J. Univ. Kuwait (Sci.) **9,** 1982

Stress distribution around rigid pile subjected to torsion

MOSAID AL-HUSSAINI

Department of Civil Engineering, University of Kuwait

ABSTRACT

A closed form solution for the determination of stress distribution around a circular pile subjected to a combined torsional and lateral loading is described in this paper. The pile is assumed to be rigid and the surrounding soil is considered to be in the elastic range. The solution is presented in the form of a dimensionless influence chart to provide generality and simplicity. The stress distribution derived in this study was compared with those obtained from an infinite strip loaded with uniform horizontal shears.

INTRODUCTION

Most piles are designed to carry axial and/or lateral loads and are rarely designed to carry torsional loads. However, there are many instances where piles are subjected to significant torsional stresses. One example is when a pile is in a pile group which is subjected to an eccentric lateral loading. Unlike piles which are subjected to axial or lateral forces, there seems to be very little study of piles subjected to torsional loading.

Probably, the earliest scientific work on the subject is that of Coulomb (1785) who studied the problem of cordage and rigging of ships and provided a solution to the problem of torsion around spars. He also introduced the friction circle construction for the analysis of a shaft as it turned in a frictional contact surface. Since that time, no serious effort was made to study the torsional response to deep foundation until 1971 when Kaldjin (1971) developed a numerical finite element procedure for determining the stress and displacement distribution of a circular footing with various embedments subjected to torsion. His analysis, which is based on the assumption that soil is homogeneous, isotropic, and elastic, compared well with the theory of elasticity solution for a circular disc on a semi-infinite mass and subjected to torsion. Stole (1972) developed an apparatus for conducting torsional field tests and carried out a number of full scale load tests. He concluded that it is cheaper and easier to determine the axial ultimate load capacity of a friction pile in clay from a torsion loading test rather than a conventional axial load test. Poulos (1975) presented a composite finite element solution to determine the response of a single rigid cylindrical pile subjected to torsion. In his analysis, he assumed that the pile shaft consisted of a number of small and equal cylindrical elements while the pile base was composed of a number of annular rings. Each element within the pile is acted upon by uniform shear stress, and the condition of soil-pile interaction is assumed to remain elastic as the pile rotates. He presented a chart for torsional flexibility of piles as a function of pile geometry and relative stiffness. An analytical solution for determining the torsional response of a pile embedded in elastic soil was published recently by Randolph (1981). His solution is based on various simplifying assumptions regarding the stress field around the torsionally loaded pile, and results relevant to the pile torsional stiffness agreed well with the numerical analysis of Poulos (1975). Scott (1981) suggested that the response of a torsionally loaded pile can be simulated by a beam on elastic subgrade subjected to torsion.

In this paper, a closed form solution is developed for determining the stress field with the soil surrounding a torsionally loaded pile.

METHOD OF ANALYSIS

Since any pile foundation loaded laterally is liable to undergo some torsion due to the eccentricity of applied loads, it is of practical importance to study piles which are subjected to both lateral and torsional loads as shown in Fig. 1a. The soil is assumed to

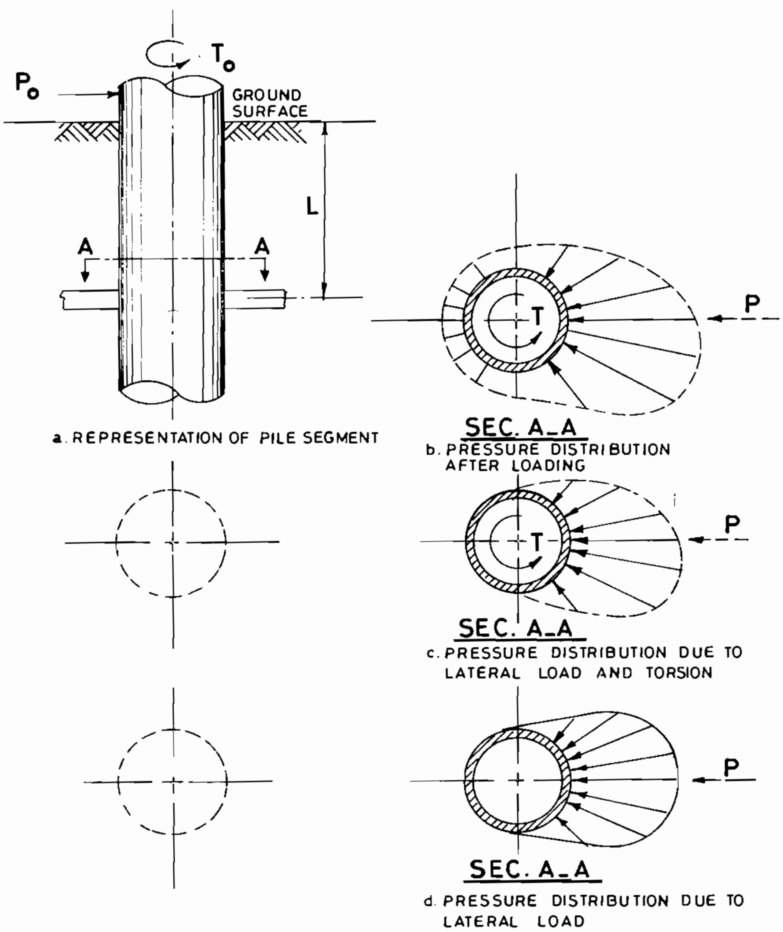

Fig. 1. Probable pressure distribution around pile subjected to lateral and torsional load.

