J. Univ. Kuwait (Sci.) **11**, 1984

Deformations of soil around laterally and torsionally loaded pile

MOSAID M. AL-HUSSAINI

Department of Civil Engineering, University of Kuwait

ABSTRACT

In this study a closed form solution is derived for the determination of deformation in soil surrounding a laterally and torsionally loaded pile. The solution is based on the assumption that the pile is relatively rigid while the soil is assumed to be homogeneous, isotropic and elastic. Results of the study are presented in dimensionless graphs to facilitate quick and easy determination of deformation in the affected soil.

INTRODUCTION

In a previous study (Al-Hussaini 1982a) a closed form solution was presented for the determination of normal and shear stresses within soil around laterally loaded pile. It was followed by another study (Al-Hussaini 1982b) where a closed form solution was developed for the determination of normal and shear stresses in the soil surrounding a torsionally loaded rigid pile. This paper is an extension of both previous studies, where a closed form solution for the determination of deformation distribution in the soil surrounding a laterally and torsionally loaded pile, is to be presented.

THEORETICAL CONSIDERATION

Consider a hypothetical pile subjected to a lateral and torsional load of P_o and T_o, respectively, at the top of the pile as shown in Fig. 1. These two forces will generate a lateral soil resistance and torsional resistance of variable magnitudes along the pile. Assume that the lateral thrust at depth L from the ground surface is equal to P and the torsional resistance is equal to T (Fig. 1). The lateral thrust P and the torsional resistance T may generate contact pressures and shear stresses, respectively, between the pile and the affected soil as shown in Fig. 2. In order to determine the deformations around this pile, the equilibrium, boundary, and compatibility conditions must be satisfied.

For plane strain condition, equations of equilibrium in Cartesian coordinate system are

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0 \tag{1a}$$

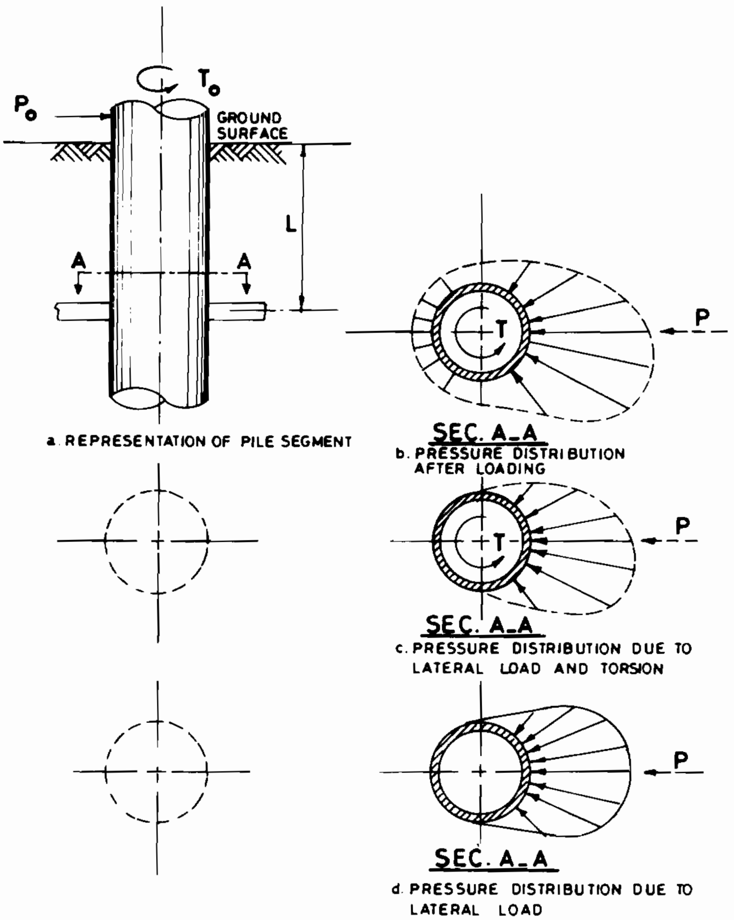

Fig. 1. Schematic diagram of pile subjected to lateral and torsional load.

$$\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}=0 \tag{1b}$$

where σ_x and σ_y are the normal stresses acting on the x and y planes, respectively, and τ_{xy} is the shear stress acting in the xy plane.

The compatibility equation for plane strain condition can be expressed as

$$\nabla^2 (\sigma_x+\sigma_y)=0 \tag{2}$$

where ∇ is the Lablacian operator.

