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Approximations and positive linear operators

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ABSTRACT

The aim of this paper is to sharpen the results of Censor (1971) and Mohapatra (1977) given on the degree of approximation by positive linear operators.

1. INTRODUCTION

Let $B=\{A^{(n)}\}=\{(a_{pm}^{(n)})\}$ be a sequence of infinite matrices such that $a_{pm}^{(n)}\geqslant 0$ for p, m, $n=1,2\ldots$ A sequence of real numbers $\{x_m\}$ is said to be B-summable to ℓ (Bell 1973) if

$$\lim_{p\to\infty}\sum_{m=1}^{\infty} a_{pm}^{(n)}\, x_m=\ell \text{ uniformly in } n=1,2\ldots$$

If, for some matrix A, $A^{(n)}=A$ for $n=1,2\ldots$; then B-summability is just matrix summability by A. If, for $n=1,2\ldots$;

$$a_{pm}^{(n)}=\frac{1}{p} \text{ for } n+1\leqslant m\leqslant n+p \text{ and } a_{pm}^{(n)}=0$$

otherwise, then B-summability reduces to almost convergence.

Recently some results of Censor (1971) and Mohapatra (1977) on the rate of convergence of sequence of positive linear operators have been unified by Swetits (1979) through the use of B-summability method. The object of this paper is to sharpen the results of Censor (1971) and Mohapatra (1977). Corresponding estimates for some special operators have been deduced also.

Let $\{L_m\}$ be a sequence of positive linear operators on $C[a,b]$ and let $\{A^{(n)}\}=B$ be a sequence of infinite matrices with non negative real entries. For $f\in C[a,b]$, let $A_p^{(n)}(f;x)$ denote the double sequence

$$A_p^{(n)}(f;x)=\sum_{m=1}^{\infty} a_{pm}^{(n)}\, L_m(f(t);x);\ p,n = 1, 2\ldots \qquad (1.1)$$

We define, following Swetits (1979), $\| A_p(f)\|$ to be

$$\sup_n \ \sup_{x\in[a,b]} |A_p^{(n)}(f;x)| \quad .$$

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We say that for $f \in C\,[a,b]$, $\{L_m(f)\}$ is B-summable to f, uniformly on $[a,b]$ if and only if $\| A_p(f)-f \| \to 0$ as $p \to \infty$.

2. THEOREM

We prove the following: Let $\{K_p\}$ be a sequence of positive numbers. Let $\{L_m\}$ be a sequence of positive linear operators on $C[a,b]$. Let $f \in C^1[a,b]$ with modulus of continuity $w(f';)$. Let $B=\{A^{(n)}\}$ be a sequence of infinite matrices with non-negative real entries. Assume that $\| A_p(e_0) \| < \infty$ where $e_0(x)=1$ for all $x \in [a,b]$. Then for, $p=1,2\ldots$

$$\| A_p(f)-f \| \leqslant \|f\| \cdot \| A_p(e_o)-1 \| + \|f'\| \;\; \| A_p(t-x) \| + w(f';\, k_p\, \mu_p)\mu_p \left\{ \| A_p(e_o) \|^{\frac{1}{2}} + \frac{1}{2k_p} \right\} \tag{2.1}$$

However, if in addition $A_p^{(n)}(e_0)\,(x)=1$ and $A_p^{(n)}\,(t)(x)=x$ then

$$\| A_p(f)-f \| \leqslant \left(1+\frac{1}{2k_p}\right) \mu_p \cdot w(f';\, K_p\mu_p) \tag{2.2}$$

where $\mu_p = \| A_p(t-x)^2(x) \|^{\frac{1}{2}}$ and $\| \cdot \|$ norm being sup norm over $[a,b]$.

