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Some fixed point theorems—II

JAGADISH ACHARI

Department of Mathematics, Science College, Nanded 431 602, Maharashtra, India

ABSTRACT

Some fixed point theorems for the mapping T of a metric space X into itself which satisfies the following inequality

$$d(Tx,Ty) < \max\{d(x,y),\ 1/2\,[d(x,Tx)+d(y,Ty)],\ 1/2\,[d(x,Ty)+d(y,Tx)]\}$$

for $x,y \in X$, $x \neq y$ have been proved.

INTRODUCTION

The well known contraction principle of Banach (1922) states that if T is a mapping of a complete metric space X into itself such that

$$d(Tx,Ty) \leq \alpha\, d(x,y) \tag{1}$$

for a non-negative number α, $0 \leq \alpha < 1$ and for each $x,y \in X$, then T has a unique fixed point. Kannan (1968) in a recent paper studied the following mapping. Let T be a self map of a complete metric space X such that

$$d(Tx,Ty) \leq \beta\,[d(Tx,x)+d(Ty,y)] \tag{2}$$

for a number β, $0 < \beta < 1/2$ and for each $x,y \in X$. Then T has a unique fixed point. Also Chatterjea (1972) studied a mapping which can be stated as follows: if T is a self mapping of a complete metric space X such that

$$d(Tx,Ty) \leq \gamma\,[d(Tx,y)+d(Ty,x)] \tag{3}$$

for a number γ, $0 < \gamma < 1/2$ and for each $x,y \in X$, then T has a unique fixed point.

Recently Zamfirescu (1972) generalised all the three mappings stated above and proved the following theorem.

Theorem (Zamfirescu). Let X be a complete metric space and T a self map on X such that it satisfies at least one of the following conditions:

$$\begin{aligned} &\text{(i)}\ d(Tx,Ty) \leq \alpha\, d(x,y) \\ &\text{(ii)}\ d(Tx,Ty) \leq \beta[d(Tx,x)+d(Ty,y)] \\ &\text{(iii)}\ d(Tx,Ty) \leq \gamma[d(Tx,y)+d(Ty,x)] \end{aligned} \tag{4}$$

for real numbers α, β, γ, $0 \leq \alpha < 1$, $0 \leq \beta, \gamma < 1/2$. Then T has a unique fixed point.

In this paper we obtain some fixed point theorems with the mapping which satisfies the following inequality:

$$d(Tx,Ty) < \max\{(d(x,y), 1/2[d(x,Tx)+d(y,Ty)], 1/2\,[d(x,Ty)+d(y,Tx)]\} \tag{5}$$

for $x,y \in X$, $x \neq y$.

It is remarkable to note that the mapping which satisfies (5) will not satisfy (4). We illustrate this fact with the following example (see Rhoades (1977)).

Example. Let $T(x) = x^2/x+1$, $x \geq 0$, then $d(Tx,Ty) < d(x,y)$ and T satisfies (5). For $x \geq 1$, $d(Tx, T(2x)) = x^2(2x+3)/(x+1)\,(2x+1)$ and

$$\max\{d(x,2x), [d(x,Tx)+d(2x,T(2x))]/2, [d(x,T(2x)+d(2x,Tx))]/2\} = x\,.$$

Given any $\alpha (0 < \alpha < 1)$, one can find values of x large enough so that

$$x(2x+3)/(x+1)\,(2x+1) > \alpha$$

and T does not satisfy (4). A survey of different contractive type mappings and their interrelations was given by Rhoades (1977).

Theorem 1. Let (X,ρ) be a non-empty compact space and d a non-negative real valued symmetric function on $X \times X$ such that $d(x,y) = 0$ implies $x = y$ $(x,y \in X)$. Suppose that T_1 and T_2 are mappings of X into itself satisfying the following conditions:

(i) If $T_1(x) = x = y = T_2(y)$ is not true, then T_1 and T_2 satisfy (5).

(ii) The function $f(x,y) = d(x,T_1x) + d(y,T_2y)$ is lower semi continuous on $(X,\rho) \times (X,\rho)$.

Then the mappings T_1 and T_2 have a common unique fixed point.

