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L^p-Projections and Schur multipliers

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ABSTRACT

In this paper, we prove that $\phi \in m(\ell^p_n \hat{\otimes} \ell^{p^*}_n)$ and $\| \phi \|_m \leq 1$ if and only if

$$\left| \sum_{i,j=1}^{n} \phi(i,j) \langle P(i)x, Q(j)y \rangle \right| \leqslant 1$$

for all pairs (P,Q) of p-orthogonal and p^*-orthogonal projections on ℓ^p_n and $\ell^{p^*}_n$ respectively, with

$$\frac{1}{p} + \frac{1}{p^*} = 1 \quad .$$

INTRODUCTION

Let ℓ^p_n denote the n-dimensional Banach space of all n-tuples $x = \{x(i)\}_{i=1}^{n}$ with the p-norm

$$\| x \|_p = \left(\sum_{i=1}^{n} | x(i) |^p \right)^{\frac{1}{p}}, \ 1 \leq p < \infty \quad .$$

The dual space of ℓ^p_n is the Banach space

$$\ell^{p^*}_n, \ \frac{1}{p} + \frac{1}{p^*} = 1 \quad .$$

The completion of the projective tensor product of ℓ^p_n with $\ell^{p^*}_n$, denoted by $\ell^p_n \hat{\otimes} \ell^{p^*}_n$, is the space of all functions F on $Z^+_n \times Z^+_n$ which have a representation

$$F(i,j) = \sum_{r=1}^{k} u_r(i)\, v_r(j) \tag{1}$$

and

$$\| F \|_{\tau_r} = \inf \left\{ \sum_{r=1}^{k} \| u_r \|_p \cdot \| v_r \|_{p^*} \right\},$$

where the infimum runs over all representations of F as in (1). For further information about tensor products of Banach spaces, we refer to Diestel & Uhl (1977) and Wong (1979).

A function $\phi \in \ell^\infty \ (Z^+_n \times Z^+_n)$ is called a Schur multiplier of $\ell^p_n \hat{\otimes} \ell^{p^*}_n$ if

$\phi\,\psi \in \ell^p_n \hat{\otimes} \ell^{p^*}_n$ for all $\psi \in \ell^p_n \hat{\otimes} \ell^{p^*}_n$, where $(\phi \cdot \psi)\,(i,j) = \phi(i,j) \cdot \psi(i \cdot j)$. The space of Schur multipliers of $\ell^p_n \hat{\otimes}\,{}^{p^*}_n$ will be denoted by M. The space M becomes a Banach space when we define

$$\| \phi \|_m = \sup_{\psi} \{ \| \phi \cdot \psi \|_{\tau_r}, \| \psi \|_{\tau_r} \leq 1 \}, \ \phi \in M \quad .$$

Schur multipliers of $\ell^p \hat{\otimes} \ell^q$ for arbitrary $p,q > 1$ were studied by Bennett (1977). The special case $p = q = 2$ was studied by Khalil (1978). The work of both authors is far from being complete, since it is still a difficult problem to determine whether a given function ϕ is a Schur multiplier of $\ell^p \hat{\otimes} \ell^q$ or not. Khalil (1980) proved the following theorem:

Theorem 0.1. The following are equivalent:

(i) $\phi \in m(\ell^2_n \hat{\otimes} \ell^2_n), \qquad \| \phi \|_m \leq 1$

(ii) $\left| \sum_{i,j=1}^{n} \phi(i,j) \langle P(i)x, Q(j)y \rangle \right| \leq 1,$

for all pairs (P,Q) of spectral measures on Z^+_n and all $x,y \in \ell^2_n$, $\| x \| = \| y \| = 1$. One can deduce the following simple corollary:

Corollary 0.1. Let ϕ: $Z^+_n \times Z^+_n \to C$ be such that

$$\sup_j \left(\sum_i | \phi(i,j) |^2 \right)^{\frac{1}{2}} \leq 1,$$

then

$$\phi \in m(\ell^2_n \hat{\otimes} \ell^2_n)$$

and

$$\| \phi \|_m \leq 1 \quad .$$

Proof. By Theorem 0.1, $\phi \in m(\ell^2_n \hat{\otimes} \ell^2_n)$, $\| \phi \|_m \leq 1$ if and only if

$$\left| \sum_{i,j=1}^{n} \phi(i,j) \langle x_i, y_j \rangle \right| \leq 1$$

for any pair of sets, $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$, of orthogonal elements such that

$$\sum_{i=1}^{n} \| x_i \|_2^2 \leq 1, \ \sum_{j=1}^{n} \| y_j \|_2^2 \leq 1 \quad .$$

Set

$$\tilde{y}_i = \sum_{j=1}^{n} \phi(i,j) y_j \quad .$$

Then

$$\left| \sum_{i,j=1}^{n} \phi(i,j) \langle x_i, y_j \rangle \right| = \left| \sum_{i=1}^{n} \langle x_i, \sum_{j=1}^{n} \phi(i,j) y_j \rangle \right|$$

$$= \left| \sum_{i=1}^{n} \langle x_i, \tilde{y}_i \rangle \right|$$

$$\leq \left(\sum_{i=1}^{n} \| x_i \|_2^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} \| \tilde{y}_i \|_2^2 \right)^{\frac{1}{2}} \text{ (by Schwartz ineq.).}$$

$$= \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} | \phi(i,j) |^2 \cdot \| y_j \|_2^2 \right) \right)^{\frac{1}{2}}$$

$$= \left(\sum_{j=1}^{n} \left(\sum_{i=1}^{n} | \phi(i,j) |^2 \right) \cdot \| y_j \|_2^2 \right)^{\frac{1}{2}}$$

$$\leq \sup_j \left| \left(\sum_{i=1}^{n} | \phi(i,j) |^2 \right)^{\frac{1}{2}} \right| \leq 1 \quad .$$

Hence $\| \phi \|_m \leq 1$.

The object of this paper is to prove similar results as Theorem 0.1 and Corollary 0.1 for $m(\ell_n^p \hat{\otimes} \ell_n^{p^*})$, $1 \leq p < \infty$.

1. L^p-PROJECTIONS AND $m(\ell_n^p \hat{\otimes} \ell_n^{p^*})$

If X is a Banach space, then a p-orthogonal projection (or an L^p-projection) in X is a linear map E: $X \to X$ such that $E^2 = E$ and

$$\| x \|^p = \| Ex \|^p + \| x\text{-}Ex \|^p$$

for any $x \in X$. If E is a p-orthogonal projection in X, then $\| E \| \leq 1$ and

$$X = \text{Range}\,(E) \otimes \ker\,(E),$$

such that if $x = x_1 + x_2$, then $\| x \|^p = \| x_1 \|^p + \| x_2 \|^p$.

For the proofs of these facts, and more information about p-orthogonal projections, we refer to Behrends *et al.* (1977).

The Banach spaces ℓ^p, $1 \leq p < \infty$, have many p-orthogonal projections. In fact, L^p-spaces can be characterized as those spaces having, in some sense, a maximal number of p-orthogonal projections.

Definition 1.1. Let $x,y \in \ell_n^p$. Then x,y are called p-orthogonal if $\| x+y \| = (\| x \|^p + \| y \|^p)^{\frac{1}{p}}$. Similarly, $x_1, \ldots, x_n$ are called p-orthogonal if

$$\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| = \left(\sum_{i=1}^{n} | \lambda_i |^p \| x_i \|^p \right)^{\frac{1}{p}}$$

for any scalars $\lambda_1, \ldots, \lambda_n$. The existence of p-orthogonal elements is an immediate consequence of the existence of p-orthogonal projections.

Lemma 1.1. Let $e_1, \ldots, e_n$ be p-orthogonal in ℓ_n^p and $f_1, \ldots, f_n$ be p^*-orthogonal in $\ell_n^{p^*}$ such that $\| e_i \| = \| f_i \| = 1$, $i = 1, \ldots, n$. Then the matrix $A(i,j) = \langle e_i, f_j \rangle$ defines a contraction on ℓ_n^p.

