J. Univ. Kuwait (Sci.) **11,** 1984

Some integral relations and their applications

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ABSTRACT

In this note some simple integral relations involving elementary special functions are established. It is also indicated how these integral relations can be applied to derive various double integrals by proper specialization of the unknown function.

1. INTRODUCTION

Many authors have worked on the problem of obtaining integral relations involving higher classes of special functions of one or more variables (e.g. Dahiya & Singh 1971, Koul 1974 and Raina & Koul 1977).

Here we derive a few other integral relations associating some elementary special functions and illustrate how they yield double integrals which may be of interest.

2. THE INTEGRAL RELATIONS

Consider Poisson's integral representation (Erdélyi *et al.* 1953, p. 81, Eq. 9) for the Bessel function

$$J_{\vartheta}(z) = \frac{(z/2)^{\vartheta}}{\sqrt{\pi}\;\Gamma(\vartheta+\frac{1}{2})}\int_0^{\pi} (\sin\phi)^{2\vartheta}\, e^{iz\cos\phi}\, d\phi,\ \mathrm{Re}(\vartheta) > -\tfrac{1}{2}\quad . \tag{2.1}$$

This implies that for $\mathrm{Re}(\vartheta) > -\frac{1}{2}$, we have

$$\int_0^{\infty}\left(r\,f(r^2)\int_0^{\pi/2}(\sin 2\theta)^{2\vartheta}\,\mathrm{e}^{ir^2\cos 2\theta}\,d\theta\right)dr$$

$$= 2^{\vartheta-1}\sqrt{\pi}\;\Gamma(\vartheta+\tfrac{1}{2})\int_0^{\infty} r^{1-2\vartheta}\,J_{\vartheta}(r^2)\,f(r^2)\,dr, \tag{2.2}$$

provided the integral on the right exists. Changing to cartesian coordinates by the transformation

$$x = r\cos\theta,\ y = r\sin\theta,$$

on the left-hand side, we are easily led to the integral relation

$$\int_0^\infty \int_0^\infty \left(\frac{xy}{x^2+y^2}\right)^{2\vartheta} e^{i(x^2-y^2)} f(x^2+y^2)\, dxdy$$
$$= 2^{-(\vartheta+2)} \sqrt{\pi}\, \Gamma(\vartheta+\tfrac{1}{2}) \int_0^\infty t^{-\vartheta} J_\vartheta(t) f(t)\, dt \quad . \tag{2.3}$$

An obvious generalisation of (2.3) can be formulated if we invoke Gegenbauer's formula (Erdélyi *et al.* 1953, p. 178. Eq. 38) instead of the formula (2.1). By proceeding on the same lines as just stated above, the generalization of (2.3) thus obtained is

$$\int_0^\infty \int_0^\infty \left(\frac{xy}{x^2+y^2}\right)^{2\vartheta} e^{i(x^2-y^2)} C_n^\vartheta\left(\frac{x^2-y^2}{x^2+y^2}\right) f(x^2+y^2)\, dxdy$$
$$= \frac{2^{-(\vartheta+2)} \sqrt{\pi}\, i^n\, \Gamma(\vartheta+\tfrac{1}{2})\Gamma(n+2)}{(n!)\,\Gamma(2\vartheta)} \int_0^\infty t^{-\vartheta} J_{\vartheta+n}(t) f(t)\, dt, \tag{2.4}$$

where the function $C_n^\vartheta(z)$ is a Gegenbauer polynomial and $f(t)$ is so chosen that the integrals exist. It may be noted that (2.4) corresponds to (2.3) in the special case when $n=0$.

Further integral relations can be motivated by suitably using known formulas (or their modified forms). Thus, the integral representations (Erdélyi *et al*, 1953, p. 191, Eq. 30 and Whittaker & Watson 1963, p. 382, Ex. 23) will yield integral relations involving a Laguerre polynomial and a Bessel function, respectively.

3. APPLICATIONS

The function f appearing in the integral relations established in the preceding section may be chosen appropriately to derive various double integrals.

For example, setting

$$f(t) = J_\mu(t), \tag{3.1}$$

in (2.3) and using Erdélyi *et al.* (1954, p. 342, Eq. 24) we obtain

$$\int_0^\infty \int_0^\infty \left(\frac{xy}{x^2+y^2}\right)^{2\vartheta} e^{i(x^2-y^2)} J_\mu(x^2+y^2)\, dxdy$$
$$= \frac{\sqrt{\pi}\, \Gamma(\vartheta+\tfrac{1}{2})\, \Gamma(\vartheta)\, 2^{-2(\vartheta+1)}}{\Gamma[(1+\mu+2\vartheta)/2]\Gamma[(1-\mu+2\vartheta)/2]},$$
$$\mathrm{Re}(\mu) > -1,\ \mathrm{Re}(\vartheta) > 0 \quad . \tag{3.2}$$

ACKNOWLEDGEMENT

The author is thankful to the referees for suggesting some alterations in the original version of the paper.

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(Received 7 September 1982, revised 12 September 1983)

بعض العلاقات التكاملية وتطبيقاتها

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خلاصة

لقد أمكن الحصول على بعض العلاقات التكاملية البسيطة المتضمنة لدوال خاصة أولية . كما ذكر أن هذه العلاقات التكاملية يمكن تطبيقها لاشتقاق تكاملات مضاعفة مختلفة بتخصيص مناسب للدالة المجهولة .