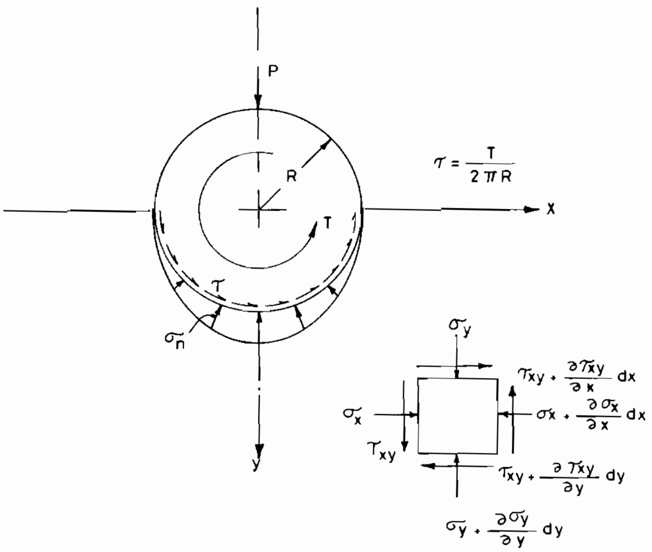

Fig. 2. Schematic drawing of pile subjected to torsional stresses.

be homogeneous and isotropic, and the pressure of the periphery of the pile is the accumulative effect of the lateral earth pressure, lateral load, and torsion as shown in Fig. 1b. If the effect of the lateral earth pressure is ignored then the pressure acting on the pile due to torsional and lateral loads at a depth L below the soil surface is as shown in Fig. 1c. In a recent study (Al-Hussaini 1982) the stress distribution around a laterally loaded pile is determined. In this paper the stress distribution due to the torsional load will be investigated.

Field tests conducted by Filatov *et al.* (1975) on a pile with a spherical end showed that the contact soil pressure materialized only on the lower half sphere away from the load while the upper half was free from any contact pressure. Using the same analogy, it is assumed that the induced normal and shear stresses P and T respectively, are distributed only along the lower half of the pile contact area with the soil as shown in Fig. 2. The equilibrium and compatibility at any point within the soil surrounding the pile satisfy the following differential equation:

$$\nabla^2(\sigma_x+\sigma_y) = \nabla^4[U(x,y)] \tag{1}$$

where σ_x and σ_y are the normal stresses acting on the x and y planes respectively, ∇^2 is the Laplacian operator, and U_{xy} is a stress function which is related to the normal and shear stresses as follows:

$$\sigma_x = \frac{\partial^2 U(x,y)}{\partial y^2} \tag{2a}$$

$$\sigma_y = \frac{\partial^2 U(x,y)}{\partial x^2} \tag{2b}$$

$$\tau_{xy} = \frac{\partial^2 U(x,y)}{\partial x \partial y} \tag{2c}$$

where τ_{xy} is the shear stress.

The solution of the biharmonic Equation 1 is obtained more easily by using complex variables and a procedure adopted by Timoshenko and Goodier (1962) to obtain the following:

$$U_{(x,y)} = \tfrac{1}{2}[\bar{Z}\Phi(Z) + Z\overline{\phi(Z)} + \chi(Z) + \overline{\chi(Z)}] \tag{3}$$

where $\phi(Z)$ and $\chi(Z)$ are suitable analytic functions, and $\overline{\phi(Z)}$ and $\overline{\chi(Z)}$ are the respective conjugates which can be determined from the appropriate boundary conditions; $Z = x + iy$ and $\bar{Z} = x - iy$. The values of σ_x, σ_y and τ_{xy} as defined in Equation 2 can be obtained directly from the second derivative of Equation 3, such that

$$\sigma_x + i\tau_{xy} = \phi'(Z) + \overline{\phi'(Z)} - Z\overline{\phi''(Z)} - \overline{\chi''(Z)} \tag{4a}$$

$$\sigma_y - i\tau_{xy} = \phi'(Z) + \overline{\phi'(Z)} + Z\overline{\phi''(Z)} + \overline{\chi''(Z)} \tag{4b}$$

By substituting $\phi'(Z)$ and $\chi''(Z)$ for $\Phi(Z)$ and $\psi(Z)$ respectively, and subsequently adding and subtracting Equations (4a) and (4b), the following expressions are obtained:

$$\sigma_x + \sigma_y = 2[\Phi(Z) + \overline{\Phi(Z)}] = 4\,\mathrm{Re}[\Phi(Z)] \tag{5a}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{Z}\Phi'(Z) + \psi(Z)] \tag{5b}$$

By choosing the functions $\Phi(Z)$ and $\psi(Z)$ appropriately, the state of stress can completely be defined in the region under consideration. Since the geometry of the problem in this case does not have a simple boundary, it is more convenient to use a curvilinear co-ordinate system using the following relationships:

$$Z = f(t) = f(r + is) \tag{6}$$

where r and s are the curvilinear co-ordinates of a t-plane. Consequently, it is permissible and convenient to express $\Phi(Z)$ and $\psi(Z)$ in terms of the complex variable t, such that

$$\Phi(Z) = \Phi[f(t)] = \Phi(t) \tag{7a}$$

$$\Phi'(Z) = \frac{\Phi'(t)}{f'(t)} \tag{7b}$$

$$\Psi(Z) = \Psi[f(t)] = \Psi(t) \tag{7c}$$

By substituting Equations 6, 7 and 8 in Equation 5, the following expressions are obtained:

