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Semi-dual continuous abelian groups

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ABSTRACT

A module M is called semi-dual continuous if for every submodule A of M, M decomposes as $M=M_1\oplus M_2$ such that $M_1\subset A$ and $M_2\cap A$ is small in M. In this note we study the semi-dual continuous abelian groups. The main result is the following: an abelian group G is semi-dual continuous if and only if G is torsion and for every prime p, the primary component G_p is either of finite rank and divisible, or $G_p=\Sigma\oplus\langle a_i\rangle$ such that for some positive integer n, depending on p, a_i is of order p^n or p^{n+1}.

INTRODUCTION

Let M be a module over a ring R. A submodule S of M is said to be small in M if $S+A\neq M$ for every proper submodule A of M. The sum of all small submodules of M is the Jacobson radical of M and is denoted by Rad M. Also Rad M is the intersection of all maximal submodules of M.

Z and **Q** will denote the rings of integer and rational numbers respectively. Let p be a prime number. $\mathbf{Z}_{p^n}$, $1\leq n\leq\infty$, will denote the cocyclic group of type p^n. If $n<\infty$, $\mathbf{Z}_{p^n}$ is a cyclic group of order p^n, while $\mathbf{Z}_{p^\infty}$ is the quasi-cyclic group generated by elements $c_1,c_2,\ldots,c_n,\ldots$, such that $p\,c_1=0$, $p\,c_2=c_1,\ldots,p\,c_{n+1}=c_n,\ldots$ (Fuchs 1960).

Let G be an abelian group. Rad G is the Fratini subgroup of G. For $a\in G$, $|(a)|$ will denote the order of the element a. The p-primary component of G will be denoted by G_p.

For further definitions and basic properties of abelian groups, we refer to Fuchs (1960).

A module M is called dual continuous (Mohamed & Singh 1977) if it satisfies the following conditions:

(i) For every submodule A of M there exists a decomposition $M=M_1\oplus M_2$ such that $M_1\subset A$ and $M_2\cap A$ is small in M.
(ii) Every epimorphism from M onto a summand of M splits.

Every quasi-projective module satisfies condition (ii) but not necessarily condition (i). Mohamed & Singh (1977) proved that a ring R is (semi)perfect if and only if every (finitely generated) quasi-projective R-module satisfies condition (i). In this paper, we initiate the study of modules satisfying condition (i). Such modules will be called semi-dual continuous (for short, sd-continuous). We will be interested only in sd-continuous **Z**-modules, that is sd-continuous abelian groups.

MAIN RESULTS

The following two lemmas are obvious.

Lemma 1. A summand of an sd-continuous module is sd-continuous.

Lemma 2. A module M is sd-continuous if and only if every submodule A of M is of the form $A = N \oplus S$ where N is a summand of M and S small in M.

Corollary 3. The **Z**-modules **Z** and **Q** are not sd-continuous.

Corollary 4. The **Z**-module $\mathbf{Z}_{p^n}$ is sd-continuous, $1 \leq n \leq \infty$.

Next we prove

Proposition 5. Let m and n be integers such that $1 \leqslant m \leqslant n \leqslant \infty$. If $n-m>1$, then $G = \mathbf{Z}_{p^m} \oplus \mathbf{Z}_{p^n}$ is not sd-continuous.

Proof. Choose $a \in \mathbf{Z}_{p^m}$ of order p^m and $b \in Z_{p^n}$ of order p^{m+1}, and let K be the subgroup of G generated by $a+b$. Then K is a cyclic group of order p^{m+1}, and hence contains no non-zero summands of G. However, K is not small in G as K is not contained in the maximal subgroup $<pa> + \mathbf{Z}_{p^n}$. Hence G is not sd-continuous by Lemma 2.

Proposition 6. Every sd-continuous abelian group G is torsion.

Proof. Let T be the torsion subgroup of G. As G is sd-continuous,

$$G = T' \oplus F,\ T' \subset T \text{ and } T \cap F \text{ is small in } G.$$

Suppose that $T \cap F \neq 0$. Then as $T \cap F$ is the torsion subgroup of F, it contains a non-zero summand of F by Fuchs (1960, Corollary 24.3). However, this is a contradiction as $T \cap F$ is small in G. Hence $T \cap F = 0$, and

$$G = T \oplus F \quad .$$

It is clear that F is torsionfree. We claim that $F=0$. Suppose not. If F is not reduced, then it contains a summand isomorphic to Q. But this would imply that Q is sd-continuous by Lemma 1, which is a contradiction to Corollary 3. Thus F is reduced. Then there exists a prime number p such that $pF \neq F$. Let $x \in F$ such that $x \notin pF$. Then $<x> \not\subset \text{Rad } F$ and so $<x>$ is not small in F. As F is sd-continuous by Lemma 1, $<x>$ contains a non-zero summand K by Lemma 2. Since $<x> \simeq \mathbf{Z}$, $K = <x>$, then **Z** is sd-continuous by Lemma 1 which is again a contradiction to Corollary 3. Therefore $F=0$ and the result follows.

Remark 7. Let G be a torsion abelian group. Then $G = \Sigma \oplus G_p$ where the summation runs over all primes p. Then using Lemma 1 and the fact that $\text{Hom}(G_p, G_q) = 0$ for $p \neq q$, one can easily see that G is sd-continuous if and only if each G_p is sd-continuous.

Using Proposition 6, the above remark shows that it is enough to study sd-continuous abelian p-groups.

