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Common fixed-point theorems for mappings and set-valued mappings

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ABSTRACT

Let (X, d) be a complete metric space and let $B(X)$ be the set of all non-empty, bounded subsets of X. The function $\delta(A,B)$ with A and B in X is defined by

$$\delta(A,B) = \sup\{d(a,b) : a\in A,\ b\in B\} \quad .$$

Now let F and G be mappings of X into $B(X)$ and let I and J be mappings of X into itself satisfying the inequality

$$\delta(Fx,Gy) \leqslant c\cdot \max\{d(Ix,Jy),\delta(Ix,Gy),\delta(Jy,Fx)\}$$

for all x,y in X, where $0 \leqslant c < 1$. It is proved that if F and I commute and the mappings G and J commute, if the range of I contains the range of F and the range of J contains the range of G and if F or I and G or J are continuous, then F, G, I and J have a unique common fixed point z.

In Fisher (1981a), mappings S, T, I and J of a complete metric space (X,d) into itself were considered satisfying the inequality

$$d(Sx,Ty) \leqslant c\cdot \max\{d(Ix,Jy),d(Ix,Ty),d(Sx,Jy)\}$$

for all x,y in X, where $0 \leqslant c < 1$. The object of this paper is to consider the replacement of S and T by set-valued mappings F and G of X into $B(X)$, the set of all nonempty, bounded subsets of X.

As in Fisher (1981b), the function $\delta(A,B)$ with A and B in $B(X)$ is defined by

$$\delta(A,B) = \sup\{d(a,b) : a\in A,\ b\in B\} \quad .$$

If A consists of a single point a we write

$$\delta(A,B) = \delta(a,B) \quad .$$

If B also consists of a single point b we write

$$\delta(A,B) = \delta(a,b) = d(a,b) \quad .$$

It follows easily from the definition that

$$\delta(A,B) = \delta(B,A) \geqslant 0,\ \delta(A,B) \leqslant \delta(A,C) + \delta(C,B)$$

for all A,B and C in $B(X)$.

Now let $\{A_n : n=1,2,\ldots\}$ be a sequence of non-empty subsets of X. We say that the sequence $\{A_n\}$ converges to the subset A of X if

(i) each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n=1,2,\ldots,$

(ii) for arbitrary $\varepsilon>0$, there exists an integer N such that

$A_n \subset A_\varepsilon$ for $n>N$, where A_ε is the union of all open spheres with centres in A and radius ε.

A is then said to be the limit of the sequence $\{A_n\}$. The following lemma was proved in Fisher (1981b).

Lemma. If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X,d) which converge to the bounded subsets A and B respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.

Now let F be a mapping of a complete metric space (X,d) into $B(X)$. We say that the mapping F is continuous at the point x in X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$. We say that F is a continuous mapping of X into $B(X)$ if F is continuous at each point x in X. We say that a point z in X is a fixed point of F if z is in Fz.

We now prove the following theorem.

Theorem 1. Let F and G be mappings of a complete metric space (X,d) into $B(X)$ and let I and J be mappings of X into itself satisfying the inequality

$$\delta(Fx,Gy) \leqslant c \cdot \max\{d(Ix,Jy),\delta(Ix,Gy),\delta(Jy,Fx)\} \tag{1}$$

for all x, y in X, where $0 \leqslant c < 1$. If the mappings F and I commute and the mappings G and J commute, if the range of I contains the range of F and the range of J contains the range of G and if F or I and G or J are continuous, then F,G,I and J have a unique common fixed point z. Further, $Fz=Gz=\{z\}$ and z is the unique common fixed point of F and I and of G and J.

Proof. We let $x=x_0$ be an arbitrary point in X and define the sequence $\{x_n\}$ inductively. Having defined the point x_{n-1}, we choose a point x_n with Ix_n in Fx_{n-1} for $n=1,2,\ldots$ This can be done since the range of I contains the range of F.