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}[\Phi(t)] \tag{8a}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\left[\overline{f(t)}\,\frac{\Phi'(t)}{f'(t)} + \Psi(t)\right] \tag{8b}$$

From Equation 8, the stresses σ_y, σ_x, and τ_{xy} can be obtained as

$$\sigma_y = \mathrm{Re}\left[2\Phi(t) + \overline{f(t)}\,\frac{\Phi'(t)}{f'(t)} + \Psi(t)\right] \tag{9a}$$

$$\sigma_x = \mathrm{Re}\left[2\Phi(t) - \overline{f(t)}\,\frac{\Phi'(t)}{f'(t)} - \Psi(t)\right] \tag{9b}$$

$$\tau_{xy} = \text{Im}\left[\overline{f(t)}\,\frac{\Phi'(t)}{f'(t)} + \Psi(t)\right] \tag{9c}$$

The state of stress at any point, therefore, is determined once the boundary conditions of the problem are imposed, and the functions $\Phi(t)$, $f'(t)$, and $\Psi(t)$ are known.

BOUNDARY CONDITIONS

Since the curvilinear potentials $\Phi(t)$ and $\Psi(t)$ are used to replace the complex potential in the z-plane, it is more convenient to express the Cartesian stresses σ_x,σ_y and τ_{xy} in terms of the curvilinear stresses σ_r, σ_s, and τ_{rs}. The relationship between the stress components in the Cartesian and curvilinear co-ordinates, according to Timoshenko and Goodier (1962), is as follows:

$$\sigma_s + \sigma_r = \sigma_y + \sigma_x \tag{10a}$$

$$(\sigma_s - \sigma_r + 2i\tau_{rs}) = (\sigma_y - \sigma_x + 2i\tau_{xy})\frac{f'(t)}{\overline{f'(t)}} \tag{10b}$$

Let σ_n and τ be the normal and shear stresses respectively, acting on a region B of the z-plane boundary, then

$$[\sigma_n + i\tau]_B = [\sigma_y + i\tau_{xy}]_B = \left[\frac{\sigma_y + \sigma_x}{2} + \frac{\sigma_y - \sigma_x + 2i\tau_{xy}}{2}\right]_B \tag{11}$$

Substituting Equation 8 in Equation 11, the boundary stresses can be expressed as

$$[\sigma_n + i\tau]_B = \left[\Phi(t) + \overline{\Phi(t)} + \overline{f(t)}\frac{\Phi'(t)}{\overline{f'(t)}} + \Psi(t)\frac{f'(t)}{\overline{f'(t)}}\right]_B \tag{12}$$

Since the boundary of the modified semi-infinite mass in the z-plane corresponds only to the real axis of the t-plane (i.e. $s = o$, and $t = r$), along the boundary of the t-plane, Equation 12 becomes

$$\sigma_n - i\tau = \Phi(r) + \overline{\Phi(r)} + f(r)\frac{\overline{\Phi'(r)}}{f'(r)} + \overline{\Psi(r)}\frac{\overline{f(r)}}{f'(r)} \tag{13a}$$

$$\sigma_n + i\tau = \overline{\Phi(r)} + \Phi(r) + \overline{f(r)}\frac{\Phi'(r)}{\overline{f'(r)}} + \Psi(r)\frac{f'(r)}{\overline{f'(r)}} \tag{13b}$$

Equation 13 represents the boundary conditions in the most general form.

SOLUTION OF THE PROBLEM

The solution of stresses around a pile subjected to torsion is generally similar to the solution of the pressure distribution around laterally loaded piles (Al-Hussaini 1982). The procedure requires definition of a function which enables the mapping of the region containing the semi circular region representing the boundary of the pile in the z-plane. This region will be mapped onto a semi-infinite region representing the t-plane as illustrated in Fig. 3. Such a transformation is as follows:

$$Z = R\,\text{Tanh}\left[\frac{\pi i}{2} - \frac{1}{4}\ln\left(\frac{t-R}{t+R}\right)\right] = f(t) \tag{14}$$

where R is the pile radius.

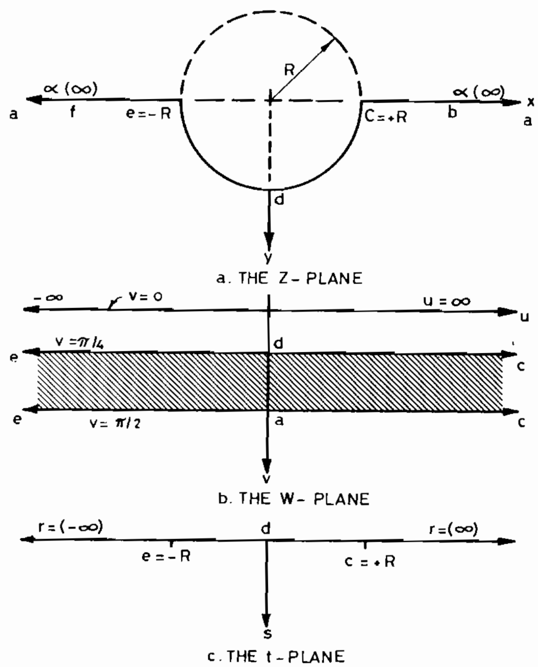

Fig. 3. Transformation of half-space with semi-circle onto half space.

