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A note on the convergence of sequences of mappings and their common fixed points in a 2-metric space. II

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ABSTRACT

We prove two convergence theorems for a triplet of sequences of mappings from a 2-metric space to itself.

1. INTRODUCTION

Let (X, d) be a 2-metric space (Gähler 1963/64) containing at least three points and P, Q, S and T self-mappings of X. Consider the following conditions for a positive number $h<1$ and for all $x, y, a \in X$:

$$d(Px, Qy, a) \leqslant h \cdot \max\{d(Tx, Ty, a), d(Px, Tx, a), d(Qy, Ty, a), d(Px, Ty, a), d(Qy, Tx, a)\}; \qquad (1.1)$$

$$d(Px, Py, a) \leqslant h \cdot \max\{d(Sx, Ty, a), d(Px, Sx, a), d(Py, Ty, a), \tfrac{1}{2}[d(Px, Ty, a)+d(Py, Sx, a)]\}; \qquad (1.2)$$

$$d(Px, Py, a) \leqslant h \cdot \max\{d(Sx, Ty, a), d(Px, Sx, a), d(Py, Ty, a), d(Px, Ty, a), d(Py, Sx, a)\}. \qquad (1.3)$$

We remark that (1.2) implies (1.3), i.e. any set of three mappings P, S and T satisfying (1.2) will also satisfy (1.3). This note is in continuation of the authors' recent work (Singh & Ram 1981) that contains convergence theorems in the spirit of (1.1). The purpose of this note is to establish convergence theorems in the spirit of (1.3). It may be mentioned that if P, S and T satisfying (1.2) or (1.3) have a common fixed point then it is unique.

2. CONVERGENCE THEOREMS

Theorem 1. Let P_n, S_n and T_n be mappings from a 2-metric space X to itself with a common fixed point u_n for each $n=1, 2, \ldots$. Let $\{P_n\}$, $\{S_n\}$ and $\{T_n\}$ converge respectively to P, S, $T: X \to X$ uniformly. If P, S and T satisfy (1.3) and u be their common fixed point, then $u_n \to u$.

Proof. Fix $\varepsilon_i > 0, i = 1, 2$. Since $P_n \to P$ and $S_n \to S$ uniformly, there exist positive integers N_1 and N_2 such that for all $a \in X$,

$$d(P_n u_n, Pu_n, a) < \varepsilon_1 \text{ for all } n \geqslant N_1$$

and

$$d(S_n u_n, Su_n, a) < \varepsilon_2 \text{ for all } n \geqslant N_2.$$

Let

$$N = \max\{N_1, N_2\} \text{ and } \varepsilon/M = \max\{\varepsilon_1, \varepsilon_2\}$$
$$\text{where } M = \max\{(2h+2)/(1-h), (3h+2)\}.$$

For any n,

$$\begin{aligned} d(u_n, u, a) &\leqslant d(Pu_n, Pu, a) + d(u_n, Pu_n, a) + d(u_n, u, Pu_n) \\ &\leqslant h \cdot \max\{d(Su_n, Tu, a), d(Pu_n, Su_n, a), \\ &\quad d(Pu_n, Tu, a), d(Pu, Su_n, a)\} + d(P_n u_n, Pu_n, a) + \\ &\quad + d(P_n u_n, Pu_n, u), \quad a \in X. \end{aligned}$$

So for all $a \in X$ and for all $n \geqslant N$, one of the following holds:

$$\begin{aligned} d(u_n, u, a) &< hd(Su_n, u, a) + 2\varepsilon/M \\ &\leqslant h[d(Su_n, S_n u_n, a) + d(Su_n, u, S_n u_n) + \\ &\quad + d(u_n, u, a)] + 2\varepsilon/M \end{aligned}$$

that is

(A) $$(1-h)d(u_n, u, a) < (2h+2)\varepsilon/M;$$

$$\begin{aligned} d(u_n, u, a) &< hd(Pu_n, Su_n, a) + 2\varepsilon/M \\ &\leqslant h[d(Pu_n, P_n u_n, a) + d(S_n u_n, Su_n, a) + \\ &\quad + d(Pu_n, Su_n, S_n u_n)] + 2\varepsilon/M \end{aligned}$$

(B) $$< (3h+2)\varepsilon/M;$$

$$\begin{aligned} d(u_n, u, a) &< hd(Pu_n, u, a) + 2\varepsilon/M \\ &\leqslant h[d(Pu_n, P_n u_n, a) + d(u_n, u, a) \\ &\quad + d(Pu_n, u, P_n u_n)] + 2\varepsilon/M \end{aligned}$$

that is

(C) $$(1-h)d(u_n, u, a) < (2h+2)\varepsilon/M;$$

and similarly $d(u_n, u, a) < hd(u, Su_n, a) + 2\varepsilon/M$ gives

(D) $$(1-h)d(u_n, u, a) < (2h+2)\varepsilon/M.$$

Clearly, in each of the cases (A)–(D), $d(u_n, u, a) < \varepsilon$ for all $a \in X$ and for all $n \geqslant N$. Hence $u_n \to u$.