Let us now assume that the sequence of real numbers $\{\delta(Fx_n,Gx)\}$ is unbounded. Then there exists an integer n such that

$$(1-c)\delta(Fx_n,Gx) > c\delta(Gx,Jx)$$

and

$$\delta(Fx_n,Gx) > \max\{\delta(Fx_r,Gx)\colon 0 \leqslant r < n\} \tag{2}$$

These inequalities imply that

$$\begin{aligned} c\delta(Fx_r,Jx) &\leqslant c[\delta(Fx_r,Gx)+\delta(Gx,Jx)] \\ &< \delta(Fx_n,Gx) \end{aligned}$$

for r$=0,1,\ldots,n$ and so

$$\delta(Fx_n,Gx) > c \cdot \max\{\delta(Fx_r,Jx)\colon 0 \leqslant r \leqslant n\} \tag{3}$$

It follows from inequalities (2) and (3) that

$$\delta(Fx_n,Gx) > c\cdot\max\{\delta(Fx_{n-1},Jx),\delta(Fx_{n-1},Gx),\delta(Fx_n,Jx)\} \quad . \tag{4}$$

However, on using inequality (1) we have

$$\begin{aligned}\delta(Fx_n,Gx) &\leqslant c\cdot\max\{d(Ix_n,Jx),\delta(Ix_nGx),\delta(Jx,Fx_n)\}\\ &\leqslant c\cdot\max\{\delta(Fx_{n-1},Jx),\delta(Fx_{n-1},Gx),\delta(Fx_n,Jx)\}\end{aligned}$$

since Ix_n is in Fx_{n-1}. This contradicts inequality (4) and so the sequence $\{\delta(Fx_n,Gx)\}$ must be bounded.

Similarly let $y=y_0$ be an arbitrary point in X and define the sequence $\{y_n\}$ inductively by choosing a point y_n with Jy_n in Gy_{n-1} for $n=1,2,\ldots$ The sequence $\{\delta(Gy_n,Fy)\}$ will of course be bounded.

Since

$$\delta(Fx_r,Gy_s)\leqslant\delta(Fx_r,Gx)+\delta(Gx,Fy)+\delta(Fy,Gy_s) \quad ,$$

it follows that

$$M=\sup\{\delta(Fx_r,Gy_s)\colon r,s=0,1,2,\ldots\}$$

is finite.

Now for arbitrary $\in>0$ choose an integer N such that

$$c^N M < \in \quad .$$

Using inequality (1) with m, $n>N$ it follows similarly to the proof of Theorem 1 in Fisher (1981a) that

$$\begin{aligned}&\delta(Fx_m,Gy_n)\leqslant\\ &\qquad\leqslant c^N\cdot\max\{\delta(Fx_r,Gy_s)\colon m-N\leqslant r\leqslant m;n-N\leqslant s\leqslant n\}\\ &\qquad\leqslant c^N M<\in\end{aligned}$$

and so

$$\begin{aligned}\delta(Fx_m,Fx_n)&\leqslant\delta(Fx_m,Gy_s)+\delta(Gy_s,Fx_r)\\ &<2\in\end{aligned}$$

for m, n, $s>N$. Thus if z_n is an arbitrary point in Fx_n for $n=1,2,\ldots$ we have

$$d(z_m,z_n)\leqslant\delta(Fx_m,Fx_n)<2\in$$

for m, $n>N$. It follows that the sequence $\{z_n\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X, the point z being independent of the particular choice of each z_n. In particular, the sequence $\{Ix_n\}$ will converge to z and further, the sequence of sets $\{Fx_n\}$ will converge to the set $\{z\}$.

Similarly, if w_n is an arbitrary point in Gy_n for $n=1,2,\ldots$, then the sequence $\{w_n\}$ converges to a point w, the point w being independent of the particular choice of each w_n. In particular, the sequence $\{Jy_n\}$ will converge to w and the sequence of sets $\{Gy_n\}$ will converge to the set $\{w\}$.

Using inequality (1) we have

$$\delta(Fx_n,Gy_n)\leqslant c\cdot\max\{d(Ix_n,Jy_n),\delta(Ix_n,Gy_n),\delta(Jy_n,Fx_n)\} \quad .$$

Letting n tend to infinity and using the lemma we have

$$d(z,w) \leqslant cd(z,w) \quad .$$

Since $c < 1$, it follows that $z = w$.

Now suppose that I is continuous. Then since every sequence $\{z_n\}$, with z_n in Fx_n, converges to z, it follows that $\{I^2x_n\}$ converges to Iz and $\{IFx_n\} = \{FIx_n\}$ converges to $\{Iz\}$. Using inequality (1) we have

$$\delta(FIx_n,Gy_n) \leqslant c \cdot \max\{d(I^2x_n,Jy_n),\delta(I^2x_n,Gy_n),\delta(Jy_n,FIx_n)\} \quad .$$

Letting n tend to infinity and using the lemma we have

$$d(Iz,z) \leqslant cd(Iz,z)$$

and it follows that $Iz = z$.