Proof. Since $f(x,y)$ is a lower semicontinuous function on the (non-empty) compact space $(X,\rho) \times (X,\rho)$, there is a point (u,v) in $X \times X$ at which f attains its infimum. We note that if any one of the following conditions is satisfied:

$$\text{(a)}\ T_1\,(T_2, v) = T_2\, v = v\,;$$

$$\text{(b)}\ u = T_1\, u = T_2\ (T,u)\,,$$

then u or v is a common fixed point of T_1 and T_2. Suppose that it is not, then by (i) we have

$$\begin{aligned} f(T_2\, v, T_1\, u) &= d(T_2v, T_1(T_2v) + d(T_1u, T_2(T_1u)) \\ &< \max\{[d(T_1(T_2v), T_2v) + d(T_2v,v)]/2, \\ &\quad d(v,T_2v), [d(T_2v,T_2v) + d(T_1(T_2v),v)]/2 \\ &\quad + \max\{[d(T_1u,u) + d(T_2(T_1u), T_1u)]/2, \\ &\quad d(T_1u,u), [d(T_1u,T_1u) + d(T_2(T_1u),u)]/2\} \end{aligned}$$

In the above expression we can assume either $d(T_1u,u)$, $d(T_2v,v)$ are maximum or

$$1/2\,[d(T_2v,v)+d(T_1T_2v,T_2v),\ 1/2\,[d(T_1u,u)+\ d(T_2(T_1u),T_1u)]$$

are maximum. In both cases we get

$$f(T_2v,T_1u)\ <\ d(T_2v,v)+d(T_1u,u)\ <f(u,v)\,.$$

This is a contradiction to the minimality of f at the point (u,v). Next we show the unicity. Let $u\neq v$ then by (1) we have

$$d(u,v)\ <\ d(u,v)$$

which is impossible. Hence $u=v$ and the proof is complete. In the above theorem one can take, for instance, as d a metric on X. We can obtain further assertions by taking $T_1=T_2=T$.

Theorem 2. If in addition to the hypothesis of Theorem 1 we have (iii) $d(Tx,u)<d(x,u)$ if $x\neq u$, where u is the unique fixed point of T, then for every $x\in X$, $\{T^n_x\}\to u$.

Proof. Let $x\in X$ and $\{T^n_x\}\supset\{T^{n_i}x\}\to v\in X$ (X is compact). Consider the sequence $\{d(T^nx,u)\}$. Now

$$d(T^nx,u)\ =\ d(T^nx,Tu)\ <\ \max\{[d(T^nx,T^{n-1}x)+dTu,u)]/2,$$
$$d(u,T^{n-1}x),\ [d(Tu,T^{n-1}x)+d(T^nx,\ u)]/2\}\,.$$

Either $d(T^{n-1}x,u)$ or $[d(u,T^{n-1}x)+d(T^nx,u)]/2$ is maximum and in both cases we get

$$d(T^nx,\ u)\ <\ d\,(T^{n-1}x,u)\,.$$

Hence

$$d(T^nx,\ u)\ <\ d(T^{n-1}x,u),\,.\,.\,.\,.\,.\,.\,.\ <\ d(x,u)\,.$$

Thus $\{d(T^nx,u)\}$ is non-increasing sequence and is therefore convergent. Also since

$$\{T^{n_i}x\}\to v,\ \{d(T^{n_i}x,u)\}\to d(u,v)$$

and

$$d(T^{n_i+1}x,u)\ \leq\ d(T^{n_i+1}x,\ Tv)+d(Tv,u)\,.$$

Since T is continuous and $\{T^{n_i}x\}\to v$, we have

$$d(u,v)\ =\ \lim_i\ d(T^{n_i}x,u)\ =\ \lim_n\ d(T^nx,u)\ =\ \lim_i\ d(T^{n_i+1}x,u)\ \leq\ d(Tv,u)\,.$$

This relation contradicts (iii) of Theorem 2 unless $u=v$.
Hence

$$\lim_n d(T^nx,u)\ =\ \lim_i d(T^{n_i}x,u)\ =\ \lim_i d(T^{n_i}x,v)\ =\ 0\,.$$

This completes the proof of the theorem.

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بعض مبرهنات نقطة ثابتة

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قسم الرياضيات بكلية العلوم ، نانده ٤٣١٦٠٢
ماهاراشترا ، الهند

خلاصة

لقد تم اثبات بعض المبرهنات المتعلقة بالتصوير T من فضاء متري X إلى نفسه ويحقق شروطا محددة .