It has been shown (Airy 1863) that Equations 1 and 2 will be satisfied if one introduces a stress function $U(x,y)$ such that the sought stress components are second partial derivatives of the stress function such as

$$\sigma_x=\frac{\partial^2 U\ (x,y)}{\partial y^2} \tag{3a}$$

$$\sigma_y=\frac{\partial^2 U\ (x,y)}{\partial x^2} \tag{3b}$$

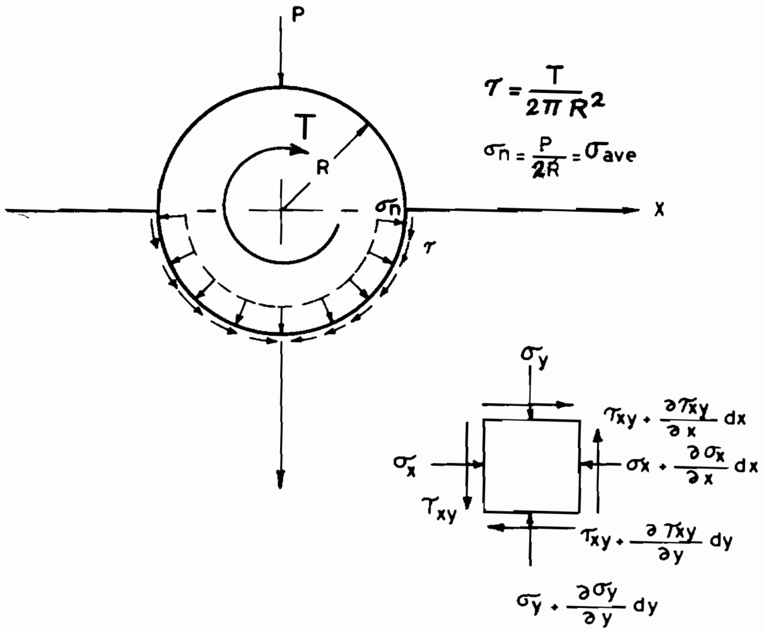

Fig. 2. Stress at a point in the vicinity of laterally and torsionally loaded pile.

$$\tau_{xy} = \frac{\partial^2 U\,(x,y)}{\partial x\,\partial y} \tag{3c}$$

When Equation 3 is substituted into the compatibility Equation 2 the latter transforms into

$$\nabla^2(\sigma_x + \sigma_y) = \nabla^4[U(x,y)] = 0 \tag{4}$$

The solution of the fourth order homogeneous partial differential Equation 4 can be simplified using a complex variable procedure (Timoshenko & Goodier (1962) to obtain

$$U(x,y) = \tfrac{1}{2}[\bar{z}\ \phi(z) + z\overline{\phi(z)} + \overline{\chi(z)}] \tag{5}$$

where $z = x + iy$; $\phi(z)$ and $\chi(z)$ are analytical functions whose complex conjugates are $\phi(z)$ and $\chi(z)$, respectively. These functions can be determined from the boundary condition of the particular problem. It has been shown (Timoshenko & Goodier 1962) that the normal and shear stresses can be expressed as

$$\sigma_x + i\tau_{xy} = \phi'(z) + \overline{\phi'(z)} - z\ \overline{\phi''(z)} - \overline{\chi''(z)} \tag{6a}$$

$$\sigma_y - i\tau_{xy} = \phi'(z) + \overline{\phi'(z)} + z\ \overline{\phi''(z)} + \overline{\chi''(z)} \tag{6b}$$

By adding and subtracting Equations 6a to 6b, then replacing i by $-i$, and substituting $\phi'(z)$ and $\chi''(z)$ for $\Phi(z)$ and $\Psi(z)$, respectively, the following expression can be obtained:

$$\sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi(z)}] = 4\mathrm{Re}[\Phi(z)] \tag{7a}$$

$$\sigma_y - \sigma_x - 2\ i\tau_{xy} = 2[z\,\overline{\Phi'(z)} + \overline{\Psi(z)}] \tag{7b}$$

$$\sigma_y - \sigma_x + 2\ i\tau_{xy} = 2[\bar{z}\,\Phi'(z) + \Psi(z)] \tag{7c}$$

COMPLEX REPRESENTATION OF DISPLACEMENTS

Starting with Hooke's Law, the stress-strain relationships can be expressed as

$$\varepsilon_x = \frac{1}{E}[\sigma_x - v(\sigma_y + \sigma_z)] \tag{8a}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - v(\sigma_z + \sigma_x)] \tag{8b}$$

$$\varepsilon_z = 0 \tag{8c}$$

where ε_x, ε_y and ε_z are the Cartesian strains in the direction of σ_x, σ_y and σ_z respectively, v and E are the Poisson's ratio and the modulus of elasticity of the material respectively. Under plane strain conditions σ_z is dependent on σ_x and σ_y according to the expression $\sigma_z = v(\sigma_x + \sigma_y)$. By substituting this value for σ_z in Equations 8a and 8b the following expressions can be obtained:

$$\frac{E}{1+v}\,\varepsilon_x = (1-v)\,\sigma_x - v\,\sigma_y = 2G\varepsilon_x \tag{9a}$$

$$\frac{E}{1+v}\,\varepsilon_y = (1-v)\,\sigma_y - v\,\sigma_x = 2G\varepsilon_y \tag{9b}$$

where G is the shear modulus of the material. By substituting Equation 9 in Equation 7, and after some algebraic manipulation the following expression is generated:

$$2G\varepsilon_x = (1\text{–}2v)\,[\Phi(z) + \overline{\Phi(z)}] - \bar{z}\,\Phi'(z) - z\,\overline{\Phi'(z)} - \overline{\Psi(z)} - \Psi(z) \tag{10a}$$

$$2G\varepsilon_y = (1-2v)\,[\Phi(z) + \overline{\Phi(z)}] + \bar{z}\,\Phi'(z) + z\,\overline{\Phi'(z)} + \overline{\Psi(z)} + \Psi(z) \tag{10b}$$

Integrating Equation 10a with respect to x and Equation 10b with respect to iy, and adding the results the following expression is obtained:

$$2G\,(u + i\,v) = (3\text{–}4v)\,\phi(z) - z\,\overline{\Phi(z)} - \overline{\psi(z)} + f(x) + ig(y) \tag{11}$$

where u and v are the displacements in the x and y directions respectively, $\Psi(z) = \psi'(z)$, $f(x)$ is a function of x only, and $g(y)$ is a function of y only.

