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On common fixed-point theorems for a pair of mappings

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ABSTRACT

Some fixed-point theorems for a pair of a class of mappings involving four points of the space under consideration have been proved.

INTRODUCTION

Recently Pittnauer (1976) has studied fixed-point theorems for contractive type mappings involving four points of the space. Achari (1979a, 1979b, 1980) has further generalised the idea of Pittnauer with a more general type of mappings with a different approach from that of Pittnauer.

The aim of this paper is to study the common fixed points for a pair of a class of mappings involving four points of the space under consideration. We have then extended this idea to finite families of mappings. Finally we have studied some results with the special case of our main theorem.

1. MAIN RESULTS

Let (X,d) be a complete metric space. Let $\phi_i; \bar{P}\to[0,\infty)$ (P is $i=1,2,3$, the range of d and $\bar{P}$ is the closure of P) be an upper semicontinuous function from the right on $\bar{P}$ and satisfy the condition

$$\phi_i(t) < t/3 \text{ for } t > 0 \text{ and } \phi_i(0) = 0,\ i=1,2,3 \quad . \tag{1.1}$$

Also let f and g be mappings of X into itself such that

$$d(fu_1,gu_2) \le \frac{\phi_1[d(u_2,gu_4)]\,[1+\phi_1(d(u_1,fu_3))]}{1+\phi_1[d(u_1,u_2)]} + \frac{\phi_2[d(u_2,fu_3)]\,[1+\phi_2(d(u_1,gu_4))]}{1+\phi_2[d(u_1,u_2)]} + \phi_3[d(u_1,u_2)] \tag{1.2}$$

for $u_1,u_2,u_3,u_4 \in X$.

Theorem 1. If f and g are mappings of X into itself satisfying (1.2) then f and g have a common unique fixed point.

Proof. Let $x,y \in X$ and we define

$$u_1 = gy,\ u_2 = fx,\ u_3 = x,\ u_4 = y \quad .$$

The expression (1.2) takes the form

$$d(fgy,gfx) \le \frac{\phi_1[d(fx,gy)]\,[1+\phi_1(d(fx,gy))]}{1+\phi_1[d(fx,gy)]} + \phi_3[d(fx,gy)]$$

$$= \phi_1[d(fx,gy)] + \phi_3[d(fx,gy)] \quad . \tag{1.3}$$

Let $x_0 \in X$ be arbitrary and we construct a sequence $\{x_n\}$ defined by

$$fx_{n-1} = x_n,\ gx_n = x_{n+1},\ fx_{n+1} = x_{n+2},\ n = 1,2,\ldots$$

Putting $x = x_{n-1}$, $y = x_n$ in (1.3) we get

$$d(f(gx_n), g(fx_{n-1})) \le \phi_1[d(fx_{n-1}, gx_n)] + \phi_3[d(fx_{n-1}, gx_n)]$$

$$\text{i.e. } d(x_{n+2}, x_{n+1}) \le \phi_1[d(x_n,x_{n+1})] + \phi_3[d(x_n,x_{n+1})] \quad . \tag{1.4}$$

Let n be even and set $\beta_n = d(x_{n-1},x_n)$. Then

$$\beta_{n+2} = d(x_{n+1},x_{n+2}) \le \phi_1[d(x_n,x_{n+1})] + \phi_3[d(x_n,x_{n+1})]$$

$$\le \phi_1(\beta_{n+1}) + \phi_3(\beta_{n+1}) \quad . \tag{1.5}$$

From (1.5) it is obvious that β_n decreases with n and hence $\beta_n \to \beta$ say, as $n \to \infty$. If possible, let $\beta > 0$. Then since ϕ_i is upper semicontinuous, we obtain in the limit as $n \to \infty$

$$\beta \le \phi_1(\beta) + \phi_3(\beta)$$

$$< 2/3\,\beta$$

which is impossible unless $\beta = 0$.

Now we will show that the sequence $\{x_n\}$ is Cauchy. We assume that it is not so. Then there exists an $\varepsilon > 0$ and sequences of integers $\{m(k)\},\{n(k)\}$ with $m(k) > n(k) \ge k$ such that

$$d_k = d(x_{m(k)}, x_{n(k)}) \ge \varepsilon,\ k = 1,2,3,\ldots \tag{1.6}$$

If $m(k)$ is the smallest integer exceeding $n(k)$ for which (1.6) holds, then from the well-ordering principle, we have

$$d(x_{m(k)-1}, x_{n(k)}) < \varepsilon \quad . \tag{1.7}$$

Then

$$d_k \le d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)})$$

$$\le \beta_{m(k)} + \varepsilon < \beta_k + \varepsilon$$

which implies that $d_k \to \varepsilon$ as $k \to \infty$. Now the following cases are to be considered:

(i) m is even and n is odd,
(ii) m and n are both odd,
(iii) m is odd and n is even,
(iv) m and n are both even.

