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Bayes estimate of reliability of *k*-out-of-*m* systems in the exponential model

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ABSTRACT

Let S be a k-out-of-m system with independent components from an exponential model. Assume θ has a gamma prior distribution. The Bayes estimate $R_n^*(t)$ of the conditional reliability function $R(t)$ is obtained. Consistency and asymptotic normality of $R_n^*(t)$ have been established. The accuracy of the Bayes estimate and the justification of the exponential prior distribution are readily seen through the results of a Monte Carlo study.

INTRODUCTION

Let S be a system with m components such that the system operates if and only if at least k of them operate successfully. This is called a k-out-of-m system. Assume that X_i, the time to failure of the ith components ($i=1, \ldots, m$), follows an exponential probability density function (PDF),

$$f(x|\theta) = \theta \exp(-\theta x), \quad x > 0, \theta > 0, \tag{1}$$

where θ is a random variable with prior PDF

$$q(\theta) = \beta^p \theta^{p-1} \exp(-\beta\theta)/\Gamma(p) \tag{2}$$

where β and p are known positive constants. For each component i, a random sample X_{ij} ($j=1, \ldots, n$) from $f(x|\theta)$ is tested and the life data are recorded. Let β_{mn} be the σ-field generated by X_{ij} for $i=1, \ldots, m$ and $j=1, \ldots, n$. Let X be the time to failure of S. Then a Bayesian may be interested in the conditional system reliability,

$$R(t) = P(X > t|\theta) = \sum_{h=k}^{m} \sum_{\alpha=0}^{m-h} (-1)^\alpha \binom{m}{h} \binom{m-h}{\alpha} \exp[-t\theta(h+\alpha)] \tag{3}$$

(cf. Basu & El-Mawaziny 1978 (p. 851)), if the random value taken by θ holds over a long period of time, perhaps for the life of the system.

The expectation of $R(t)$ with respect to the prior distribution, $E[R(t)]$, is called the system reliability. It is a prior estimate for the reliability of the system. However, a Bayesian tries to modify this estimate after he obtains a random sample of observations on the system. So he would like to estimate $R(t)$ by the posterior mean $R_n^*(t) = E[R(t)|\beta_{mn}]$. Our aim is to find the Bayes estimate, $R_n^*(t)$, for $R(t)$, and to study its properties.

On the other hand, a non-Bayesian may be interested in estimating $P(X>t|\theta)$ if θ is considered as an unknown constant (non-random). Basu & El-Mawaziny (1978) have shown that the maximum likelihood estimator (MLE) of $R(t)$ is

$$\tilde{R}_n(t) = \sum_{h=k}^{m} \sum_{\alpha=0}^{m-h} (-1) \binom{m}{h} \binom{m-h}{\alpha} \exp[-t(h+\alpha)/X..] \tag{4}$$

where

$$X.. = \sum_{j=1}^{n} \sum_{i=1}^{m} X_{ij}/nm .$$

They have also derived the minimum variance unbiased estimator (MVUE), of $R(t)$. In their work, the parameter θ is considered an unknown constant. The asymptotic normality of the MLE and of the MVUE, $\tilde{R}_n$ and $\hat{R}(t)$ respectively, have been derived.

THE BAYES ESTIMATE

Let S be a k-out-of-m system. Let $X_i (i=1, \ldots, m)$ be the time to failure of the ith component of S, and let X_i have PDF given by (1). Assume θ has a prior PDF given by (2). Given the set-up explained in the introduction, it is desired to derive the Bayes estimate of $R(t)$. The posterior mean of $R(t)$ given the σ-field generated by $X_{ij}(i=1, \ldots, m$ and $j=1, \ldots, n)$ is

$$R_n^*(t) = E[R(t)|\beta_{mn}]$$

$$= \frac{\int R(t) q(\theta) \prod_{j=1}^{n} \prod_{i=1}^{n} f(x_{ij}|\theta) d\theta}{\int q(\theta) \prod_{j=1}^{n} \prod_{i=1}^{m} f(x_{ij}|\theta) d\theta} . \tag{5}$$

Hence

$$R_n^*(t) = \sum_{h=k}^{m} \sum_{\alpha=0}^{m-h} (-1)^{\alpha} \binom{m}{h} \binom{m-h}{\alpha} \left(1 - \frac{t(h+\alpha)}{\beta + t(h+\alpha) + mnX..}\right)^{mn+p} \tag{6}$$

where

$$X.. = \sum_{j=1}^{n} \sum_{i=1}^{m} X_{ij}/mn .$$

PROPERTIES OF THE BAYES ESTIMATE

It is clear from (5) that $0 \leqslant R_n^*(t) \leqslant 1$ since $0 \leqslant R(t) \leqslant 1$. Using (5), (3) and (2), we get

$$ER_n^* = E\{E[R(t)|X_{ij}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n]\}$$

$$= E[R(t)]$$

$$= \sum_{h=k}^{m} \sum_{\alpha=0}^{m-h} (-1) \binom{m}{h} \binom{m-h}{\alpha} E\{\exp[-t(h+\alpha)\theta]\}$$

$$= \sum_{h=k}^{m} \sum_{\alpha=0}^{m-h} (-1)^{\alpha} \binom{m}{h} \binom{m-h}{\alpha} \left(1 - \frac{t(h+\alpha)}{\beta + t(h+\alpha)}\right)^{p} \tag{7}$$

i.e. R_n^* is integrable.