Taking the derivative of Equation 11 and comparing it with Equation (10) the following conclusion can be reached:

$$f'(x) + g'(y) = 0 \tag{12}$$

It follows that $f'(x) = -g'(y) = C$ where C is a constant. Thus the functions $f(x)$ and $g(y)$ represent rigid body displacement in the z-plane and they do not influence the stresses or strains. The constant C attains definite value if one assumes an initial rigid body displacement of the region under consideration. If $f(x)$ and $g(y)$ are discarded, then Equation 11 takes the following form:

$$2G\,(u + i\,v) = (3\text{–}4v)\,\phi(z) - z\,\overline{\Phi(z)} - \overline{\psi(z)} \tag{13}$$

REPRESENTATION IN CURVILINEAR COORDINATES

Since the plane of contact between the pile and the surrounding soil contains a curved boundary (Fig. 2), the solution will be simplified by mapping the curve geometry of the pile onto a simpler geometry such as a semi infinite space. A function which facilitates such a transformation can be represented by the following:

$$z = x + iy = f(t) = f(r + i\,s) \tag{14}$$

where r and s are curvilinear coordinates in the t-plane, such that $t = r + i\,s$.

It is more appropriate to transform the functions $\phi(z)$, $\Phi(z)$ and $\psi(z)$ from the z-plane to the corresponding functions in the t-plane. This can be accomplished as follows:

$$\phi(z) = \phi[f(t)] = \phi(t) \tag{15a}$$

$$\phi'(z) = \frac{\phi'(t)}{f'(t)} = \phi(t) \tag{15b}$$

$$\Psi(z) = \Psi[f(t)] = \Psi(t) \tag{15c}$$

$$\Psi'(z) = \frac{\Psi'(t)}{f'(t)} = \psi(z) \tag{15d}$$

It has also been shown (Timoshenko & Goodier 1962) that stresses and deformations in the z-plane and the corresponding one in the t-plane can be expressed as

$$\sigma_r + \sigma_s = \sigma_x + \sigma_y \tag{16a}$$

$$\sigma_s - \sigma_r + 2i\,\tau_{rs} = (\sigma_y - \sigma_x + 2i\,\tau_{xy})\frac{f'(t)}{\overline{f'(t)}} \tag{16b}$$

$$(\zeta + i\eta) = (u + iv)\frac{f'(t)}{\overline{f'(t)}} \tag{16c}$$

where ξ and η are components of deformations in the r and s directions of the t-plane.

Replacing the functions in the z-plane by the corresponding one in the t-plane, Equation 13 may be reduced to the following expression:

$$2G\,(u + i\,v) = (3\text{–}4v)\,\phi(t) - \overline{\Phi(t)} - \overline{\psi(t)} \tag{17}$$

From Equation 17, the value of u and v can be obtained as

$$u = \frac{1}{2G}\,\mathrm{Re}\,[(3\text{–}4v)\,\phi(t) - \overline{\Phi(t)} - \overline{\psi(t)}] \tag{18a}$$

$$v = \frac{1}{2G}\,\mathrm{Im}\,[(3\text{–}4v)\,\phi(t) - \overline{\Phi(t)} - \overline{\psi(t)}] \tag{18b}$$

In a similar manner the strains as expressed in Equation 9 can be written as

$$\varepsilon_x = \frac{1-2v}{G}\,\mathrm{Re}[\Phi(t)] - \frac{1}{2G}\left[\overline{f(t)}\,\frac{\Phi'(t)}{f'(t)} + \Psi(t)\right] \tag{19a}$$

$$\varepsilon_y = \frac{1-2v}{G}\,\mathrm{Re}[\Phi(t)] + \frac{1}{2G}\left[\overline{f(t)}\,\frac{\Phi'(t)}{f'(t)} + \Psi(t)\right] \tag{19b}$$

BOUNDARY CONDITION

Let σ_n and τ be the normal and shear stresses, respectively, applied to the boundary of the problem (Fig. 3) in the z-plane, thus

$$(\sigma_n+i\tau)=(\sigma_y+i\tau_{xy})=\left(\frac{\sigma_x+\sigma_y}{2}+\frac{\sigma_y-\sigma_x+2i\tau_{xy}}{2}\right) \tag{20}$$