Case (i)

$$d_k = d(x_m,x_n) \le d(x_m,x_{m+1})+d(x_{m+1}, x_{n+1})+d(x_n,x_{n+1})$$
$$\le \beta_{m+1}+\beta_{n+1}+d(gx_m,fx_n)$$
$$\le \beta_{m+1}+\beta_{n+1}+\frac{\phi_1[d(x_m,gx_{m-1})]\,[1+\phi_1[d(x_n,fx_{n-1})]]}{1+\phi_1[d(x_n,x_m)]}+$$
$$+\frac{\phi_2[d(x_m,fx_{n-1})]\,[1+\phi_2(d(x_n,gx_{m-1}))]}{1+\phi_2[d(x_n,x_m)]}+\phi_3(d(x_m,x_n))$$

(By putting $u_1=x_n$, $u_2=x_m$, $u_3=x_{n-1}$, $u_4=x_{m-1}$ in (1.2))

$$\le \beta_{m+1}+\beta_{n+1}+\phi_2(d_k)+\phi_3(d_k)$$

letting $k\to\infty$ we get $\varepsilon < 2/3\,(\varepsilon)$.

This is a contradiction if $\varepsilon > 0$.

Case (ii)

We have

$$d_k=d(x_m,x_n) \le d(x_m,x_{m+1})+d(x_{m+2},x_{m+1})+d(x_{m+2},x_{n+1})+d(x_n,x_{n+1})$$
$$\le \beta_{m+2}+\beta_{m+1}+\beta_{n+1}+d(gx_{m+1},fx_n)$$
$$\le \beta_{m+2}+\beta_{m+1}+\beta_{n+1}+\frac{\phi_1[d(x_{m+1},gx_m)]\,[1+\phi_1(d(x_n,fx_{n-1}))]}{1+\phi_1[d(x_n,x_{m+1})]}+$$
$$+\frac{\phi_2[d(x_{m+1},fx_{n-1})]\,[1+\phi_2(d(x_n,gx_m))]}{1+\phi_2[d(x_n,x_{m+1})]}+\phi_3(d(x_n,x_{m+1}))$$

(By putting $u_1=x_n$, $u_2=x_{m+1}$, $u_3=x_{n-1}$, $u_4=x_m$ in (1.2))

$$\le \beta_{m+2}+\beta_{m+1}+\beta_{n+1}+\phi_2(d(x_n,x_{m+1}))+\phi_3(d(x_{m+1},x_n)) \quad .$$
$$\le \beta_{m+2}+\beta_{m+1}+\beta_{n+1}+\phi_2\,[d_k+\beta_{m+1}]+\phi_3[d_k+\beta_{m+1}] \quad .$$

Letting $k\to\infty$ in the above inequality we obtain $\varepsilon < 2/3(\varepsilon)$, which is a contradiction if $\varepsilon > 0$. Similarly the cases (iii) and (iv) may be disposed of. This leads us to conclude that the sequence $\{x_n\}$ is Cauchy and since X is complete, there exists a point $w\varepsilon X$ such that $x_n\to w$ as $n\to\infty$. Next we will show that $gw=w=fw$. Putting $u_1=x_{n-1}, u_2=w, u_3=x_{n+1}$, $u_4=x_n$ in (1.2) we have

$$d(fx_{n-1},gw) \le \frac{\phi_1[d(w,gx_n)]\,[1+\phi_1(d(x_{n-1},fx_{n+1}))]}{1+\phi_1[d(x_{n-1},w)]}+$$
$$+\frac{\phi_2[d(w,fx_{n+1})]\,[1+\phi_2(d(x_{n-1},gx_n))]}{1+\phi_2[d(x_{n-1},w)]}+\phi_3[d(x_{n-1},w)]$$