Now it is clear that the conditional reliability function given θ, $R(t)$, the Bayes reliability function (conditional reliability given the observations) $R_n^*(t)$ and the prior mean reliability function (the system reliability) $ER_n^*(t)$ have the same properties, namely, that each one of them lies between zero and one, is equal to one when $t=0$, is decreasing in t, and has limit equal to zero as $t \to \infty$.

Theorem 1. The MLE $\tilde{R}_n(t)$, the MVUE $\hat{R}_n(t)$, and the Bayes estimate $R_n^*(t)$ of $R(t)$ are a.s. equivalent as $n \to \infty$.

Proof. Basu & EL-Mawaziny (1978, Theorem 2) have shown that $\hat{R}_n(t)$ and $\tilde{R}_n(t)$ are a.s. equivalent as $n \to \infty$. It remains to show that $\tilde{R}_n(t)$ and $R_n^*(t)$ are a.s. equivalent as $n \to \infty$. It can be shown that, as $n \to \infty$,

$$\left(1 - \frac{t(h+\alpha)}{\beta + t(h+\alpha) + mnX..}\right) - \exp\left(-\frac{t(h+\alpha)}{X..}\right) \xrightarrow{\text{a.s.}} 0. \tag{8}$$

Using (6), (4) and (8) we get $R_n^*(t) - \tilde{R}_n(t) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$.

Theorem 2. The Bayes estimate $R_n^*(t)$ is a strongly consistent estimator for $R(t)$.

Proof. By the strong law of large numbers it follows that conditionally on θ, for each i,

$$n^{-1} \sum_{j=1}^{n} X_{ij} \xrightarrow{\text{a.s.}} \theta^{-1}, \quad \text{as } n \to \infty.$$

Hence

$$(nm)^{-1} \sum_{j=1}^{n} \sum_{i=1}^{m} X_{ij} \xrightarrow{\text{a.s.}} \theta^{-1}, \quad \text{as } n \to \infty; \tag{9}$$

i.e. $X..$ is a strongly consistent estimate for θ^{-1} and thus

$$\exp[-t(h+\alpha)/X..] \xrightarrow{\text{a.s.}} \exp[-t\theta(\mathrm{h}+\alpha)], \text{ as } n \to \infty. \tag{10}$$

Using (10), (4) and (3) we get $\tilde{R}_n(t) \xrightarrow{\text{a.s.}} R(t)$, as $n \to \infty$. This result, together with Theorem 1, implies that $R_n^*(t)$ is a strongly consistent estimate for $R(t)$.

ASYMPTOTIC NORMALITY OF THE BAYES ESTIMATE

Basu & El-Mawaziny (1978, Theorem 1) have shown that when θ is non-random $n^{\frac{1}{2}}[\tilde{R}_n(t) - R(t)]$ and $n^{\frac{1}{2}}[\hat{R}_n(t) - R(t)]$ converge in distribution to a normal random variable.

From a Bayesian point of view, one would like to have a similar result for the Bayesian estimate $R_n^*(t)$ conditionally on the observations (X_{ij}, $i=1, 2, \ldots, m, j=1, 2, \ldots, n$).

Theorem 3. Given the observations (X_{ij}, $i=1, \ldots, m; j=1, \ldots, n$),

$$n^{\frac{1}{2}} \sigma^{-1}[R(t)-R_n^*(t)] | \beta_{mn} \xrightarrow{d} N(0, 1)$$

where

$$\sigma = \theta t\, m^{-\frac{1}{2}} \sum_{h=k}^{m} \sum_{\alpha=0}^{m-h} (-1)^\alpha \binom{m}{h} \binom{m-h}{\alpha} (h+\alpha) e^{-t\,\theta(h+\alpha)}.$$

Proof. The proof depends on the following lemma which is a direct extension of Theorem 7 given by Awad (1980).

Lemma (Awad): Assume that
(a) $Y_1, \ldots, Y_n | \phi$ are i.i.d. with mean ϕ and finite variance σ^2,
(b) ϕ is square integrable random variable,
(c) $E(\phi|\mathscr{F}_n) = a_n \bar{Y}_n + b_n$, where

$$\bar{Y}_n = \sum_{i=1}^{n} Y_i/n,$$

$a_n \sim 1$, $b_n - b_{n-1} \sim c\, n^{-2}$ for some constant c, and $\mathscr{F}_n$ is the σ-field generated by ($Y_1, \ldots, Y_n$).