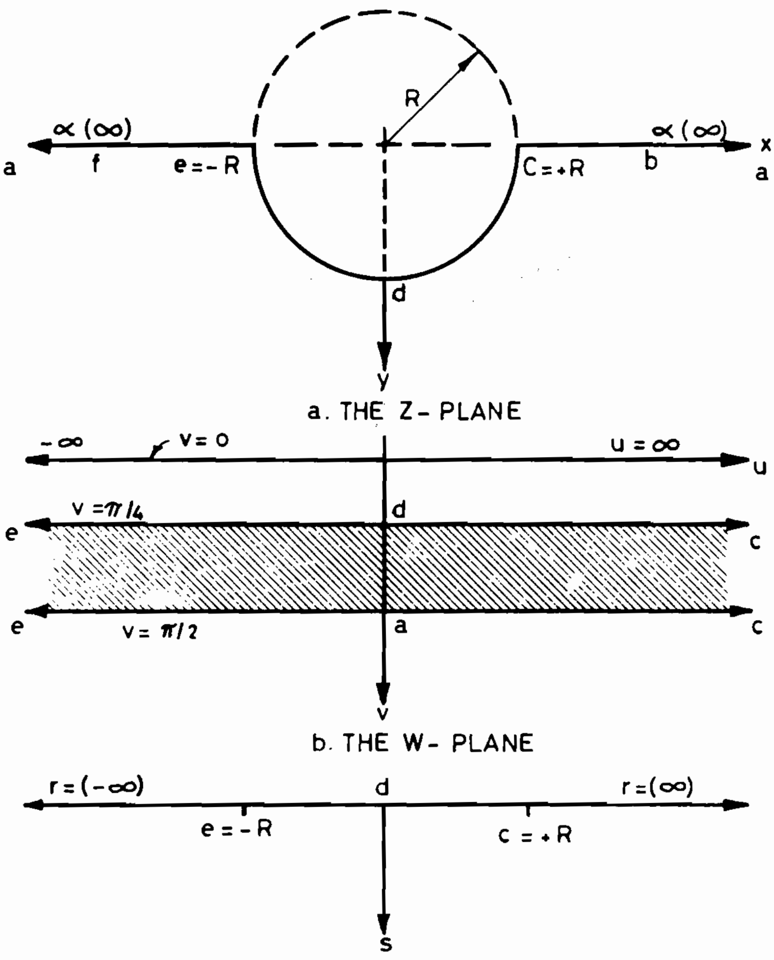

Fig. 3. Transformation of half space with semicircle onto half space.

By utilizing Equations 7, 15 and 16 and after proper substituting in Equation 20 the following is obtained:

$$\sigma_n+i\tau=\Phi(t)+\overline{\Phi(t)}+\overline{f(t)}\,\frac{\Phi'(t)}{\overline{f'(t)}}+\frac{f'(t)}{\overline{f'(t)}}\,\Psi(t) \tag{21}$$

Since the boundary of a semi-infinite mass in the z-plane (Fig. 3) corresponds only to the real axis of the t-plane (i.e. $t=r$ for $s=0$), the boundary condition along the t-plane as expressed by Equation 21, can be expressed as

$$\sigma_n + i\tau = \Phi(r) + \overline{\Phi(r)} + \overline{f(r)}\,\frac{\Phi'(r)}{\overline{f'(r)}} + \frac{f'(t)}{\overline{f'(r)}}\,\Psi(r) \tag{22a}$$

$$\sigma_n - i\tau = \Phi(r) + \overline{\Phi(r)} + f(r)\,\frac{\overline{\Phi'(r)}}{f'(r)} + \frac{\overline{f'(t)}}{f'(t)}\,\Psi(r) \tag{22b}$$

SOLUTION OF THE PROBLEM

The solution of stresses around a laterally loaded pile (Fig. 1) has been previously developed (Al-Hussaini 1982a). This solution is based on the determination of a function that maps the z-plane, represented by the soil influenced by the lateral thrust of the pile (Fig. 3) onto a semi-infinite region represented by the t-plane. The function which accomplishes this transformation is expressed as

$$z = R\tanh\left[\frac{\pi i}{2} - \frac{1}{4}\ln\left(\frac{t-R}{t+R}\right)\right] \tag{23}$$

where R is the radius of the pile.

Since the transformation function between the z-plane and the t-plane is found (Equation 23) the functions $\Phi(t)$ and $\Psi(t)$ need to be determined prior to arriving at a solution for the distribution of deformations. These functions can be determined by applying the Cauchy integral formula (Churchill 1974) to the boundary conditions as expressed in Equation 22. After performing the integration, the values of Φ, Φ', and $\psi(t)$ can be expressed as

$$\Phi(t) = \frac{\sigma_n - i\tau}{2\pi i}\ln\left(\frac{t-R}{t+R}\right) \tag{24a}$$

$$\Phi'(t) = \frac{\sigma_n - i\tau}{\pi i}\left(\frac{R}{t^2-R^2}\right) \tag{24b}$$

$$\Psi(t) = \frac{\tau}{\pi}\ln\left(\frac{t-R}{t+R}\right) - \frac{f(t)\,\Phi'(t)}{f'(t)} \tag{24c}$$

Equation 19 in conjunction with Equation 24 are sufficient for determining strains within any point surrounding the pile. However the deformations u and v cannot be determined unless the value of $\phi(t)$ and $\psi(t)$ are known.