Since the semi-infinite t-plane represents a simple closed curve, and the functions in Equation 13 are the boundary values of a harmonic function of the t-plane which vanish at infinity, the Cauchy integral formula (Churchill 1974) can be applied to determine stresses at any point within the t-plane. Starting with Equation 13a, it follows that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\sigma_n - i\tau}{r-t}dr = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\Phi(r)}{r-t}dr + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{\Phi(r)}}{r-t}dr$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{\Phi'(r)}}{f'(r)}\frac{f(r)}{r-t}dr + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{f(r)}}{f'(r)}\frac{\overline{\Psi(r)}}{r-t}dr \quad (15)$$

Noting that σ_n and τ exist within the interval $-R \le r \le R$, after eliminating zero terms, Equation 15 becomes

$$\frac{1}{2\pi i}\int_{-R}^{R}\frac{\sigma_n - i\tau}{r-t}dr = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\Phi(r)}{r-t}dr = \Phi(t) \quad (16)$$

The value of $\Phi(t)$ following integration, becomes

$$\Phi(t) = \frac{\sigma_n - i\tau}{2\pi i}\, ln\left(\frac{t-R}{t+R}\right) \quad (17)$$

and

$$\Phi'(t) = \frac{\sigma_n - i\tau}{\pi i}\left(\frac{R}{t^2 - R^2}\right) \tag{17b}$$

In a similar manner the Cauchy integral formula is applied to Equation 13b to obtain the function $\Psi(t)$.

$$\frac{1}{2\pi i}\int_{-R}^{R}\frac{\sigma_n + i\tau}{r - t}dr = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{\Phi(r)}}{r - t}dr + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\Phi(r)}{r - t}dr$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{f(r)}}{\overline{f'(r)}}\frac{\Phi'(r)}{r - t}dr + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{f'(r)}}{\overline{f'(r)}}\frac{\Psi(r)}{(r - t)}dr \tag{18}$$

Since at the boundary of t-plane $f(r) = \overline{f(r)}$ and $f'(r) = \overline{f(r)}$, then Cauchy integral can be applied on the last two terms, the term involving $\overline{\Phi(r)}$ is equal to zero, and by integration, Equation 18a becomes

$$\frac{\sigma_n + i\tau}{2\pi i}\ln\frac{t - R}{t + R} = \Phi(t) + \frac{f(t)}{f'(t)}\Phi'(t) + \Psi(t) \tag{19}$$

Substituting Equation 17a in Equation 19, the value of $\Psi(t)$ becomes

$$\Psi(t) = \frac{\tau}{\pi}\ln\frac{t - R}{t + R} - \frac{f(t)}{f'(t)}\Phi'(t) \tag{20}$$

If the effect of lateral pressure is ignored, then σ_n must be set equal to zero and values of $\Phi(t)$, $\Phi'(t)$ as expressed in Equation 17 become

$$\Phi(t) = -\frac{\tau}{2\pi}\ln\left(\frac{t - R}{t + R}\right) \tag{21a}$$

$$\Phi'(t) = -\frac{\tau}{\pi}\left(\frac{R}{t^2 - R^2}\right) \tag{21b}$$

GRAPHICAL SOLUTION

All functions required for determining σ_x, σ_y and τ_{xy} as expressed in Equation 9 are known. The numeric evaluation is carried out by a digital computer since manual computation would be extremely lengthy and difficult. To maintain maximum generality and simplicity of the solution, the results are presented in a dimensionless form which permits a description of the state of stress at any point within the affected soil around the pile. Isobars of the induced stresses are given as a fraction of the average torsional stress τ acting along the contact area between the soil and pile. Isobars for σ_y, σ_x and τ_{xy} are presented in Figs 4, 5 and 6 respectively.

COMPARISON WITH ANOTHER SOLUTION

The stress distribution derived in this study was compared with those derived by Florin (1959) for stresses beneath an infinite strip loaded with a uniform horizontal load.