Proof of Theorem. We know that

$$f(t)-f(x)=f'(x)(t-x)+\int_x^t ((f'(\xi)-f'(x))\, d\xi \;. \tag{2.3}$$

Since

$$|f'(\xi)-f'(x)| \leqslant w(f';|\xi-x|) \leqslant \left\{1+\frac{|\xi-x|}{\delta}\right\} w(f';\,\delta) \text{ where } \delta>0. \tag{2.4}$$

From (2.3), (2.4), the inequalities

$$|A_p^{(n)}\,(f)| \leqslant A_p^{(n)}\,(|f|)$$

and

$$A_p^{(n)}(f \cdot g) \leqslant \{A_p^{(n)}\,(f^2)\}^{\frac{1}{2}} \cdot \{A_p^{(n)}\,(g^2)\}^{\frac{1}{2}}$$

we get that

$$\begin{aligned} &|A_p^{(n)}\,(f)\,(x)-f(x)\,A_p^{(n)}(e_o)(x)| \\ &\leqslant |f'(x)\,A_p^{(n)}(t-x)(x)| + \left| A_p^{(n)}\left\{\int_x^t (f'(\xi)-f'(x))d\xi\right\}(x)\right| \qquad (2.5)\\ &\leqslant |f'(x)| \cdot |A_p^{(n)}(t-x)(x)| + w(f';\,\delta)\; A_p^{(n)}\left\{\left|\int_x^t \left(1+\frac{|\xi-x|}{\delta}\right)d\xi\right|\right\}(x) \;, \\ &\leqslant |f'(x)| \cdot |A_p^{(n)}(t-x)\,(x)| + w(f';\,\delta)\; A_p^{(n)}\left\{|t-x|+\frac{(t-x)^2}{2\delta}\right\}(x) \;, \\ &\leqslant \|f'\| \cdot \| A_p(t-x) \| + w(f';\,\delta)\left\{\mu_p \cdot \| A_p(e_o) \|^{\frac{1}{2}} + \frac{(\mu_p)^2}{2\delta}\right\} \;. \end{aligned}$$

Choosing $\delta = K_p\mu_p$, this reduces to

$$|A_p^{(n)}(f)(x) - (f(x)\, A_p^{(n)}(e_o)(x)|$$

$$\leqslant \| f' \| \cdot \| A_p(t-x) \| + w(f'; K_p\mu_p)\cdot \mu_p \left\{ \| A_p(e_o) \|^{\frac{1}{2}} + \frac{1}{2K_p} \right\} \tag{2.6}$$

Clearly

$$|-f(x) + f(x)\cdot A_p^{(n)}(e_o)(x)| \leqslant \| f \| \ \| A_p(e_o) - 1 \| \tag{2.7}$$

On adding (2.6) and (2.7), we get (2.1).
In case $\mu_p = 0$, then for every $\delta > 0$ we get from (2.6) that

$$A_p^{(n)}(f)(x) = f(x)\, A_p^{(n)}(e_o)(x).$$

So

$$|A_p^{(n)}(f)(x) - f(x)| = |f(x)\, A_p^{(n)}(e_o)(x) - f(x)| \leqslant \| f \| \cdot \| A_p(e_o) - 1 \| \quad .$$

Again if $A_p^{(n)}(e_o)(x) = 1$ and $A_p^{(n)}(t)(x) = x$; then $A_p^{(n)}(t-x)(x) = 0$ and from (2.5) we get (2.2).

This completes the proof.

3. REMARKS

We note that by choosing $a_{pm}^{(n)} = \delta_m^p$ in the above theorem, one obtains an estimate sharper than that of Censor (1971, Theorem 6). On the other hand in the case of $a_{pm}^{(n)} = \frac{1}{p}$, $n+1 \leqslant m \leqslant n+p$; $a_{pm}^{(n)} = 0$ otherwise, our theorem is sharper than that of Mohapatra (1977, Theorem 4).

We have deduced the following estimates by choosing $a_{pm}^{(n)} = \delta_m^p$ in each of the cases given below.