DETERMINATION OF $\phi(t)$ AND $\psi(t)$

The value of the function $\phi(t)$ can be obtained directly from Equation 24a.

$$\phi(t) = \int \Phi(t)\,\mathrm{d}t = \int \frac{\sigma_n - i\tau}{2\pi i}\ln\left(\frac{t-R}{t+R}\right)\mathrm{d}t + C$$

$$\phi(t) = \frac{\sigma_n - i\tau}{2\pi i}\left[(t-R)\ln(t-R) - (t+R)\ln(t+R) + 2\mathrm{R}\right] + C \tag{25}$$

In a similar manner we can integrate Equation 24c using the following procedure:

$$\psi(t)=\int \Psi(t)\,\mathrm{dt} = \left[\frac{\tau}{\pi}\ln\left(\frac{t-R}{t+R}\right)-\frac{f(t)\,\Phi'(t)}{f'(t)}\right]\mathrm{dt}+C_1 \qquad (26)$$

where

$$f(t)=R\tanh\left[\frac{\pi i}{2}-\frac{1}{4}\ln\left(\frac{t-R}{t+R}\right)\right]$$

$$f'(t)=-\frac{R^2}{2(t^2-R^2)}\operatorname{sech}^2\left[\frac{\pi i}{2}-\frac{1}{4}\ln\left(\frac{t-R}{t+R}\right)\right]$$

$$\Phi'(t)=\frac{\sigma_n-i\tau}{\pi i}\left(\frac{R}{t^2-R^2}\right)$$

After performing the integration and rearranging the terms the value of $\psi(t)$ can be written as

$$\psi(t)=\frac{\sigma_n-i\tau}{\pi i}\sqrt{t^2-R^2}+\frac{\tau}{\pi}\left[(t-R)\ln(t-R)-(t+R)\ln(t+R)-2R\right]+C_1 \qquad (27)$$

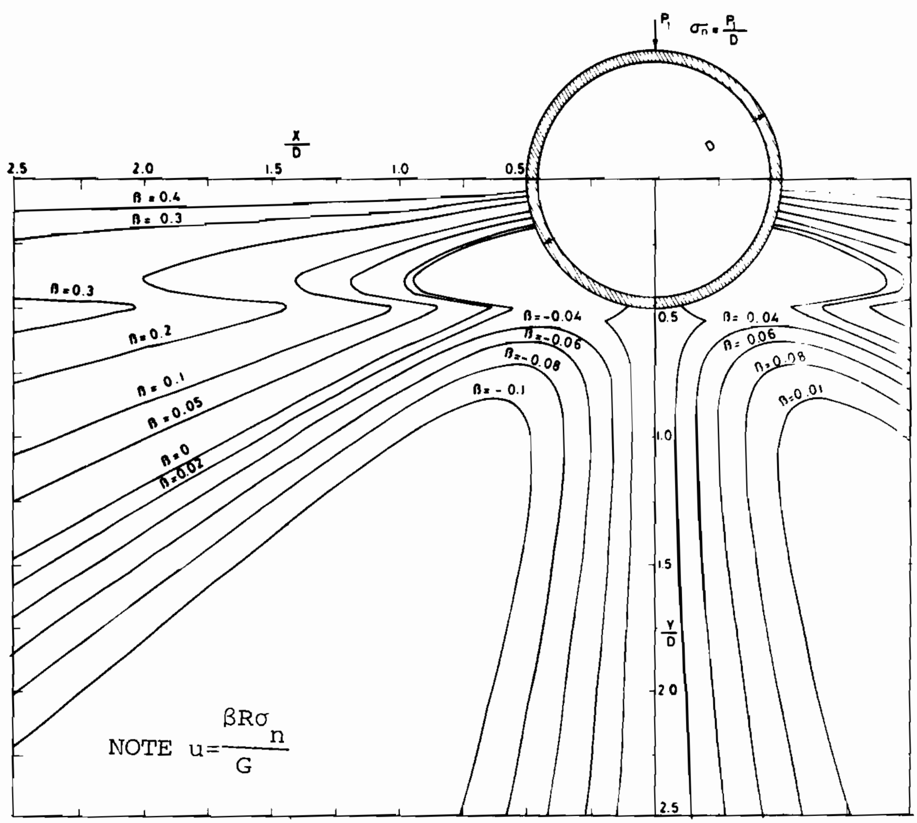

Fig. 4. Isobars of u induced by radial stress σ_n.

By knowing the functions $\Phi(t)$, $\phi(t)$, and $\psi(t)$ the solution of Equation 18 is essentially completed. To obtain a finite displacement the integration was carried within a region $(o<|t|<10R)$; beyond this region displacements are considered negligible.