Proof. Let $x \in \ell_n^p$. Then

$$\| Ax \| = \sup_y | \langle Ax,y \rangle | \, , \; y \in \ell_n^{p^*}, \; \| y \| \leq 1,$$

$$= \sup_y \left| \sum_{i,j=1}^{n} \langle e_i, f_j \rangle x(i) y(j) \right|$$

$$= \sup_y \left| \langle \sum_{i=1}^{n} x(i) e_i, \sum_{j=1}^{n} y(j) f_j \rangle \right|$$

$$\leq \sup_y \left\| \sum_{i=1}^{n} x(i) e_i \right\| \cdot \left\| \sum_{j=1}^{n} y(j) f_j \right\|$$

$$\leq \sup_y \left(\sum_{i=1}^{n} | x(i) |^p \cdot \| e_i \|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{n} | y(j) |^{p*} \cdot \| f_j \|^{p*} \right)^{\frac{1}{p^*}}$$

$$\leq \left(\sum_{i=1}^{n} | x(i) |^p \right)^{\frac{1}{p}} = \| x \| \quad .$$

Hence $\| Ax \| \leq \| x \|$ and $\| A \| \leq 1$.

Now, we prove

Theorem 1.1. Let $\phi \in \ell^\infty(Z_n^+ \times Z_n^+)$. Then the following are equivalent:

(i) $\phi \in m(\ell_n^p \hat{\otimes} \ell_n^{p^*})$ and $\| \phi \|_m \leq 1$, $1 < p < \infty$

(ii) $\left| \sum_{i,j=1}^{n} \phi(i,j) \langle x_i, y_j \rangle \right| \leq 1$, for any pair of sets,

$\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ of p-orthogonal and p^*-orthogonal elements in ℓ_n^p and $\ell_n^{p^*}$ respectively such that

$$\left(\sum_{i=1}^{n} \| x_i \|^p \right)^{\frac{1}{p}} \leq 1$$

and

$$\left(\sum_{i=1}^{n} \| y_j \|^{p*} \right)^{\frac{1}{p^*}} \leq 1 \quad .$$

Proof. (i)→(ii). Let $\phi \in m(\ell_n^p \hat{\otimes} \ell_n^{p^*})$ and $\| \phi \|_m \leq 1$. Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be as in (ii).

Set

$$e_i = \begin{cases} \dfrac{x_i}{\| x_i \|} & \text{if } x_i \neq 0 \\ 0 & \text{if } x_i = 0 \end{cases}$$

$$f_j = \begin{cases} \dfrac{y_j}{\| y_j \|} & \text{if } y_j \neq 0 \\ 0 & \text{if } y_j = 0 \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, \ldots, n$.

Then $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are two sets of p-orthogonal and p^*-orthogonal in ℓ_n^p and $\ell_n^{p^*}$ respectively and $\| e_i \| = \| f_j \| = 1$, $i,j = 1, \ldots, n$. Put $u(i) = \| x_i \|$,

$v(j) = \| y_j \|$ and $A(i,j) = \langle e_i, f_j \rangle$, $i,j = 1, \ldots, n$. Then $u \in \ell_n^p$, $v \in \ell_n^{p^*}$ and $\| u \| \leq 1$, $\| v \| \leq 1$. Further, $A \in (\ell_n^p \hat{\otimes} \ell_n^{p^*})^*$, and $\| A \| \leq 1$ by Lemma 1.1. Hence

$$\left| \sum_{i,j=1}^{n} \phi(i,j) \langle x_i, y_j \rangle \right|$$

$$= \left| \sum_{i,j=1}^{n} \phi(i,j) u(i) v(j) \langle e_i, f_j \rangle \right|$$

$$= \left| \sum_{i,j=1}^{n} \phi(i,j) u(i) v(j) A(i,j) \right| \leq 1$$

by (i).