Case 1. For $f \in C\,[0, 1]$, let

$$L_m(f)(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k}{m}\right)$$

be the Bernstein operator of order m. We know that

$$L_m(1)(x) = 1,\ L_m(t)(x) = x,\ L_m(t-x)^2(x) = \frac{x(1-x)}{m}$$

So for

$$m \geqslant 1,\ \mu_m = \frac{1}{2\sqrt{m}} \quad .$$

By choosing $K_m = 2$ in (2.2), one obtains for $f \in C^1[0,1]$

$$\| L_m(f) - f \| \leqslant \tfrac{5}{8}\cdot \frac{1}{\sqrt{m}}\, w\left(f'; \frac{1}{\sqrt{m}}\right), \quad (m \geqslant 1)$$

which is sharper than the corresponding estimate of Lorentz (1953). Again by choosing

$$K_m = \frac{2}{\sqrt{m}}$$

in (2.2), we obtain from $f \in C^1\,[0,1]$

$$\| L_m(f) - f \| \leqslant \left\{ \frac{1}{2\sqrt{m}} + \frac{1}{8} \right\} w\left(f', \frac{1}{m} \right), \; (m \geqslant 1). \tag{3.1}$$

The result (3.1) is due to Schurer *et al.* (1976).

Case 2. For $f \in C[0, \infty)$, let

$$L_m^\lambda\,(f)(x) = e^{-mx}\,\mathrm{Sech}\,(2\lambda\sqrt{mx}) \sum_{k=0}^{\infty} \frac{(-1)^k H_{2k}\,(i\lambda)}{k}\,(mx)^k f\left(\frac{k}{m}\right)$$

where

$$\frac{H_{2k}(i\lambda)}{k} = \sum_{v=0}^{k} \frac{(-1)^v (2i\lambda)^{2k-2v}}{v \quad 2k-2v}\,, \; \lambda \text{ real}$$

be the positive linear operator of order m introduced by Meir & Sharma (1967). We know that

$$L_m^\lambda(1)(x) = 1,$$

$$L_m^\lambda(t-x)(x) = \lambda \sqrt{\frac{x}{m}}\,\tanh\,(2\lambda\sqrt{mx}) \text{ and}$$

$$L_m^\lambda(t-x)^2(x) = \frac{(\lambda^2+1)x}{m} + \frac{\lambda\sqrt{x}}{2m\sqrt{m}}\,\tanh\,(2\lambda\sqrt{mx})$$

So for $m \geqslant 1$, and for all $x \in [o,a]$

$$\mu_m = \sqrt{\frac{(\lambda^2+1)a}{m} + \frac{\lambda\sqrt{a}}{2m\sqrt{m}}\,\tanh\,(2\lambda\sqrt{ma})}$$

$$= \frac{\sqrt{D_m}}{m} \text{ (say)} \quad .$$

By choosing

$$K_m = \frac{1}{\sqrt{D_m}}$$

in (2.2), we get for $f \in C^1[o,a]$, $m \geqslant 1$,

$$\| L_m f - f \| \leqslant \frac{\sqrt{a}}{\sqrt{m}}\,\tanh\,(2\lambda\sqrt{ma}) \cdot \| f' \| + \left(D_m + \frac{D_m^2}{2} \right) \cdot \frac{1}{\sqrt{m}} \cdot w\left(f'; \frac{1}{\sqrt{m}} \right) \quad . \tag{3.2}$$

We note that by choosing $\lambda=0$ in (3.2), one obtains for Szâsz operators

$$\| L_m f - f \| \leqslant \left(\sqrt{a} + \frac{a}{2}\right) \cdot \frac{1}{\sqrt{m}} \cdot w\left(f', \frac{1}{\sqrt{m}}\right)$$

Which is sharper than the corresponding estimate of Stancu (1969)

$$\| L_m f - f \| \leqslant (\sqrt{a} + a) \cdot \frac{1}{\sqrt{m}} \cdot w\left(f'; \frac{1}{\sqrt{m}}\right) \quad .$$

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خلاصة

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