The conversion of Equation 18 to an explicit equation would result in a very long expression, thus the numerical evaluation of the deformations was carried by a digital computer. In order to simplify the solution, results are presented in a dimensionless form that allows easy determination of the deformation in the affected soil around a laterally and torsionally loaded pile. The solution is presented in the form of isobars of deformations where the ordinate and abscissa are normalized in terms of the pile diameter, D. Isobars for the induced deformations are given in terms of the average contact pressure σ_n (where $\sigma_n = P/D$) or the average contact shear at the level of interest τ (where $\tau = 2T/\pi D^2$). The horizontal deformation u and the vertical deformation v were calculated on the basis that Poisson's ratio, ν, of the soils is equal to 0·48. Isobars for v and u, due to normal stress alone, are presented (Figs. 4 and 5) as $\alpha R\sigma_n/G$ and $\beta R\sigma_n/G$ respectively, where α and β are deformation factors and G is the shear modulus of the soil. Isobars for v and u, due to shear stress alone, are presented (Figs. 6 and 7) as $\gamma R\tau/G$ and $\delta R\tau/G$, where γ and δ are deformation factors, and τ is the average shear stress along the contact surface between the pile and the affected soil.

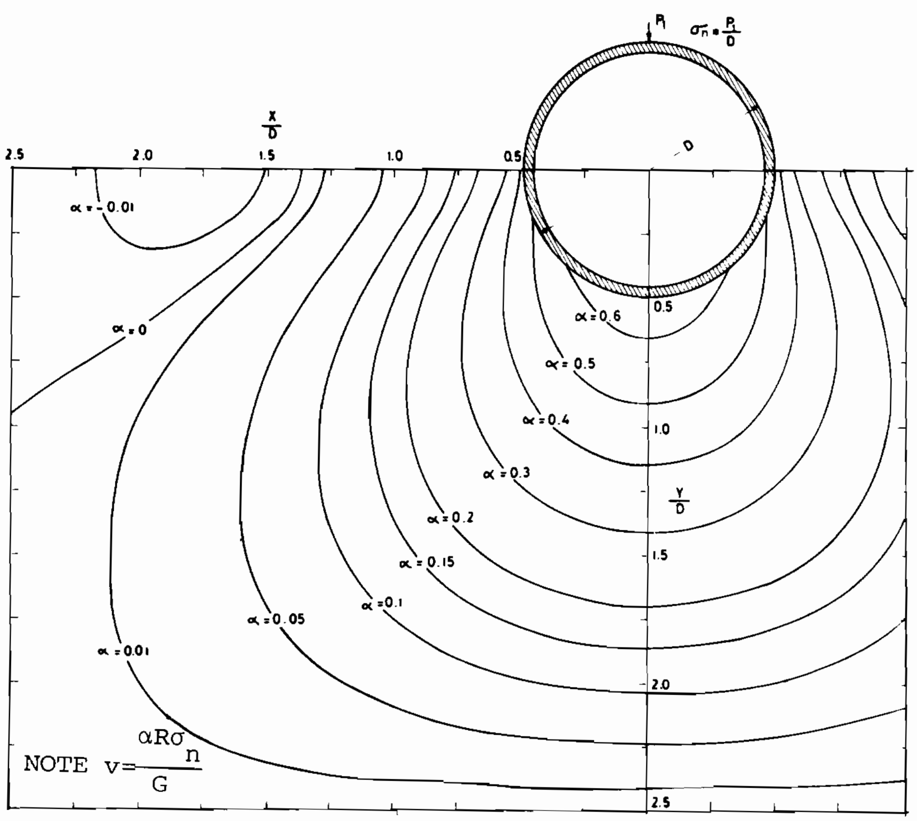

Fig. 5. Isobars of v induced by radial stress σ_n.

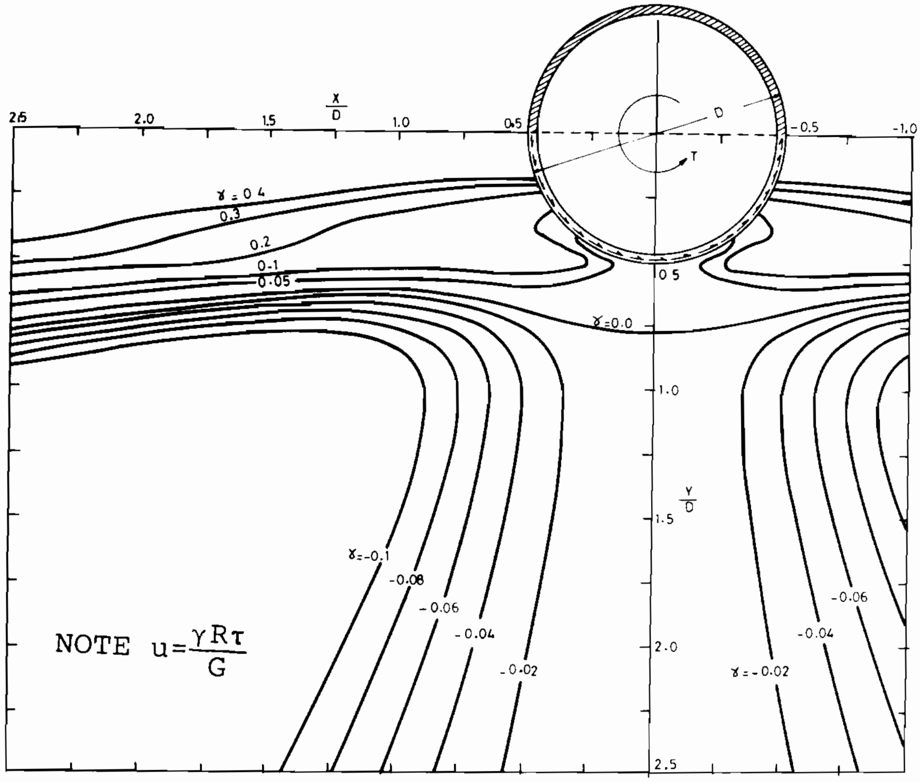

Fig. 6. Isobars of *u* induced by torsional shear stress τ.