Conversely, (ii)→(i). By definition

$$\| \phi \|_m = \sup \left| \sum_{i,j=1}^{n} \phi(i,j) u(i) v(j) A(i,j) \right|,$$

where the supremum is taken over all $u \otimes v$ in the unit ball of $\ell_n^p \hat{\otimes} \ell_n^{p^*}$ and A in the unit ball of the dual space $(\ell_n^p \hat{\otimes} \ell_n^{p^*})^*$ which is isometrically isomorphic to the space of all bounded linear maps on ℓ_n^p (Wong 1977). Since every point of the unit ball of ℓ_n^p is an extreme point (Köthe 1969), it is enough to take A to be an isometry in the above supremum. But if $A: \ell_n^p \to \ell_n^p$ is an isometry, then one can easily show that A takes p-orthogonal elements to p-orthogonal elements in ℓ_n^p.

Let $(\delta_i)_{i=1}^n$ be the canonical basis of ℓ_n^p (and of $\ell_n^{p^*}$). Then $\delta_1, \ldots, \delta_n$ are p-orthogonal in ℓ_n^p (and p^*-orthogonal in $\ell_n^{p^*}$). Consequently if A is an isometry of ℓ_n^p, then $A\delta_1, \ldots, A\delta_n$ is p-orthogonal in ℓ_n^p. Then the matrix representing A is given by $A(i,j) = \langle A\delta_i, \delta_j \rangle$. Hence if $u \otimes v$ is an element of the unit ball of $\ell_n^p \hat{\otimes} \ell_n^{p^*}$, then

$$\left| \sum_{i,j=1}^{n} \phi(i,j) u(i) v(j) A(i,j) \right|$$

$$= \left| \sum_{i,j=1}^{n} \phi(i,j) u(i) v(j) \langle A\delta_i, \delta_j \rangle \right|$$

$$= \left| \sum_{i,j=1}^{n} \phi(i,j) \langle x_i, y_j \rangle \right|,$$

where $x_i = u(i) A\delta_i$, $y_j = v(j)\delta_j$. Clearly $x_1, \ldots, x_n$ are p-orthogonal in ℓ_n^p and $y_j, \ldots, y_n$ are p^*-orthogonal in $\ell_n^{p^*}$. Further

$$\sum_{i=1}^{n} \| x_i \|^p = \left(\sum_{i=1}^{n} | u(i) |^p \cdot \| A\delta_i \|^p \right) \leq 1,$$

and

$$\sum_{j=1}^{n} \| y_j \|^{p^*} = \sum_{i=1}^{n} | v(j) |^{p^*} \| \delta_j \|^{p^*} \leq 1 \quad .$$

It follows by (ii) that

$$\left| \sum_{i,j=1}^{n} \phi(i,j) \langle x_i, y_j \rangle \right| \leqslant 1 \quad .$$

Thus $\phi \in m(\ell_n^p \hat{\otimes} \ell_n^{p^*})$ and $\| \phi \|_m \leq 1$. This completes the proof of the theorem.

Remark. For $p=1$, $\ell_n^1 \hat{\otimes} \ell_n^\infty = \ell_n^1(\ell_n^\infty)$, the n-dimensional Banach space of vector-valued n-tuples $x=\{x(i)\}_{i=1}^n$, $x(i) \in \ell_n^\infty$,

$$\| x \|_1 = \sum_{i=1}^n \| x(i) \|_\infty, [3] \quad .$$

It is very easy to show that

$$m(\ell^1(\ell_n^\infty)) = \ell_n^\infty(\ell_n^\infty) = \{x = \{x(i)\}_{i=1}^n, x(i) \in \ell_n^\infty \; \| x \| = \sup_i \| x(i) \|_\infty .\}$$

As a corollary we obtain the following:

Corollary 1.2. $(\ell_n^p \hat{\otimes} \ell_n^{p^*})^* \subseteq m(\ell_n^p \hat{\otimes} \ell_n^{p^*})$ and $\| \phi \|_m \leq \| \phi \|_{L(\ell_n^p)}$, the norm of ϕ as an operator on ℓ_n^r.

