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On K^λ-summability of Fourier series

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ABSTRACT

Vuĉkoviĉ (1965) established a theorem on K^λ-summability of Fourier series. Kathal (1969) generalised the theorem of Vuĉkoviĉ. Here, in this paper, we have generalised further the theorem of Kathal under very general conditions.

1. DEFINITION AND NOTATION

Let us define, for $n=0,1,2,\ldots$, the numbers $[{n \atop m}]$, for $0 \leqslant m \leqslant n$, by

$$\prod_{v=0}^{n-1} (x+v) = \sum_{m=0}^{n} [{n \atop m}]\, x^m , \tag{1.1}$$

where

$$\prod_{v=0}^{n-1} (x+v) = x(x+1)(x+2)\ldots(x+n-1) . \tag{1.2}$$

The numbers $[{n \atop m}]$ are known as the absolute values of the Stirling numbers of the first kind.

Let $\{S_n\}$ be the sequence of partial sums of an infinite series $\Sigma\, a_n$ and let

$$S_n^\lambda = \frac{\Gamma\lambda}{\Gamma\lambda+n} \sum_{m=0}^{n} [{n \atop m}]\, \lambda^m S_m , \tag{1.3}$$

denote the n-th K^λ—mean of order $\lambda>0$. If

$$S_n^\lambda \rightarrow S , \tag{1.4}$$

as $n\rightarrow\infty$, where S is a fixed finite quantity, then the sequence $\{S_n\}$ or the series $\Sigma\, a_n$ is said to be summable by Karamata's methods K^λ of order $\lambda>0$ to the sum S. The methods K^λ are regular for $\lambda>0$.

Let $f(t)$ be a 2π—periodic and Lebesgue—integrable function of t in the interval $(-\pi, \pi)$. Then the Fourier series of $f(t)$ is given by

$$f(t) \sim \tfrac{1}{2}\, a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) \tag{1.5}$$

Let us write

$$\phi(t) = f(x+t)+f(x-t)-2f(x)$$

$$K_n(t) = \frac{\sum_{m=0}^{n} [^n_m]\lambda^m \sin(m+\frac{1}{2})t}{\lfloor\overline{\lambda+n} \, . \, \sin\frac{t}{2}}$$

$$\tau = [1/t] = \text{the integral part of } 1/t.$$

2. INTRODUCTION AND KNOWN RESULTS

The methods K^λ were first introduced by Karamata (1935). Lototsky (1953) reintroduced the special case $\lambda=1$. Only after the paper of Agnew (1957), did an intensive study of those and similar methods take place.

Vuĉkoviĉ (1965) established the following theorem on the summability of Fourier series by Karamata's methods.

Theorem A. If

$$\phi(t) = 0\left[\frac{1}{\log\frac{1}{t}}\right], t\to+0, \tag{2.1}$$

then the series (1.5) is summable K^λ to the sum $f(x)$ at the point $t=x$, for every $\lambda>0$.

Kathal (1969) generalized Theorem A by proving the following:

Theorem B. If

$$\Phi(t) = \int_0^t |\phi(u)|\, du = 0\left[\frac{t}{\log\frac{1}{t}}\right], \tag{2.2}$$

as $t\to+0$, then the series (1.5) is summable K^λ $(\lambda>0)$ to the sum $f(x)$ at the point $t=x$.

3. THE MAIN RESULT

In this paper, we generalise Theorem B under very general conditions by establishing the following:

Theorem. Let $\{p_n\}$ be a non-negative monotonic non-increasing sequence of constants such that

$$\sum_{v=0}^{n} p_v = P_n\to\infty, \tag{3.1}$$

as $n\to\infty$. Let $\varepsilon(t)$ and $\mu(t)$ be two positive functions of t such that $\varepsilon(t)$, $\mu(t)$ and $\frac{t\varepsilon(t)}{\mu(t)}$ increase monotonically with t and

$$\epsilon(n)\log n = O[\mu(P_n)] \tag{3.2}$$

If

$$\Phi(t) = \int_0^t |\phi(u)|\, du = 0\left[\frac{t\in(1/t)}{\mu(P_\tau)}\right], \tag{3.3}$$

as $t \to +0$, then the series (1.5) is summable K^λ $(\lambda > 0)$ to the sum $f(x)$, at the point $t = x$.

It is important to note that our theorem includes Theorems A and B as special cases.

4. LEMMA

For the proof of our theorem, we shall use the following:

Lemma [Vuĉkoviĉ (1965)]: Let $\lambda > 0$ and $0 < t < \frac{\pi}{2}$. Then

$$\frac{|I_m \sqrt{\lambda e^{it}+n}|}{\sqrt{\lambda \cos t + n}\, \sin \frac{t}{2}} = \frac{|\sin(\lambda \log n \cdot \sin t)|}{\sin \frac{t}{2}} + O(1),$$

as $n \to \infty$, uniformly in t.