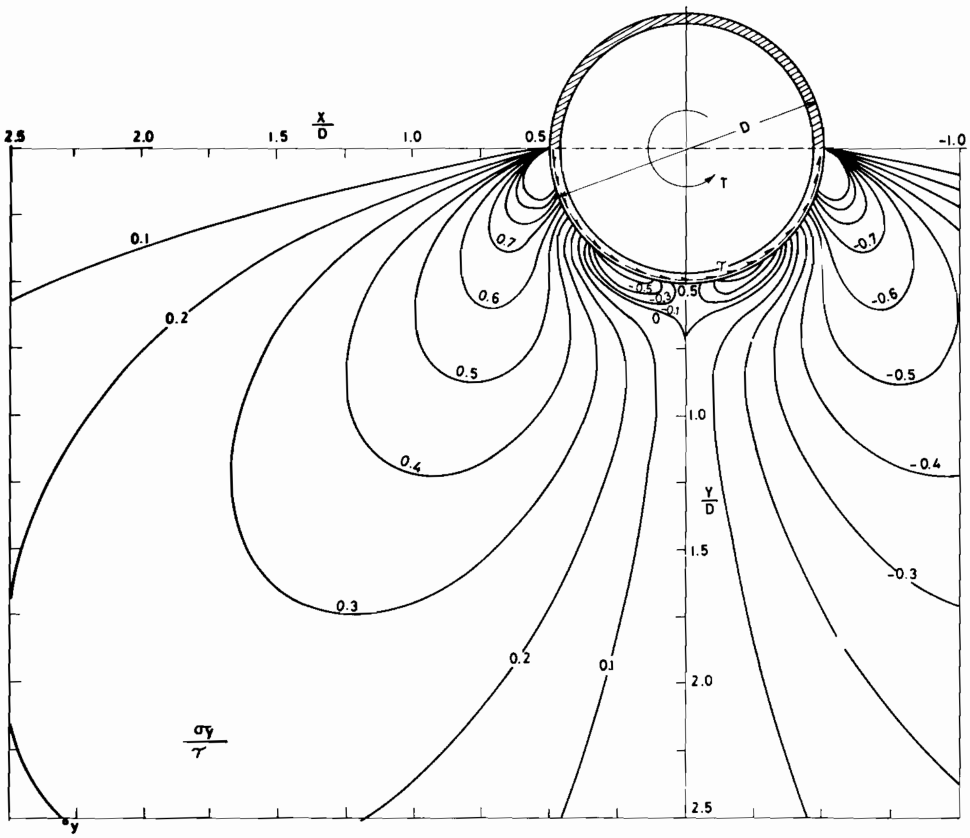

Fig. 4. Isobars of σ_y induced by torsional shear stress τ.

According to Florin (1959), the stresses σ_y, σ_x and τ_{xy} beneath an infinite strip of width 2a loaded with a uniform shear stress of intensity q can be expressed as

$$\sigma_y = \frac{q}{\pi}\left[\frac{4axy^2}{(a^2+x^2+y^2)^2-4a^2x^2}\right] \tag{22a}$$

$$\sigma_x = \frac{q}{\pi}\left[ln\frac{(a+x)^2+y^2}{(a-x)^2+y^2}-\frac{4axy^2}{(a^2+x^2+y^2)^2-4a^2x^2}\right] \tag{22b}$$

$$\tau_{xy} = \frac{q}{\pi}\left[(\text{arctg}\frac{a-x}{y}+\text{arctg}\frac{a+x}{y}-\frac{2ay(a^2-x^2+y^2)}{(a^2+x^2+z^2)^2-4a^2x^2}\right] \tag{22c}$$

The stresses σ_y, σ_x and τ_{xy} were also presented in dimensionless form after substituting $2a$ by D and q by τ to facilitate a direct comparison with the results of this study. Isobars of σ_y, σ_x, and τ_{xy} given as a fraction of the applied shear stress τ are illustrated in Figs 7, 8 and 9 respectively. The difference between the isobars for the pile (Figs 4, 5 & 6) and those for the corresponding loaded semi-infinite strip (Figs 7, 8 & 9) results from the difference between the geometry of the pile surface and the planar geometry of the loaded semi-infinite strip. It must be noted that, as D becomes very small, both the

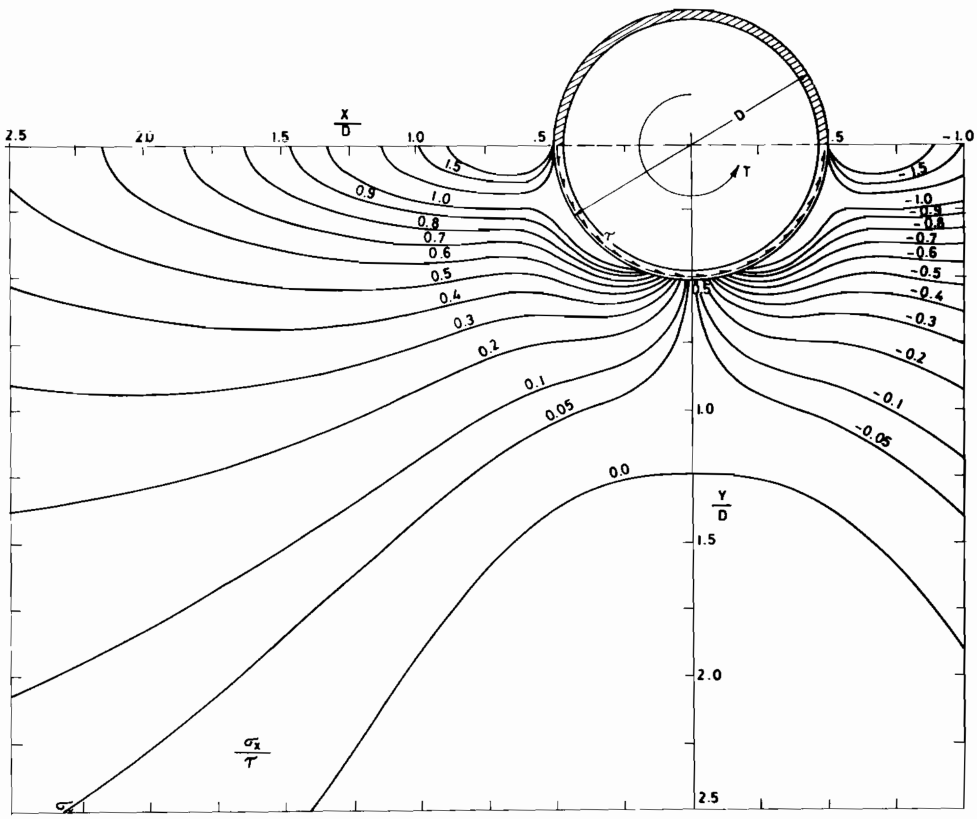

Fig. 5. Isobars of σ_x induced by torsional shear stress τ.

solution obtained in this study and that of Florin (1959) gave identically similar numerical results.