As $n\to\infty$ we get $d(w,gw) \le 0$ which is a contradiction and hence $w=gw$. In the same way it is possible to show that $fw=w$. Thus w is a common fixed point of f and g. If possible let there be another fixed point $v(\neq w)$ such that $fv=v=gv$. Then putting $u_1=u_4=w$, $u_2=u_3=v$ in (1.2) we get

$$d(w,v) = d(fw,gv) \leq \frac{\phi_1[d(w,v)]\,[1+\phi_1(d(w,v))]}{1+\phi_1[d(w,v)]} +$$

$$+\frac{\phi_2[d(d(v,v)]\,[1+\phi_2(d(w,w))]}{1+\phi_2[d(w,v)]} + \phi_3(d(w,v))$$

$$\leq \phi_1(d(w,v)) + \phi_3(d(w,v)) < 2/3\; d(w,v)$$

which is a contradiction. Hence $w = v$. Also it can be easily proved that f and g cannot have any other fixed point apart from the common fixed point w. This completes the proof of the theorem.

Theorem 2. Let f_k $(k = 1,2 \ldots n)$ be a family of mappings of X into itself. If f_k satisfy the conditions

(a) $f_1 f_2 \ldots f_n$ commutes with every f_k,

(b) $d(f_1 f_2 \ldots f_n u_1, f_n f_{n-1} \ldots f_1 u_2) \leq$

$$\leq \frac{\phi_1[d(u_2, f_n f_{n-1} \ldots f_1 u_4)]\,[1+\phi_1(d(u_1, f_1 f_2 \ldots f_n u_3))]}{1+\phi_1[d(u_1,u_2)]} +$$

$$+\frac{\phi_2[d(u_2, f_1 f_2 \ldots f_n u_3)]\,[1+\phi_2(d(u_1, f_n f_{n-1} \ldots f_1 u_4))]}{1+\phi_2[d(u_1,u_2)]} + \phi_3[d(u_1,u_2)]$$

for $u_1,u_2,u_3,u_4 \in X$ and $\phi_i(t), i = 1,2,3$ satisfies the condition (1.1). Then $\{f_k\}$ have a unique common fixed point.

Proof. Let us put $f = f_1 f_2 \ldots f_n$ and $g = f_n f_{n-1} \ldots f_1$ then (b) takes the form

$$\text{(c)} \quad d(fu_1, gu_2) \leq \frac{\phi_1[d(u_2, gu_4)]\,[1+\phi_1(d(u_1, fu_3))]}{1+\phi_1[d(u_1,u_2)]} +$$

$$+\frac{\phi_2[d(u_2, fu_3)]\,[1+\phi_2(d(u_1, gu_4))]}{1+\phi_2[d(u_1,u_2)]} + \phi_3(d(u_1,u_2)) \quad .$$

By Theorem 1, f and g have a unique common fixed point w. Then $fw = gw = w$. For any $f_k, f_k\,(fw) = f_k W$. By the assumption, $f(f_k w) = f_k w$. So $f_k w$ is a fixed point of f and w is a fixed point of g. By putting $u_1 = u_3 = f_k\,w$ and $u_2 = u_4 = w$ in (c) we have

$$d(f_k w,\, gw) = d(ff_k w, gw) \leq \frac{\phi_1[d(w,gw)][1+\phi_1(d(f_k w, ff_k w))]}{1+\phi_1[d(f_k w, w)]}$$

$$+\frac{\phi_2[d(w, ff_k w)]\,[1+\phi_2(d(f_k w, gw))]}{1+\phi_2[d(f_k w, w)]} + \phi_3(d(f_k w, w))$$

$$\leq \phi_2(d(w, f_k w) + \phi_3(d(w, f_k w))$$
$$< 2/3\; (w, f_k w)$$

which is a contradiction. Hence $f_k w = w, (k = 1 \ldots n)$. This means that w is the common fixed point of the family $\{f_k\}$. The unicity of the fixed point w can easily be proved.

2. A SPECIAL CASE

In this section we shall show that our theorem contains a special case and we will prove some results with this special case.

If we define the functions $\phi_i(t)$ by $\phi_1(t)=a\cdot t, \phi_2(t)=b\cdot t$, $\phi_3(t)=c\cdot t$ with $0<a+b+c<1, u_1=u_3,=x, u_2=u_4=y$ then we have the following expression:

$$d(fx,gy) \leq \frac{a\cdot d(y,gy)\,[1+a\cdot d\,(x,fx)]}{1+a\cdot d(x,y)} + \frac{b\cdot d(y,fx)\,[1+b\cdot d(x,gy)]}{1+b\cdot d(x,y)} + c\cdot d(x,y) \tag{2.1}$$

for $x,y \in X$.

