J. Univ. Kuwait (Sci.) **10,** 1983

On $|N,p_n|_k$ summability factors of infinite series

HÜSEYIN BOR*

Department of Mathematics, University of Ankara, Turkey

ABSTRACT

Through this paper, it is proved that, if $s_n = a_1 + a_2 + a_3 + \dots + a_n$,

$$\sum_{n=1}^{m} |s_n - a_1|^k |\lambda_n|^k |\phi_n|^k \frac{p_n}{Pn} = 0(1)$$

$$\sum_{n=1}^{m} |s_n - a_1|^k |\lambda_n|^k |\Delta\phi_n|^k \left(\frac{P_n}{p_n}\right)^{k-1} = 0(1),$$

and

$$\sum_{n=1}^{m} |s_n - a_1|^k |\Delta\lambda_n|^k |\phi_{n+1}|^k \left(\frac{P_n}{p_n}\right)^{k-1} = 0(1) \text{ as } n \to \infty,$$

then the series $\Sigma a_n \lambda_n \phi_n$ is summable $|\bar{N},p_n|_k$ $(k \geqslant 1)$.

For the special case $k = 1$, the theorem given by Daniel (1964) is obtained.

1. INTRODUCTION

Let

$$\sum_{n=1}^{\infty} a_n$$

be a given infinite series with partial sums s_n and (p_n) be a sequence of positive real constants such that

$$P_n = p_0 + p_1 + p_2 + \dots + p_n \to \infty \quad \text{as } n \to \infty, (P_{-1} = p_{-1} = 0). \tag{1.1}$$

We write

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v. \tag{1.2}$$

* Present address: Department of Mathematics, Erciyes University, Sumer, Kayseri, Turkey.

The series Σa_n is said to be absolutely summable $(\bar{N},p_n)$ with index k, or simply summable $|\bar{N},p_n|_k$ $(k \geqslant 1)$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty \text{ ([1]).}$$

If we take $k=1$, then $|\bar{N},p_n|_k$ summability is identical with $|\bar{N},p_n|$summability.

2. DANIEL'S THEOREM

Daniel (1964) proved the following theorem:

Theorem A. If $s_n = a_1 + a_2 + \ldots + a_n$ and

$$\sum_{n=2}^{\infty} |s_n - a_1|\,|\lambda_n|\,|\Delta\phi_n| < \infty \tag{2.1}$$

$$\sum_{n=2}^{\infty} |s_n - a_1|\,|\lambda_n|\,|\phi_n| - \frac{p_n}{P_n} < \infty \tag{2.2}$$

and

$$\sum_{n=2}^{\infty} |s_n - a_1|\,|\phi_{n+1}|\,|\Delta\lambda_n| < \infty, \tag{2.3}$$

then the series $\Sigma a_n \lambda_n \phi_n$ is summable $|\bar{N},p_n|$.

3. EXTENSION OF THEOREM A

The object of this paper is to extend Theorem A for $|\bar{N},p_n|k$ summability in the form of the following theorem:

Theorem. If $s_n = a_1 + a_2 + a_3 + \ldots + a_n$, and

$$\sum_{n=1}^{m} |s_n - a_1|^k |\lambda_n|^k |\phi_n|^k \frac{p_n}{P_n} = 0(1) \tag{3.1}$$

$$\sum_{n=1}^{m} |s_n - a_1|^k |\lambda_n|^k |\Delta\phi_n|^k \left(\frac{P_n}{p_n}\right)^{k-1} = 0(1) \tag{3.2}$$

and

$$\sum_{n=1}^{m} |s_n - a_1|^k |\Delta\lambda_n|^k |\phi_{n+1}|^k \left(\frac{P_n}{p_n}\right)^{k-1} = 0(1) \text{ as } m \to \infty, \tag{3.3}$$

then the series $\Sigma a_n \lambda_n \phi_n$ is summable $|\bar{N},p_n|_k$ $(k \geqslant 1)$.