Although the case of pile subjected to pure torsion is much less common than pile subjected to both lateral force and torsional moment, nevertheless such pile has the potential for field testing technique. Thus few comments on rigid pile subjected to pure torsion, and its interaction with the surrounding soil are in order.

RIGID PILE WITH UNIFORM SHEAR STRESS

Assume a circular rigid pile to be subjected to pure torsional moment at its top, then this torsion will cause horizontally directed shearing stress at its surface and its base. One might expect the shear stress to be uniform at any cross section of the pile but it increases with the angle of twist in a manner resembling in appearance the shearing stress strain curve, for the same material, obtained from direct shear test. Careful field tests (Stole 1972) indicated that the relationship between the applied torque and resulting pile rotation is fairly linear up to failure where the torque remained constant with rotation; the linear relationship was also observed on unloading. This indicates that the assumption of linear elasticity for describing pile response under torsion is reasonably correct.

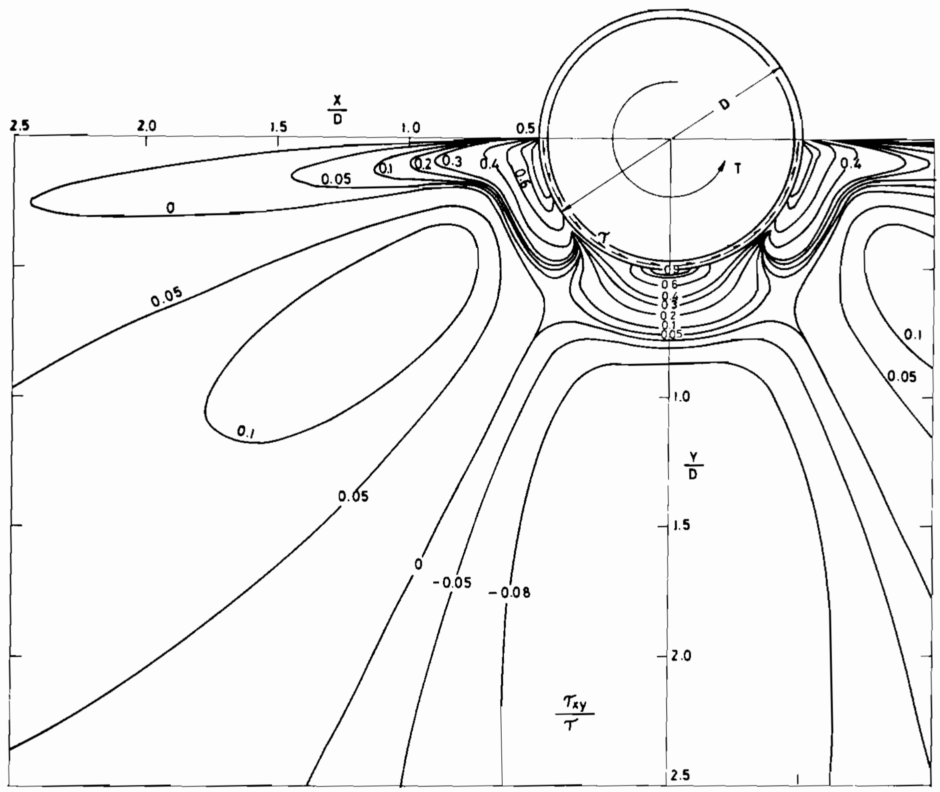

Fig. 6. Isobars of τ_{xy} induced by torsional shear stress τ.

To maintain simplicity of the analysis, it is assumed that the shear stress τ due to torsion is uniformly distributed around the periphery and the base of the pile. Thus the resisting torque generated around the surface of the pile T_s and its base T_t can be expressed as follows:

$$T_s = \frac{1}{2}\pi\tau D^2 L \tag{23a}$$

$$T_t = \frac{1}{16}\pi\tau D^3 \tag{23b}$$

where D and L are the diameter and length of the pile respectively. If we assume that no slippage occurs between the pile and the surrounding soil, then the total resisting torque of the pile T can be obtained by summing Equations 23a and 23b to get

$$T = T_s + T_t = \frac{\pi\tau D^3}{16}\left(\frac{8L}{D}+1\right) \tag{24}$$

Equation 24 will enable the determination of the shear stress around the pile from its geometry and the torque acting upon it.