ILLUSTRATED EXAMPLE

A 30 in (0·76 m) diameter and 44 ft (13·42 m) long drilled shaft is embedded to a depth of 42 ft (12·81 m) in clay (Fig. 1). Force of 30 ton (267 KN) applied to the top of the shaft induced a maximum lateral thrust of 0·5 ton/in (175 KN/m) at a depth of 3·13 ft (0·95 m) below the ground surface. Compute the soil deformation along the radial direction of the induced thrust. Assume the average shear modulus and Poisson's ratio of the soil to be 4000 psi (13,780 KN/m^2) and 0·48 respectively.

SOLUTION

Let the diameter of the shaft be designated by D, its radius by R, and the induced lateral thrust at a distance L below the ground surface designated by P as shown in Fig. 1. The average contact pressure on the contact surface of the shaft can be determined as

$$\sigma_{\text{ave}} = \sigma_n = \frac{P}{D} = \frac{0{\cdot}5 \times 2000}{30} = 33{\cdot}34 \text{ psi } (230 \text{ KN/m}^2)$$

Since σ_n is determined, the deformation u along the radial direction of P can be calculated from Fig. 4. This can be done by selecting radial distances Y/D and reading the corresponding value of α of the selected points. The value of u is simply calculated as $u = \beta R\sigma_n/G$ as shown in Table 1.

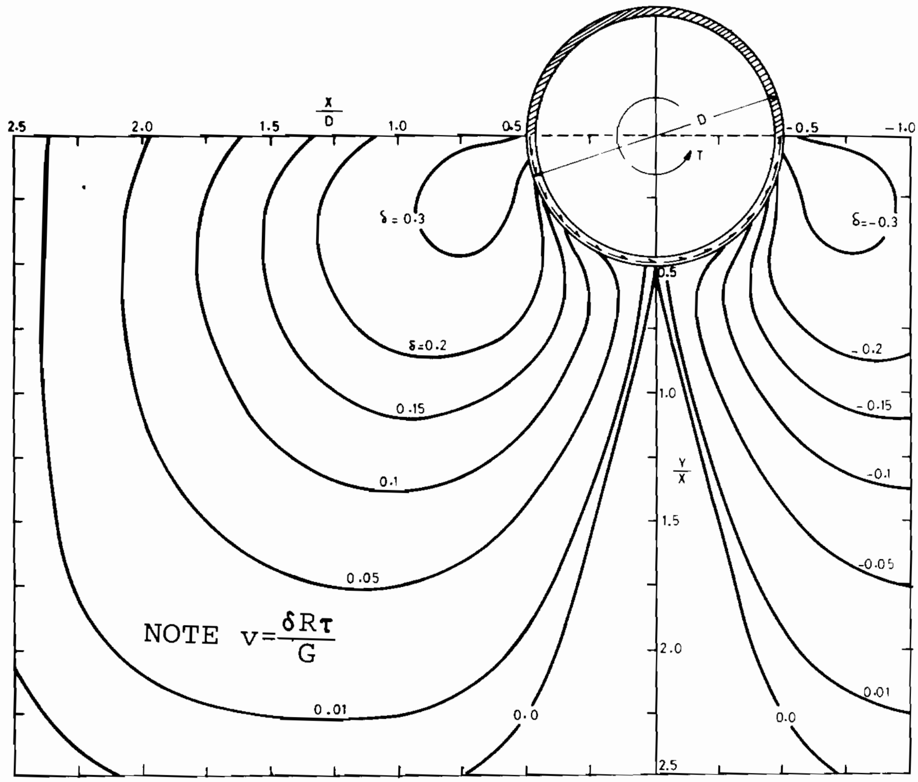

Fig. 7. Isobars of v induced by torsional shear stress τ.

Table 1. Calculation of deformations around circular pile

Radial distance Y(in)	Dimensionless ratio Y/D	Contour reading α	Deformation $u=\beta R\sigma_n/G$ (in)	Deformation u (mm)
15·0	0·50	0·64	0·080	2·03
22·5	0·75	0·57	0·071	1·78
30·0	1·00	0·46	0·058	1·47
37·5	1·25	0·36	0·045	1·14
45·0	1·50	0·26	0·033	0·84
60·0	2·00	0·11	0·014	0·35

The variation of the deformation v along the radial direction of P is presented in Fig. 8.

In this study the soil is assumed to be linearly elastic, therefore the principle of superposition is valid. Consequently the deformations due to lateral load can be added to the deformations due torsional moment when the pile is subjected to both lateral force and torsional moment.