Theorem 2. Let P_n, S_n and T_n be mappings from a 2-metric space (X, d) to itself with a common fixed point u_n for each $n=1, 2, \ldots$, and d continuous. Let self-mappings P, S and T of X be pointwise limits of $\{P_n\}$, $\{S_n\}$ and $\{T_n\}$ respectively. If u is a common fixed point of P, S and T, and

$$d(P_nx, P_ny, a) \leqslant h\cdot\max\{d(S_nx, T_ny, a), d(P_nx, S_nx, a), d(P_ny, T_ny, a), d(P_nx, T_ny, a), d(P_ny, S_nx, a)\} \quad (2.1)$$

for all x, y, a in X, $h \in (0, 1)$, $n=1, 2, \ldots$, then $u_n \to u$.

Proof. For all $a \in X$ and any n,

$$\begin{aligned} d(u_n, u, a) &\leqslant d(P_nu_n, P_nu, a)+d(P_nu, Pu, a)+d(u_n, Pu, P_nu) \\ &\leqslant h\cdot\max\{d(u_n, T_nu, a), d(P_nu, T_nu, a), \\ &\quad d(P_nu, u_n, a)\}+b_n(P)+c_n(P), \end{aligned}$$

where $b_n(P)=d(P_nu, Pu, a)$ and $c_n(P)=d(P_nu, Pu, u_n)$.

So for all $a \in X$ and any n, one of the following holds:

$$\begin{aligned} d(u_n, u, a) &\leqslant hd(u_n, T_nu, a)+b_n(P)+c_n(P) \\ &\leqslant h[d(u_n, u, a)+d(Tu, T_nu, a)+ \\ &\quad +d(u_n, T_nu, Tu)]+b_n(P)+c_n(P) \end{aligned}$$

that is

$$\text{(E)} \qquad (1-h)d(u_n, u, a) \leqslant h[b_n(T)+c_n(T)]+b_n(P)+c_n(P);$$

$$\begin{aligned} d(u_n, u, a) &\leqslant hd(P_nu, T_nu, a)+b_n(P)+c_n(P) \\ &\leqslant h[d(P_nu, Pu, a)+d(Tu, T_nu, a)+ \\ &\quad +d(P_nu, T_nu, Tu)]+b_n(P)+c_n(P) \end{aligned}$$

$$\text{(F)} \qquad = (h+1)b_n(P)+c_n(P)+hb_n(T) + d(T_nu, Tu, P_nu);$$

and $d(u_n, u, a) \leqslant hd(P_nu, u_n, a)+b_n(P)+c_n(P)$ gives

$$\text{(G)} \qquad (1-h)d(u_n, u, a) \leqslant (h+1)[b_n(P)+c_n(P)].$$

Since P_n and T_n are pointwise convergent to P and T respectively, we have

$$\lim_n b_n(P) = 0 = \lim_n b_n(T) \text{ for all } a \text{ in } X,$$

and so, since d is continuous,

$$\lim_n c_n(P) = \lim_n c_n(T) = \lim_n d(T_nu, Tu, P_nu) = 0.$$

Thus in each of the cases (E)–(G)

$$\lim_n d(u_n, u, a) = 0 \text{ for all } a \in X.$$

Consequently $u_n \to u$.

REMARKS

If d is continuous and making $n \to \infty$, it follows from (2.1) that the limit mappings P, S and T satisfy (1.3). If S and T are continuous with respect to the direct argument, P commuting with each of S and T, $P(X) \subseteq S(X) \cap T(X)$, d continuous and X complete and bounded, then P, S and T under the condition (1.3) (the space need not be bounded under the condition (1.2)) have a unique common fixed point (Singh & Lohani 1981). Results analogous to the above theorems in metric spaces have been studied by Singh & Kulshrestha (1981). Convergence results established by Rhoades (1979) and Singh (1979) may be obtained as corollaries to Theorems 1 and 2.

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