Further

$$\delta(Fz,Gy_n) \leqslant c \cdot \max\{d(Iz,Jy_n),\delta(Iz,Gy_n),\delta(Jy_n,Fz)\} \quad .$$

Letting n tend to infinity it follows that

$$\delta(Fz,z) \leqslant c\delta(z,Fz)$$

and so $Fz = \{z\}$.

Similarly, the continuity of J implies that $Jz = z$ and $Gz = \{z\}$.

Now suppose that F is continuous. Then the sequence $\{FIx_n\}$ converges to Fz and on using inequality (1) we have

$$\begin{aligned}\delta(FIx_n,Gy_n) &\leqslant c \cdot \max\{d(I^2x_n,Jy_n),\delta(I^2x_n,Gy_n),\delta(Jy_n,FIx_n)\} \\ &\leqslant c \cdot \max\{\delta(FIx_{n-1},Jy_n),\delta(FIx_{n-1},Gy_n),\delta(Jy_n,FIx_n)\}\end{aligned}$$

since Ix_n is in Fx_{n-1} and so I^2x_n is in $IFx_{n-1} = FIx_{n-1}$. Letting n tend to infinity and using the lemma we have

$$\delta(Fz,z) \leqslant c\delta(Fz,z) \quad .$$

It follows that $Fz = \{z\}$.

There must now exist a point u such that

$$Iu = z$$

since the range of I contains the range of F. Using inequality (1) we have

$$\delta(Fu,Gy_n) \leqslant c \cdot \max\{d(Iu,Jy_n),\delta(Iu,Gy_n),\delta(Jy_n,Fu)\}$$

and on letting n tend to infinity we get

$$\begin{aligned}\delta(Fu,z) &\leqslant c \cdot \max\{d(Iu,z),\delta(z,Fu)\} \\ &= c\delta(z,Fu) \quad .\end{aligned}$$

It follows that $Fu = \{z\}$ and so

$$\{z\} = Fz = FIu = IFu = \{Iz\} \quad .$$

Thus $Iz = z$.

Similarly, the continuity of G implies that $Gz = \{z\}$ and $Jz = z$.

We have therefore proved that if F or I and G or J are continuous then

$$Iz = Jz = z, \; Fz = Gz = \{z\}.$$

That z is the unique common fixed point of F and I and of G and J follows easily and the proof is similar to that given in the proof of Theorem 1 in Fisher (1981a).

The corollaries below now follow easily.

Corollary 1. Let F and G be mappings of a complete metric space (X,d) into $B(X)$ satisfying the inequality

$$\delta(Fx,Gy) \leqslant c \cdot \max\{d(x,y),\delta(x,Gy),\delta(y,Fx)\}$$

for all x, y in X, where $0 \leqslant c < 1$. Then F and G have a unique common fixed point z. Further, $Fz = Gz = \{z\}$ and z is the unique fixed point of F and of G.

Corollary 2. Let S and T be mappings of a complete metric space (X,d) into itself satisfying the inequality

$$d(Sx,Ty) \leqslant c \cdot \max\{d(x,y),d(x,Ty),d(y,Sx)\}$$

for all x, y in X, where $0 \leqslant c < 1$. Then S and T have a unique common fixed point z. Further, z is the unique common fixed point of S and of T.

The result of this corollary was given in Fisher (1980).

Theorem 1 of Fisher (1981a) and its corollaries also follow easily as does Theorem 2. It is also easily seen that Theorem 3 of Fisher (1981a) and its corollary hold without the condition that X be bounded.

To see that the conditions of Theorem 1 cannot be weakened, see the examples given in Fisher (1981b).

For an example satisfying the conditions of Theorem 1, let X be the set of real numbers $x \geqslant 0$ with the usual metric. Define mappings F, G, I and J by

$$Fx = [x/4,x/2],\ Gx = [x/2,x],\ Ix = 2x,\ Jx = 4x \quad .$$

Inequality (1) is satisfied with $c = \frac{1}{2}$.

We finally prove an analogous theorem for compact metric spaces.

Theorem 2. Let F and G be continuous mappings of a compact metric space (X,d) into $B(X)$ and let I and J be continuous mappings of X into itself satisfying the inequality

$$\delta(Fx,Gy) < \max\{d(Ix,Jy),\delta(Ix,Gy),\delta(Jy,Fx)\}$$

for all x, y in X for which the right-hand side of the inequality is positive. If the mappings F and I commute and the mappings G and J commute and if the range of I contains the range of F and the range of J contains the range of G, then F, G, I and J have a unique common fixed point z. Further, $Fz = Gz = \{z\}$ and z is the unique common fixed point of F and I and of G and J.