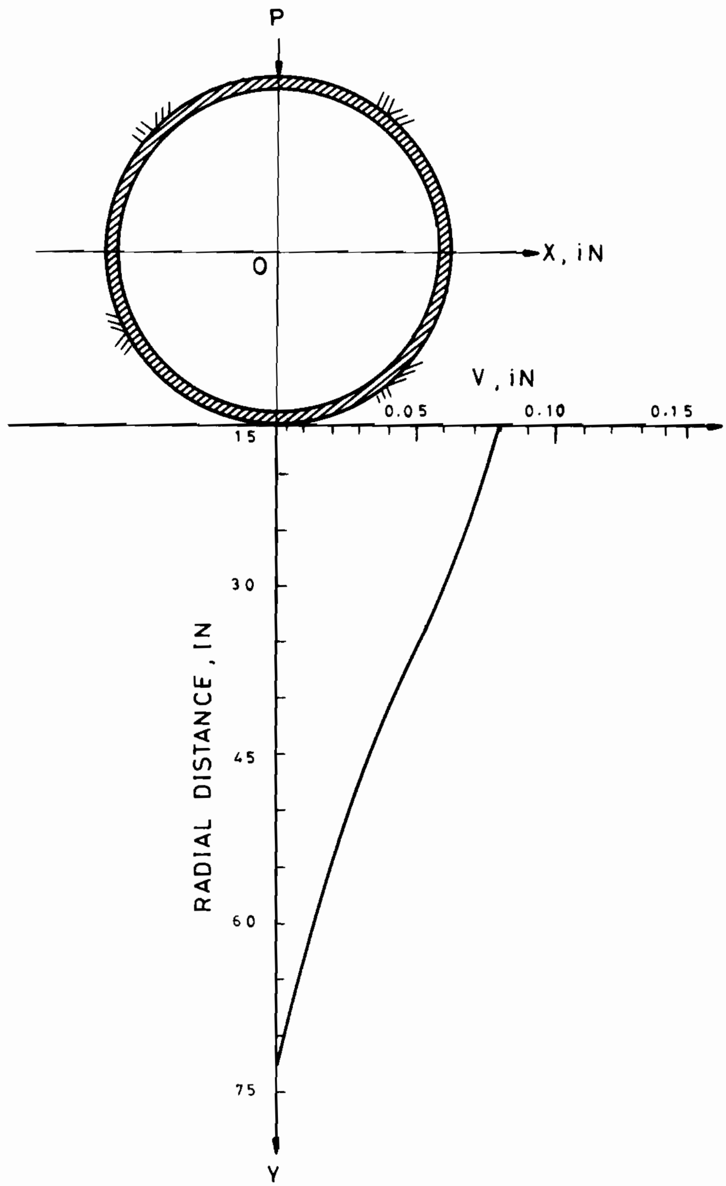

Fig. 8. Variation of radial deformation v in the direction of the induced lateral thrust (1 in = 25·4 mm).

CONCLUSION

In this study a closed form solution is presented for determining deformations around laterally and torsionally loaded pile. The solution is based on the assumption that the pile is relatively rigid, and the contact pressure and shear stresses are uniformly distributed. It is further assumed that the soil medium is linearly elastic and isotropic which is approximately correct for soils under low stress level. The results presented are meant to enhance our understanding of the deformation distribution around laterally and torsionally loaded pile, and may provide an idea of the order of magnitude of the expected deformations. Results of the study are presented in dimensionless graphs to enable the determination of deformation with minimum calculations.

ACKNOWLEDGEMENTS

The study reported herein was supported by Kuwait Foundation for the Advancement of Science under Grant no. (81-08-08). The writer gratefully acknowledges the assistance provided by Dr G.M. Naief in generating the numerical results and to the helpful suggestions provided by Professor W.A. Bassali.

REFERENCES

Airy, G.B. 1863. On the strains in the interior of beams. Roy. Soc. London Phil. Trans. **153:** 49–53.

Al-Hussaini, M. 1982a. Pressure distribution around a laterally loaded pile. J. Univ. Kuwait (Sci.) **9(1):** 63–75.

Al-Hussaini, M. 1982b. Stress distribution around rigid pile subjected to torsion. J. Univ. Kuwait (Sci.) **9(2):** 209–21.

Churchill, R.V. 1974. Complex variables and applications, pp. 129–30. McGraw-Hill, New York.

Timoshenko, S. & Goodier, N.J. 1962. Theory of elasticity, pp 168–79. McGraw-Hill, New York.

(Received 27 February 1983, revised 21 November 1983)

توزيع التغير الشكلي حول ركيزة أساسات
تحت تأثير قوى جانبية
وقوى اللي

مساعد الحسيني
قسم الهندسة المدنية بجامعة الكويت

خلاصة

لقد استنبطنا في هذا البحث حلا رياضيا لايجاد توزيع التغير الشكلي داخل التربة المحيطة بركيزة أساسات مسلط عليها قوى جانبية وقوى ليّ . في هذا الحل اعتبرت التربة متجانسة ، متماثلة الخواص ، وذات مرونة خطية . قدم الحل الرياضي في صورة خطوط بيانية لتجعل عملية ايجاد التغير الشكلي سهلة وسريعة .