Proof. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be p-orthogonal in ℓ_n^p and p^*-orthogonal in $\ell_n^{p^*}$ respectively such that

$$\sum_{i=1}^n \| x_i \|^p \leq 1$$

and

$$\sum_{j=1}^n \| y_j \|^{p*} \leq 1 \quad .$$

Let

$$x_i = \sum_{k=1}^n \lambda_{ik} \, \delta_k, \; y_j = \sum_{k=1}^n \xi_{jk} \, \delta_k \quad .$$

Then

$$\langle x_i, y_j \rangle = \sum_{k=1}^n \lambda_{ik} \, \xi_{jk} = \sum_{k=1}^n f_k(i) \, g_k(j),$$

where

$$f_k(i) = \lambda_{ik}, \; g_k(j) = \xi_{jk}.$$

So

$$\psi(i,j) = \langle x_i, y_j \rangle \in \ell_n^p \hat{\otimes} \ell_n^{p^*}$$

and

$$\| \psi \|_{\tau_r} \leq \sum_{k=1}^n \| f_k \| \cdot \| g_k \|$$

$$\leq \left(\sum_{k=1}^n \| f_k \|^p \right) 1/p \cdot \left(\sum_{k=1}^n \| g_k \|^{p*} \right)^{\frac{1}{p^*}} \text{(by Hölder ineq.).}$$

$$\leq 1.$$

Hence

$$\left| \sum_{i,j=1}^n \phi(i,j) \, \langle x_i, y_j \rangle \right|$$

$$\leq \left| \sum_{i,j=1}^{n} \phi(i,j)\psi(i,j) \right|$$

$$\leq \| \phi \|_{L(\ell^p_n)} \cdot \| \psi \|_{\tau_r} = \| \phi \|_{L(\ell^p_n)} .$$

It follows from Theorem 1.1 that $\| \phi \|_m \leq \| \phi \|_{L(\ell^p_n)}$.

Also, one can prove the following results whose proof is similar to that of the case when $p=2$, and so it will be omitted.

Corollary 1.3. Let $\phi(i,j) \in \ell^\infty(Z^+_n \times Z^+_n)$ such that

$$\sup_i \left(\sum_{j=1}^{n} | \phi(i,j) |^p \right)^{\frac{1}{p}} \leq 1.$$

Then $\phi \in m(\ell^p_n \hat{\otimes} \ell^{p^*}_n)^*$ and $\| \phi \|_m \leq 1$.

Corollary 1.4. Let α,β be any two maps on the natural numbers Z^+_n. Then the function $(\phi \circ \alpha \otimes \beta)(i,j) = \phi(\alpha(i), \beta(j)) \in m(\ell^p_n \hat{\otimes} \ell^{p^*}_n)$ if and only if $\phi \in m(\ell^p_n \hat{\otimes} \ell^{p^*}_n)$, and in that case $\| \phi \|_m = \| \phi \circ \alpha \otimes \beta \|_m$.

It follows immediately, that if $\tilde{\phi}$ is another matrix obtained from ϕ by repeating finitely many rows finitely many times, then $\tilde{\phi} \in m(\ell^p \hat{\otimes} \ell^{p*})$ if and only if $\phi \in m(\ell^p \hat{\otimes} \ell^{p*})$ and $\| \phi \|_m = \| \tilde{\phi} \|_m$. Bennett (1977) proved this result for $m(\ell^p \hat{\otimes} \ell^q)$ $p,q > 1$, using different methods.

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اسقاطات الرتبة P وضوارب شور

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خلاصة

في هذا البحث نبرهن أن العبارات التالية عبارات متكافئة :

(١) $\phi \in m(\ell_n^p \hat{\otimes} \ell_n^{p^*})$ and $\| \phi \|_m \leq 1$

(٢) $\left| \sum_{i,j=1}^{n} \phi(i,j) \langle P(i)x, Q(j)y \rangle \right| \leq 1$

لكل زوج (P,Q) حيث P هو اسقاط عمودي للرتبة P وحيث Q اسقاط عمودي للرتبة P^* على $\ell_n^p, \ell_n^{p^*}$ بالترتيب وحيث $\frac{1}{p} + \frac{1}{p^*} = 1$.