Then $n^{\frac{1}{2}} \sigma^{-1}[\phi - E(\phi|\mathscr{F}_n)] | \mathscr{F}_n \xrightarrow{d} N(0, 1)$.

Now, set $\phi = \theta^{-1}$, then it can be shown that

$$E(\phi|\beta_{mn}) = (mn+p-1)^{-1}(mn\, X.. + \beta)$$

and

$$\mathrm{Var}\,(\phi|\beta_{mn}) = (mn+p-2)^{-1} E^2(\phi|\beta_{mn}).$$

Therefore the Awad lemma implies that

$$n^{\frac{1}{2}} \phi^{-1}[\phi - E(\phi|\beta_{mn})] | \beta_{mn} \xrightarrow{d} N(0, 1).$$

This together with the results in Rao (1965, p. 386) with

$$g(\phi) = m^{\frac{1}{2}} \exp[-t(h+\alpha)/\phi] \quad \text{and } T_n = X..$$

imply that

$$n^{\frac{1}{2}} \tau^{-1}[e^{-t\,\theta(h+\alpha)} - e^{-t\,(h+\alpha)/X..}] | \beta_{mn} \xrightarrow{d} N(0, 1)$$

where $\tau = t\theta(h+\alpha) \exp[-t\theta(h+\alpha)] m^{-1}$.
Multiplying by $(-1)^\alpha \binom{m}{h}\binom{m-h}{\alpha}$, summing over α from zero to $m-h$, then summing over h from k to m, and using Theorem 1 we obtain the required result.

SPECIAL CASES

It is known that if $k=m=1$, then S is a one-component system. In this case

$$R(t) = \exp(-t\theta)$$

and

$$R_n^*(t) = [1 - t/(t+\beta+n\bar{X})]^{n+p}$$

where

$$\bar{X} = n^{-1} \sum_{j=1}^{n} X_j.$$

If $k=m$, then S has m components in series. For this system

$$R(t) = \exp(-tm\theta)$$

and

$$R_n^*(t) = \left(1 - \frac{tm}{\beta + tm + mnX..}\right)^{mn+p}.$$

If $k=1$, $m>1$, then S has m components in parallel. In this case

$$\sum_{j=1}^{m} (-1)^{j+1} \binom{m}{j} \exp(-t\theta j) = R(t)$$

and

$$R_n^*(n) = \sum_{j=1}^{m} (-1)^{j+1} \binom{m}{j} \left(1 - \frac{tj}{\beta + tj + mnX..}\right)^{mn+p}.$$

The asymptotic normality of each of these estimates follows from Theorem 3.

EXAMPLE

To illustrate the accuracy of the Bayes estimates for the reliability function $R(t)$, some numerical calculations have been done. Random samples of size n have been generated from a k-out-of-m system with

$m=3$
$k=1$, 2 and 3
$\theta=1$, 0·2, and 0·05
$n=3$, 4, 5 and 6

It is assumed that the prior distribution is exponential with $p=\beta=1$.

Using a Monte Carlo study with 1000 runs the estimates $R^*(t)$, (for $t=0{\cdot}5$, 1, and 5) are given in Tables 1, 2 and 3, together with the mean square error of these estimates.

Comparing $R_n^*(t)$ with the true value $R(t)$ we note that the Bayes estimates are close to $R(t)$. This remark also justifies the prior distribution of θ.

Table 1. Bayes estimates for $R(t)$ with their mean square error for 1-out-of-3 system

		θ								
		1			0·2			0·05		
n	t	$R(t)$	$R^*(t)$	MSE	$R(t)$	$R^*(t)$	MSE	$R(t)$	$R^*(t)$	MSE
3	0·5	0·9391	0·9247	0·0025	0·9992	0·9979	†	0·9999	0·9999	†
	1·0	0·7474	0·6937	0·0227	0·9940	0·9876	0·0002	0·9999	0·9997	†
	5·0	0·0201	0·0525	0·0030	0·7474	0·6734	0·0248	0·9892	0·9789	0·0004
4	0·5	0·9391	0·9250	0·0024	0·9992	0·9981	†	0·9999	0·9999	†
	1·0	0·7474	0·7466	0·0117	0·9940	0·9886	0·0001	0·9999	0·9997	†
	5·0	0·0201	0·0522	0·0036	0·7474	0·6598	0·0271	0·9892	0·9779	0·0005
5	0·5	0·9391	0·9187	0·0026	0·9992	0·9984	†	0·9999	0·9999	†
	1·0	0·7474	0·7109	0·0127	0·9940	0·9893	†	0·9999	0·9997	†
	5·0	0·0201	0·0479	0·0025	0·7474	0·6885	0·0184	0·9892	0·9782	0·0005
6	0·5	0·9391	0·9218	0·0023	0·9992	0·9985	†	0·9999	0·9999	†
	1·0	0·7474	0·7184	0·0117	0·9940	0·9885	0·0002	0·9999	0·9998	†
	5·0	0·0201	0·0416	0·0019	0·7474	0·7137	0·0128	0·9892	0·9806	0·0003

† At least the first four digits are zero.