5. PROOF OF THE THEOREM

As in Vuĉkoviĉ (1965), we have

$$\begin{aligned} S_n^\lambda(x) - f(x) &= \frac{\sqrt{\lambda}}{2\pi} \int_0^\pi \phi(t) \cdot K_n(t) dt \\ &= O\,[\{\int_0^{1/n} + \int_{1/n}^\pi\}|\phi(t)|\; |K_n(t)| dt] + 0(1) \\ &= O\,(I_1) + O(I_2) + 0(1), \text{ say}. \end{aligned} \tag{5.1}$$

Let us first consider I_1. Now by (1.1)

$$K_n(t) = \frac{I_m\left\{e^{it/2} \dfrac{\sqrt{\lambda e^{it}+n}}{\sqrt{\lambda e^{it}}}\right\}}{\sqrt{\lambda+n}\, \sin \frac{t}{2}}$$

Since

$$\frac{\left|I_m\left\{e^{it/2} \dfrac{\sqrt{\lambda e^{it}+n}}{\sqrt{\lambda e^{it}}}\right\}\right|}{\sin \frac{t}{2}}$$

$$= O\left[\frac{|I_m \sqrt{\lambda e^{it}+n}|}{\sin \frac{t}{2}}\right] + O\,[Re \sqrt{\lambda e^{it}+n}],$$

we obtain

$$|K_n(t)|$$

$$= O\left[\frac{\dfrac{|\lambda \cos t+n}{|\lambda+n}\,\Big|\, I_m\, |\lambda e^{it}+n|}{|\lambda \cos t+n \sin \frac{t}{2}}\right] + O\left[\frac{|\lambda \cos t+n}{|\lambda+n}\right].$$

For $0<t<1/n$,

$$\frac{|\lambda \cos t+n}{|\lambda+n} = O\,[n^{-\lambda(1-\cos t)}]$$

$$= O\,[e^{-\lambda(1-\cos t)\log n}]$$

$$= O\,[e^{-(\lambda/3)t2\log n}],$$

since, for $0<t<1/n$;

$$0 < \cos t-1 < -\frac{t^2}{3}\cdot$$

Therefore

$$I_1 = \int_0^{1/n} |\,\phi(t)\cdot|K_n(t)|dt$$

$$= O\left[\int_0^{1/n} \frac{e^{-\lambda(1-\cos t)\log n}}{\sin \frac{t}{2}}\cdot\frac{|I_m|\lambda e^{it}+n}{|\lambda \cos t+n}|\phi(t)|dt\right]$$

$$+ O\left[\int_0^{1/n} e^{-(\lambda/3)t2\log n}\cdot|\phi(t)|\,dt\right]$$

$$= O\,(I_{1.1})+O(I_{1.2}), \text{ say}$$

Now, applying the Lemma, we have

$$I_{1.1} = \int_0^{1/n} \frac{e^{-\lambda(1-\cos t)\log n}}{\sin \frac{t}{2}}|\sin(\lambda \log n \sin t)||\phi(t)|\,dt$$

$$= O\left[\int_0^{1/n} \frac{e^{-\lambda \log n(t2/2)}}{\sin \frac{t}{2}}|\sin(\lambda \log n\cdot t)|\,|\phi(t)|\,dt\right]$$

$$= O\,(\lambda \log n)\cdot\int_0^{1/n} |\phi(t)|\,dt$$

$$= O\,(\lambda \log n)\cdot 0\left[\frac{\in(n)}{n\cdot\mu(P_n)}\right]$$

$$= 0(1), \text{ by } (3.2), \tag{5.3}$$

as $n \to \infty$. Next, we have

$$I_{1.2} = \int_0^{1/n} e^{-(\lambda/3)t^2 \log n} |\phi(t)| \, dt$$

$$= O(1) \int_0^{1/n} |\phi(t)| \, dt$$

$$= 0\left[\frac{\epsilon(n)}{n \cdot \mu(P_n)}\right]$$

$$= 0(1), \tag{5.4}$$

as $n \to \infty$. From (5.2), (5.3) and (5.4), it follows that

$$I_1 = 0(1). \tag{5.5}$$

Let us now consider I_2. As, for $1/n < t < \pi$, $\phi(t)$ is bounded and

$$|K_n(t)| = O\left[\frac{n^{-\lambda(1-\cos(1/n))}}{\sin \frac{1}{2n}}\right]$$

$$= 0(1), \tag{5.6}$$

as $n \to \infty$, we obtain

$$I_2 = \int_{1/n}^{\pi} |\phi(t)| \cdot |K_n(t)| \, dt$$

$$= 0(1) \cdot \int_{1/n}^{\pi} |\phi(t)| \, dt, \text{ by (5.6)}$$

$$= 0(1), \tag{5.7}$$

as $n \to \infty$. From (5.1), (5.5) and (5.7) it follows that

$$S_n^\lambda(x) - f(x) = 0(1),$$

as $n \to \infty$. This completes the proof of our theorem.

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REFERENCES

Agnew, R.P. 1957. The Lototsky method for evaluation of series. Michigan Math. J. **4:** 105–28.

Karamata, J. 1935. Theorèms sur la sommabilitée exponentielle et d'autres sommabilitées s'y rattachant. Mathematica (Cluj) **9:** 164–78.

Kathal, P.D. 1969. A new criterion for Karamata summability of Fourier series. Riv. Mat. Univ. Parma, Italy **10(2):** 33–38.

Lototsky, A.V. 1953. On a linear transformation of sequences (Russian). Ivanov Gos. Ped. Institute Uchen Zap. **4:** 61–91.

Vuĉkoviĉ, V. 1965. The summability of Foureir series by Karamata methods. Math. Zeits. **89:** 192–95.

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حول قابلية الجمع K^λ لمتسلسلة فورييه

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خلاصة

كان كاثال قد عمم مبرهنة ڤوكوڤتش ، وقد أمكن في هذا البحث تعميم مبرهنة كاثال تحت شروط عامة جدا .