4. PROOF OF THE THEOREM

Let T_n denote the $(\bar{N}, p_n)$ mean of the series $\Delta a_n \lambda_n \phi_n$. Then

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r \phi_r. \tag{4.1}$$

$$\begin{aligned}
T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} \lambda_v \phi_v \\
&= -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (s_v - a_1) p_v \lambda_v \phi_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (s_v - a_1) P_v \lambda_v \Delta\phi_v + \\
&\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (s_v - a_1)\, P_v \Delta\lambda_v \phi_{v+1} - \frac{1}{P_n}(s_n - a_1)\, P_n \lambda_n \phi_n \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.}
\end{aligned} \tag{4.2}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k = 0(1),\ m \to \infty, \text{ for } r = 1, 2, 3, 4. \tag{4.3}$$

Now, applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,1}|^k &= \sum_{n=2}^{m-1} \left(\frac{P_n}{p_n}\right)^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (s_v - a_1) p_v \lambda_v \phi_v \right|^k \\
&\leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} |s_v - a_1|^k |\lambda_v|^k |\phi_v|^k p_v \times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right)^{k-1} \\
&= 0(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} |s_v - a_1|^k |\lambda_v|^k |\phi_v|^k p_v \\
&= 0(1) \sum_{V=1}^{m} |s_v - a_1|^k |\lambda_v|^k |\phi_v|^k p_v \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
&= 0(1) \sum_{v=1}^{m} |s_v - a_1|^k |\lambda_v|^k |\phi_v|^k \frac{p_v}{P_v} = 0(1), \text{ as } m \to \infty \text{ by (3.1).}
\end{aligned}$$

Also

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k = \sum_{n=2}^{m-1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}(s_v-a_1)P_v\lambda_v\Delta\phi_v\right|^k$$

$$= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}(s_v-a_1)\frac{P_v}{p_v}p_v\lambda_v\Delta\phi_v\right|^k$$

$$\leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}|s_v-a_1|^k\left(\frac{P_v}{p_v}\right)^k|\lambda_v|^k|\Delta\phi_v|^k p_v \times \left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}$$

$$= 0(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}|s_v-a_1|^k|\lambda_v|^k|\Delta\phi_v|^k\left(\frac{P_v}{p_v}\right)^k p_v$$

$$= 0(1)\sum_{v=1}^{m}|s_v-a_1|^k|\lambda_v|^k|\Delta\phi_v|^k\left(\frac{P_v}{p_v}\right)^k p_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$

$$= 0(1)\sum_{v=1}^{m}|s_v-a_1|^k|\lambda_v|^k|\Delta\phi_v|^k\left(\frac{P_v}{p_v}\right)^{k-1} = 0(1), \text{ as } m\to\infty \text{ by (3.2).}$$

Similarly, using (3.3) we get

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k = 0(1), \text{ as } m\to\infty.$$

Finally, we have

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,4}|^k = \sum_{n=1}^{m}|s_n-a_1|^k|\lambda_n|^k|\phi_n|^k\frac{p_n}{P_n} = 0(1), \text{ as } m\to\infty \text{ by (3.1).}$$

Therefore, we get

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}|^k = 0(1), \text{ as } m\to\infty, \text{ for } r = 1, 2, 3, 4.$$

which completes the proof of the theorem.

The special case of this theorem for $k=1$ gives us Theorem A.

REFERENCES

Bor, H. 1982. On $|\bar{N},p_n|_k$ summability method and $|\bar{N},p_n|_k$ summability factors of infinite series. PH.D.Thesis, University of Ankara.

Daniel, E.C. 1964. On absolute summability factors of infinite series. Japan Acad. J. **40:** 65–9.

(Received 3 April 1982)

حول $|N,p_n|_k$ قابلية الجمع للمتسلسلات اللانهائية

حسين بور
قسم الرياضيات بجامعة أنقرة ، تركيا

خلاصة

في هذا البحث تم اثبات أن المتسلسلة $\Sigma a_n \lambda_n \phi_n$ قابلة للجمع $|\bar{N},p_n|_k, (k \geqslant 1)$. وفي الحالة الخاصة $k = 1$ ، أمكن استنتاج مبرهنة دانيال .