We state the following theorem without proof as the proof can be established by simple routine calculations.

Theorem 3. Let f and g be mappings of a non-empty complete metric space X into itself satisfying the inequality (2.1). Then f and g have a unique common fixed point.

Theorem 4. Let X be a metric space with two metrices d and δ. If X satisfies the following conditions:

(a) $d(x,y)\leq\delta(x,y)$ for every x,y in X,
(b) X is complete with respect to d,
(c) two mappings f and g of X into itself are continuous with respect to the metric d and

$$\delta(fx,gy) \leq \frac{a(y,gy)[1+a\delta(x,fx)]}{1+a\delta(x,y)} + \frac{b\delta(y,fx)[1+b\delta(x,gy)]}{1+b\delta(x,y)} + c\delta(x,y)$$

for all x,y in X and $0< a+b+c<1$. Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary and define a sequence

$$x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}, n=0,1,2\ldots$$

Then by a routine calculation it can be shown that

$$\delta(x_{2n+2},x_{2n+1}) \leq \left(\frac{c}{1-a}\right)^{2n+1} \delta(x_0,x_1) \quad .$$

Now since $d\leq\delta$ we have

$$d(x_{2n+1},x_{2n}) \leq \alpha^{2n+1}\delta(x_0,x_1), \alpha=\frac{c}{1-a} \quad .$$

This shows that the sequence $\{x_n\}$ is a Cauchy sequence with respect to d, since X is complete with respect to d. The sequence $\{x_n\}$ has a limit w in X. Hence by the continuity of f with respect to the metric d,

$$w = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = f\lim_{n\to\infty} x_{2n} = fw \quad .$$

Similarly we have $w=gw$. Therefore w is a common fixed point of f and g. The uniqueness can easily be established.

Theorem 5. Let X be a complete metric space and f_i: $X \to X$ and f_j: $X \to X$, $i=1,2,\ldots n, j=1,2\ldots m$. Let $x,y \in X$, $0<a+b+c<1$, there exist sequences of integers $t_1,t_2,\ldots t_n,q_1,q_2\ldots q_m$ such that

$$d(f_1^{t_1} f_2^{t_2}\ldots f_n^{t_n} x, f_1^{q_1} f_2^{q_2}\ldots f_m^{q_m} y) \leq$$

$$\leq \frac{a\cdot d(y, f_1^{q_1} f_2^{q_2}\ldots f_m^{q_m} y)\,[1+a\cdot d(x, f_1^{t_1} f_2^{t_2}\ldots f_n^{t_n} x)]}{1+a\cdot d(x,y)} +$$

$$+\frac{b\cdot d(y, f_1^{t_1} f_2^{t_2}\ldots f_n^{t_n} x)\,[1+b\cdot d(x, f_1^{q_1} f_2^{q_2}\ldots f_m^{q_m} y)]}{1+b\cdot d(x,y)}$$

$$+c\cdot d(x,y) \qquad (2.2)$$

$$f_i \text{ commutes with } f_1^{t_1} f_2^{t_2}\ldots f_n^{t_n},\ i=1,2,\ldots n \qquad (2.3)$$

$$f_j \text{ commutes with } f_1^{q_1} f_2^{q_2}\ldots f_m^{q_m},\ j=1,2\ldots m \qquad (2.4)$$

Then f_i, $i=1,2\ldots n$ and $f_j, j=1,2\ldots m$ have a unique common fixed point (2.2)

Proof. Let us put $f=f_1^{t_1} f_2^{t_2}\ldots f_n^{t_n}$ and $g=f_1^{q_1} f_2^{q_2}\ldots f_m^{q_m}$ then (2.2) reduces to (2.1).

Hence by Theorem 3, f and g have a unique common fixed point w. Then $fw=gw=w$. For any f_i, $f_i(fw)=f_iw$. By condition (2.3) we have $f(f_iw)=f_iw$ $i=1,2\ldots n$. So f_iw is a fixed point of f and since f has a unique fixed point w we get $f_iw=w$. Similarly it can be shown that $f_jw=w$. It is easily seen that w is the unique common fixed point of f_i, $i=1,2,\ldots n$ and $f_j, j=1,2\ldots m$. This completes the proof.

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خلاصة

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