Lemma 8. An sd-continuous abelian p-group G is either divisible or reduced.

Proof. $G = D \oplus C$ where D is the maximal divisible subgroup of G and C is reduced. Suppose that $D \neq 0$ and $C \neq 0$. Then D contains a non-zero summand U isomorphic to $\mathbf{Z}_{p\infty}$ and C contains a non-zero summand V isomorphic to $\mathbf{Z}_{p^n}$ for some positive integer n. Since $U \oplus V$ is a summand of G, $U \oplus V$ is sd-continuous by Lemma 1. However, this is a contradiction to Proposition 5. Therefore, $D = 0$ or $C = 0$. This completes the proof.

Theorem 9. Let G be a divisible abelian p-group. Then G is sd-continuous if and only if G is a finite direct sum of copies of $\mathbf{Z}_{p\infty}$.

Proof. By Fuchs (1960, Theorem 19.1),

$$G = \sum_{i \varepsilon I} \oplus A_i$$

where $A_i \simeq \mathbf{Z}_{p\infty}$ for all $i \in I$. We need only to prove that G is sd-continuous if and only if the set I is finite.

Assume that I is finite. Let H be a subgroup of G. Then $H = D \oplus F$ where D is the maximal divisible subgroup of H and F is reduced. In fact F is finite and hence small in G. As D is a summand of G, we get G is sd-continuous by Lemma 2.

Conversely, assume that I is not finite. Let J be a countable subset of I, and let

$$G' = \sum_{j \varepsilon J} \oplus A_j \quad .$$

From each A_j select an element x_j of order p^j and let

$$K = \sum_{j \varepsilon J} \oplus <x_j> \quad .$$

Since K is unbounded, there exists an epimorphism of K onto A_1 which extends to a map from G'. This shows that K is not small in G'. Also K, being reduced, does not contain a non-zero summand of G'. Hence G' is not sd-continuous by Lemma 2. But since G' is a summand of G, it follows by Lemma 1 that G is not sd-continuous.

Theorem 10. Let G be a reduced abelian p-group. Then G is sd-continuous if and only if there exists a positive integer n such that G is a direct sum of cyclic groups of order p^n or p^{n+1}.

Proof. Let G be sd-continuous. As G is a p-group, it has a basic subgroup say B (Fuchs 1960, Theorem 29.2). By Lemma 2, $B = A \oplus S$ where A is a summand of G and S small in G. Suppose $S \neq 0$, then S contains a non-zero cyclic summand K. Now K is pure in G and hence is a summand of G. This contradicts that S is small in G. Hence $S = 0$. Then B is a summand of G. As G/B is divisible and G is reduced, we get $G = B$. Therefore

$$G = \sum_{i \varepsilon I} \oplus <a_i>$$

such that $|(a_i)| = p^{\alpha_i}$. Now, Lemma 1 and Proposition 5 imply that $|\alpha_i - \alpha_j| \leq 1$ for all $i,j \in I$. Thus if n is the least element in the set $\{\alpha_i : i \in I\}$, then any α_i is n or $n+1$.

Conversely, let

$$G = \sum_{i \in I} \oplus <a_i>$$

such that there exists an integer n such that for any i, $|(a_i)|$ is n or $n+1$. Let $b \in G$ such that $<b>$ is not small in G. Now

$$b = \lambda_1 a_{i_1} + \ldots + \lambda_t a_{i_t} \quad .$$

Let

$$M = \sum_{j=1}^{t} \oplus <a_{i_j}> \quad .$$

Then M is a summand of G and $<b>$ is not small in M. It is clear that for some k, $<\lambda_{i_k} a_{i_k}> = <a_{i_k}>$. Among such k's there is one for which a_{i_k} is of largest possible order. It is obvious that

$$M = <b> + \sum_{j \neq k} \oplus <a_{i_j}> \quad .$$

Since the order of any a_{i_j} is either p^n or p^{n+1}, we get

$$|(b)| = |(\lambda_k a_{i_k})| = |(a_{i_k})| \quad .$$

Then it is immediate that

$$<b> \cap \sum_{j \neq k} \oplus <a_{i_j}> = 0.$$

Thus $<b>$ is a summand of M, and hence a summand of G. This proves that if a subgroup of G is not small in G, then it contains a non-zero summand of G.

Now, let H be a subgroup of G. Let K be a maximal pure subgroup of G contained in H. As G is bounded, K is a summand of G (Fuchs 1960, Theorem 24.1). Then $G = K \oplus L$ for some subgroup L of G. If $H \cap L$ is not small in G, then $H \cap L$ contains a non-zero summand U of G; but then $K \oplus U$ is a pure subgroup of G contained in H which contradicts the choice of K. Thus $H \cap L$ is small in G. Therefore, G is sd-continuous.

The following is an immediate consequence of results (6) through (10).

Theorem 11. An abelian group G is sd-continuous if and only if G is torsion and for each prime p, either G_p is a finite direct sum of copies of $\mathbf{Z}_{p^\infty}$ or

$$G_p = \sum_{i \in I} \oplus <a_i>$$

such that for some positive integer n, depending on p, a_i is of order p^n or p^{n+1}.

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REFERENCES

Fuchs, L. 1960. Abelian groups. Pergamon Press, New York.

Mohamed, S. & Singh, S. 1977. Generalizations of decomposition theorems known over perfect rings. J. Austral. Math. Soc. Ser. A. **24:** 496–510.

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الزمر الابدالية شبه المتصلة بالمقابل

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خلاصة

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