Proof. Suppose first of all that there exists $c < 1$ for which F, G, I and J satisfy inequality (1) for all x, y in X for which the right-hand side of the inequality is positive. Then when the right-hand side of the inequality is zero we must have

$$Fx = Gy = \{Ix\} = \{Jy\}$$

which implies that the left-hand side of the inequality must also be zero. Inequality (1) will therefore be satisfied for all x, y in X and the result follows from Theorem 1.

Now suppose that no such $c < 1$ exists. Then if $\{c_n\}$ is a monotonically increasing sequence of real numbers which converges to one, we can find sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\delta(Fx_n,Gy_n) > c_n \cdot \max\{d(Ix_n,Jy_n),\delta(Ix_n,Gy_n),\delta(Jy_n,Fx_n)\} \tag{5}$$

for $n=1,2,\ldots$ Since X is compact we may assume, by taking subsequences if necessary, that the sequences $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively. Then on letting n tend to infinity in inequality (5) we have, since F, G, I and J are continuous

$$\delta(Fx,Gy) \geqslant \max\{d(Ix,Jy),\delta(Ix,Gy),\delta(Jy,Fx)\} \quad .$$

This is possible only if

$$Fx = Gy = \{Ix\} = \{Jy\}$$

and so

$$F^2x = FIx = IFx = \{I^2x\} \quad .$$

Now suppose that $FIx \neq Ix$. Then

$$\begin{aligned}\delta(FIx,Ix) &= \delta(FIx,Gy)\\ &< \max\{d(I^2x,Jy),\delta(I^2x,Gy),\delta(Jy,FIx)\}\\ &= \delta(FIx,Ix) \quad ,\end{aligned}$$

giving a contradiction. It follows that $Ix=z$ is a fixed point of F and $Fz=\{z\}$. Further

$$\{Iz\} = \{I^2x\} = FIx = Fz = \{z\}$$

and so z is also a fixed point of I.

Similarly, we can show that $Gy=z$ is a common fixed point of G and J and $Gz=\{z\}$.

It is easily proved that z is the unique fixed point of F and I and of G and J. This completes the proof of the theorem.

The corollaries below now follow easily.

Corollary 1. Let F and G be continuous mappings of a compact metric space (X,d) into $B(X)$ satisfying the inequality

$$\delta(Fx,Gy) < \max\{d(x,y),\delta(x,Gy),\delta(y,Fx)\}$$

for all x, y in X for which the right-hand side of the inequality is positive. Then F and G have a unique common fixed point z. Further, $Fz=Gz=\{z\}$ and z is the unique fixed point of F and of G.

Corollary 2. Let S and T be continuous mappings of a compact metric space (X,d) into itself satisfying the inequality

$$d(Sx,Ty) < \max\{d(x,y),d(x,Ty),d(y,Sx)\}$$

for all x, y in X for which the right-hand side of the inequality is positive. Then S and T have a unique common fixed point z. Further, z is the unique fixed point of S and of T.

Theorem 5 of Fisher (1981a) and its corollaries also follow easily.

For further related results on set-valued mappings, see Fisher (1981c).

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مبرهنات النقاط الثابتة المشتركة لتطبيقات وتطبيقات مجموعي القيم

برايان فيشر
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خلاصة

لنفرض أن (X,d) هو فضاء قياس كامل وان $B(X)$ هي مجموعة المجموعات المحدودة غير الخالية من X . ان الدالة $\delta(A,B)$، حيث B,A مجموعتان جزئيتان من X ، تعرف كما يأتي :

$$\delta(A,B) = \sup \{d(a,b): a \in A,\ b \in B\}.$$

لنفرض ان G,F هما تطبيقان من X إلى $B(X)$ وان J,I تطبيقان من X إلى X بحيث يكون :

$$\delta(Fx,Gy) \leqslant c \cdot \max \{d(Ix,Jy),\delta(Ix,Gy),\delta(Jy,Fx)\}$$

لكل قيم x,y في X حيث $0 \leqslant c < 1$

لقد تم اثبات المبرهنة الآتية :

اذا كان F يبادل I وكان G يبادل J وكان مدى I يحتوي على مدى F ومدى J يحتوى على مدى G وكان كل من F أو I متصلا و G أو J متصلا فيكون للتطبيقات G, F, J, I نقطة ثابتة مشتركة وحيدة z .