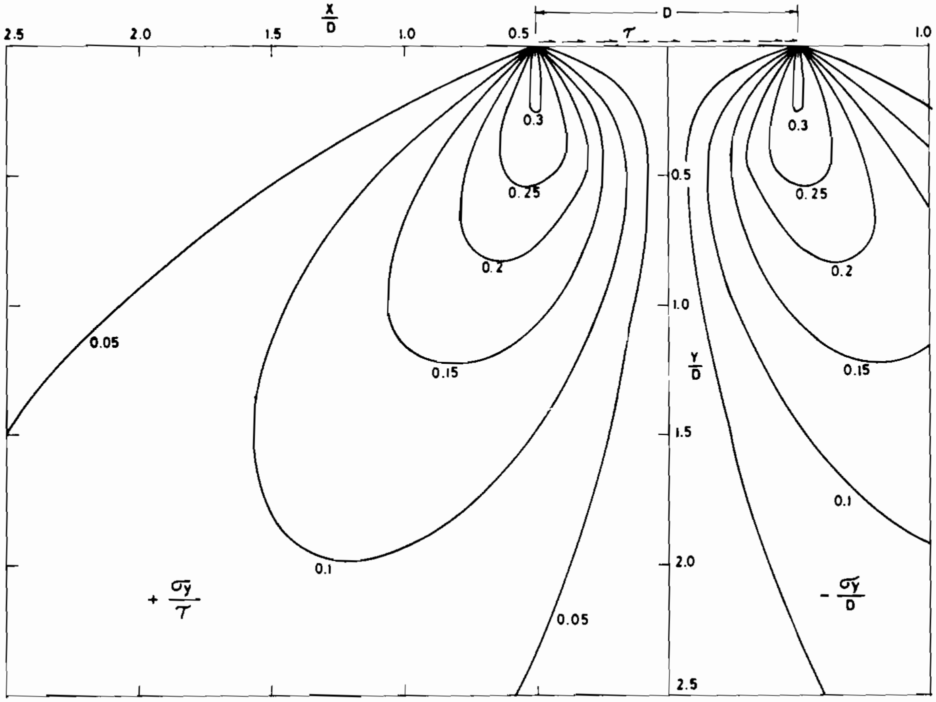

Fig. 7. Isobars of σ_y in a semi-infinite mass under uniform shear load.

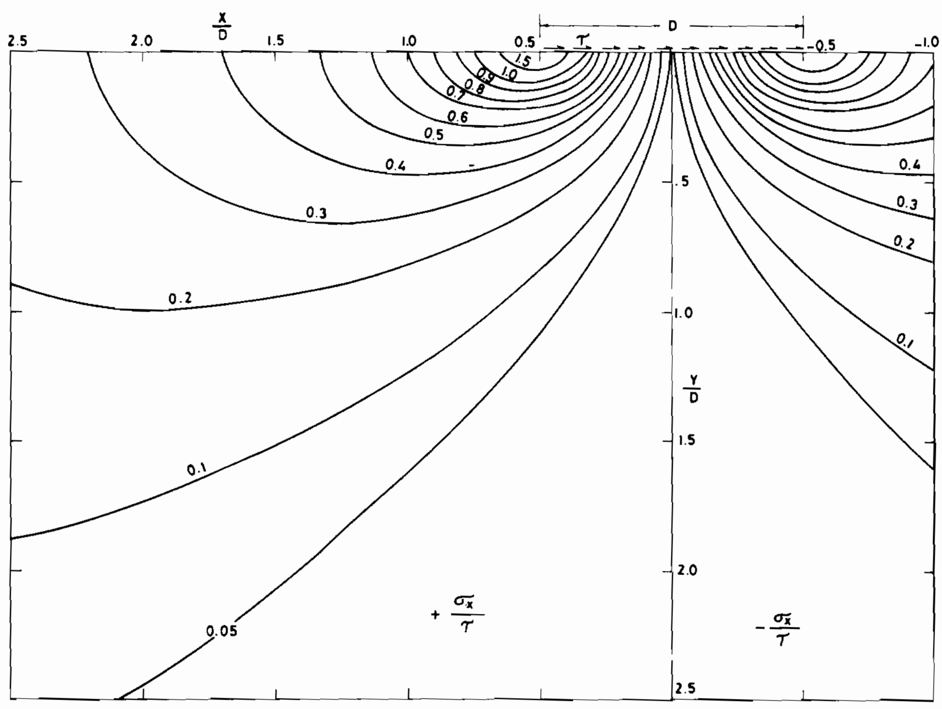

Fig. 8. Isobars of σ_x in a semi-infinite mass under uniform shear load.

