L^* = R^* = L_* = R_* = T_* = G_2$$

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حول المثل الصغرى والعظمى عموميا

أنور اللحام
قسم الرياضيات بكلية العلوم ، جامعة دمشق ،
سوريا

خلاصة

نبرهن في هذا البحث على أن المثل الأيسر L لنصف الزمرة S هو مثل أصغر أيسر عموميا لـ S إذا وفقط إذا كان $LH=HL=L$ من أجل جميع المثل اليسرى لـ S . كما نبرهن أنه يوجد لنصف الزمرة S مثل أصغر أيسر عموميا L ومثل أصغر أيمن عموميا R إذا وفقط إذا كان لنصف الزمرة S نواة T_* وكانت هذه النواة تشكل زمرة ، وفي هذه الحالة يكون $L_*=R_*=T_*$. ونبين أن المثل A ذا الجانبين للوحدية S التي لا تحتوي على عناصر متزايدة هو مثل أصغر أيمن وأيسر عموميا وذو جانبين لـ S إذا وفقط إذا كانت $S—A$ زمرة جزئية من S .

كما نبين انه إذا كانت نصف الزمرة S مع L_*, R_*, L^*, R^* تحقق الخاصية $L^* = R^* = L_*$ فإن S هي اتحاد زمرتين منفصلتين أو أن L_* هي مجموعة أحادية لها عنصر غير جامد .