Table 2. Bayes estimates for $R(t)$ with their mean square error for 2-out-of-3 system

		θ								
		1			0·2			0·05		
n	t	$R(t)$	$R^*(t)$	MSE	$R(t)$	$R^*(T)$	MSE	$R(t)$	$R^*(t)$	MES
3	0·5	0·6574	0·6620	0·0135	0·9746	0·9621	0·0007	0·9983	0·9970	†
	1·0	0·3064	0·3149	0·0197	0·9134	0·8839	0·0042	0·9931	0·9886	†
	5·0	0·0001	0·0040	†	0·3064	0·2911	0·0155	0·8749	0·8408	0·0070
4	0·5	0·6574	0·6556	0·0122	0·9746	0·9630	0·0005	0·9983	0·9973	†
	1·0	0·3064	0·3553	0·0166	0·9134	0·8882	0·0038	0·9931	0·9888	†
	5·0	0·0001	0·0036	†	0·3064	0·2709	0·0158	0·8749	0·8332	0·0079
5	0·5	0·6574	0·6359	0·0133	0·9746	0·9664	0·0003	0·9983	0·9971	†
	1·0	0·3064	0·3079	0·0113	0·9134	0·8881	0·0027	0·9931	0·9909	†
	5·0	0·0001	0·0026	†	0·3064	0·2888	0·0127	0·8749	0·8321	0·0076
6	0·5	0·6574	0·6377	0·0103	0·9746	0·9679	0·0003	0·9983	0·9976	†
	1·0	0·3064	0·3002	0·0103	0·9134	0·8859	0·0043	0·9931	0·9907	†
	5·0	0·0001	0·0019	†	0·3064	0·3087	0·0124	0·8749	0·8382	0·0049

† At least the first four digits are zero.

Table 3. Bayes estimates for $R(t)$ with their mean square error for 3-out-of-3 system

		θ								
		1			0·2			0·05		
n	t	$R(t)$	$R^*(t)$	MSE	$R(t)$	$R^*(t)$	MSE	$R(t)$	$R^*(t)$	MSE
3	0·5	0·2231	0·2615	0·0098	0·7408	0·7125	0·0061	0·9277	0·9149	0·0008
	1·0	0·0498	0·0730	0·0032	0·5488	0·5228	0·0101	0·8607	0·8384	0·0028
	5·0	†	0·0002	†	0·0498	0·0631	0·0019	0·4724	0·4561	0·0124
4	0·5	0·2231	0·2503	0·0078	0·7408	0·7127	0·0059	0·9277	0·9173	0·0006
	1·0	0·0498	0·0820	0·0030	0·5488	0·5273	0·0103	0·8607	0·8389	0·0030
	5·0	†	0·0002	†	0·0498	0·0542	0·0015	0·4724	0·4400	0·0124
5	0·5	0·2231	0·2329	0·0074	0·7408	0·7202	0·0036	0·9277	0·9147	0·0006
	1·0	0·0498	0·0626	0·0015	0·5488	0·5200	0·0081	0·8607	0·8491	0·0013
	5·0	†	0·0001	†	0·0498	0·0570	0·0012	0·4724	0·4331	0·0109
6	0·5	0·2231	0·2286	0·0049	0·7408	0·7263	0·0039	0·9277	0·9214	0·0003
	1·0	0·0498	0·0586	0·0012	0·5488	0·5182	0·0100	0·8607	0·8466	0·0011
	5·0	†	0·0001	†	0·0498	0·0622	0·0015	0·4724	0·4347	0·0077

† At least the first four digits are zero.

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تقدير بيز لفاعلية نظام ذي ك من م من المركبات الخاضعة لنموذج أسي

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خلاصة

يعالج هذا البحث نظاما ذا ك من م من المركبات الخاضعة لنموذج أسي معلمته θ ، حيث θ متغير عشوائي يخضع لتوزيع أولي جاما ، وقد تم اشتقاق تقدير بيز للفاعلية واثبات اتساقه وتقارب توزيعه من التوزيع الطبيعي . وقد دلت الدراسة العددية على قرب القيمة التقديرية لفاعلية النظام من القيمة الحقيقية لها ، وهذا ما يبرر ملاءمة التوزيع الأولي المستعمل .