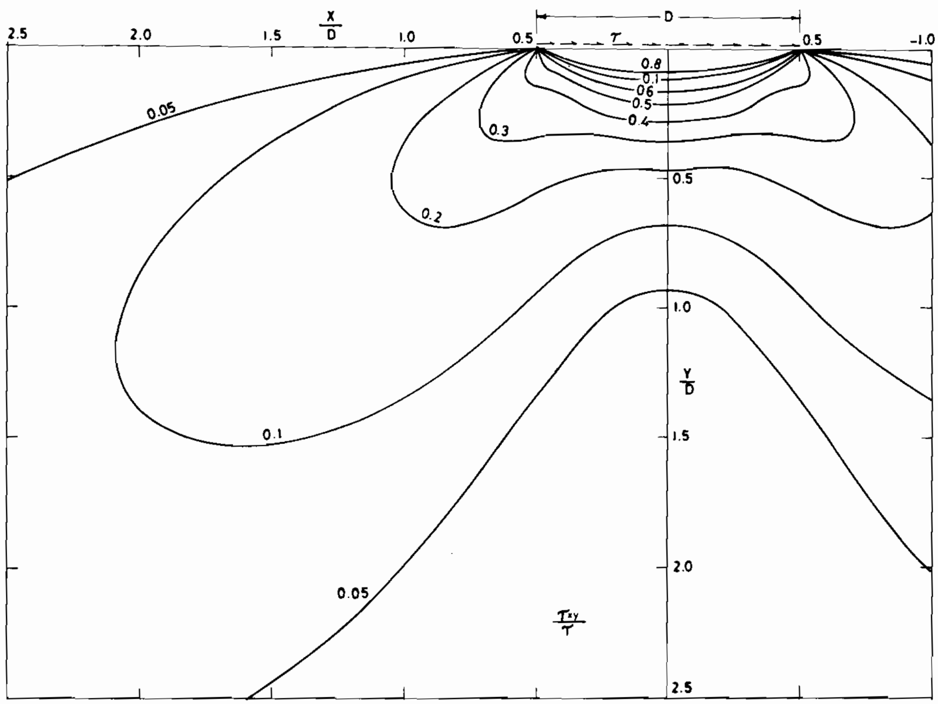

Fig. 9. Isobars of τ_{xy} in a semi-infinite mass under uniform load.

RIGID PILE WITH VARIABLE SHEAR STRESS

The assumption of uniform shear stress is only applicable to piles in some over-consolidated clays. However, piles embedded in other soils such as sands usually experience an increase in shear stress with depth. For these soils the shear stress around the pile may be expressed as follows:

$$\tau = \tau_o + K(\gamma z)^n \tag{25}$$

where τ_o, K and n are material constants which could be determined from laboratory or field tests. Once τ between the soil and pile is known then a differential equation expressing the torsional resistance of the pile, in a manner similar to Equation 24, can be generated.

Both Equations 24 and 25 assume that the horizontally directed shear stress at any depth is uniformly distributed around its periphery. The normal and shear stresses within the soil around the pile under uniform shear could be superimposed on those induced by the lateral thrust as indicated in Equation 9 even though the boundary conditions of the two cases are slightly different.

CONCLUSION

The review of literature published so far indicates that there is a limited amount of research conducted on piles subjected to torsion. The reason could be attributed to the complex nature of shear stress in soil-pile interaction, and to the lack of field tests that could be used in the analysis of this problem. The present study is an attempt to determine the normal and shear stresses within the soil surrounding a cylindrical rigid pile subjected to torsional and lateral loading. So far, the solution is limited to the case of linear elastic soil under plane strain conditions where no slippage is permitted between the soil and the pile. This study showed that the geometry of the pile influences the stress distribution around the pile. The solution presented could be expanded to other and more general cases.

ACKNOWLEDGMENTS

The author would like to acknowledge the Kuwait Foundation for the Advancement of Sciences for its support of this research.

REFERENCES

Al-Hussaini, M. 1982. Pressure distribution around a laterally loaded pile. J. Univ. Kuwait (Sci.) **9**(1): 63–75.

Churchill, R.V. 1974. Complex variables and applications, pp. 129–30. McGraw-Hill, New York.

Coulomb, C.A. 1785. Theories of simple mechanics. Mathematical dissertation, Roy. Acad. Sci., Paris–10, pp. 161–332.

Filatov A.V., Biko, N.V. & Bykov, V.I. 1975. Investigation of the bearing capacity of short situ-cast piles with an enlarged base, pp. 161–64. Plenum, New York.

Florin, V.A. 1959. Fundamentals of soil mechanics, Vol. 1, pp. 110–12. Government Press, Leningrad.

Kaldjin, M.J. 1971. Torsional stiffness of embedded footing. J. Soil Mech. Found. Eng. ASCE **97,** SM7: 969–79.

Poulos, H.G. 1975. Torsional response of piles. J. Geotech. Eng. Div, ASCE **101,** GT10: 1019–33.

Randolph, M.F. 1981. Piles subjected to torsion. J. Geotech. Eng. Div., ASCE **107,** GT8: 1095–1111.

Scott, R.F. 1981. Foundation analysis, pp. 296–97. Prentice-Hall, Englewood Cliffs.

Stole, U.W. 1972. Torque shear test on cylindrical friction piles. Civil Engineer, ASCE **42** (4): 63–64.

Timoshenko S. & Goodier, J.N. 1962. Theory of elasticity, pp. 187–97. McGraw-Hill, New York.

(Received 3 May 1982)

توزيع الضغوط حول ركائز أساسات صلبة معرضة لقوى اللي

مساعد الحسيني
قسم الهندسة المدنية بجامعة الكويت

خلاصة

يحتوي البحث على حل رياضي لايجاد توزيع الضغوط في التربة المحيطة بركائز أساسات اسطوانية مسلط عليها قوى اللي وقوى جانبية .
في هذا البحث اعتبرنا أن ركائز الأساسات صلبة غير قابلة للانحناء وأن التربة المحيطة بها في حالة مرونة متجانسة . قدمت النتائج على شكل خطوط بيانية غير ذات أبعاد ليكون الحل شاملا ومبسطا .