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On weakly σ-spaces

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ABSTRACT

Weakly σ-spaces are introduced by Al-Nashef (1990), where a space X is called a weakly σ-space if every open cover $\mathscr{U}$ of X has a weakly dissectable open refinement. This means an open refinement $\mathscr{V}$ that has a closed σ-closure preserving refinement $\mathscr{C} = \bigcup_{n \in N} \mathscr{C}_n$ such that each member of $\mathscr{V}$ is a union of members of $\mathscr{C}$. Every regular σ-space and every D-paracompact space is a weakly σ-space.

To obtain other characterizations of weakly σ-spaces we introduce the concepts of strong subparacompactness and subexpandability. We call a space X strongly subparacompact if for each open cover $\mathscr{U}$ of X there exists a sequence $(\mathscr{V}_n)_{n \in N}$ of open refinements of $\mathscr{U}$ such that, for each $x \in X$ there exists $n(x) \in N$ such that $\operatorname{ord}(x, \mathscr{V}_n) = 1$ for each $n \geqslant n(x)$. We also call a space X subexpandable if, given a discrete collection $\mathscr{F} = \{F_\alpha : \alpha \in I\}$ of closed subsets of X and its open expansion $\mathscr{U} = \{U_\alpha : \alpha \in I\}$ that satisfies $F_\alpha \cap U_\beta = \phi$ for $\alpha, \beta \in I$ with $\alpha \neq \beta$, we can find a weakly dissectable open collection $\mathscr{V} = \{V_\alpha : \alpha \in I\}$ such that $F_\alpha \subset V_\alpha \subset U_\alpha$ for each $\alpha \in I$.

We prove that the following four statements are equivalent for a space X:

(1) X is a weakly σ-space
(2) X is strongly subparacompact and subexpandable.
(3) X is subparacompact and subexpandable.
(4) X is submetacompact and subexpandable.

In studying subspaces of weakly σ-spaces, we prove that every normally situated subspace is a weakly σ-space.

1. INTRODUCTION

A space X is called a weakly σ-space (Al-Nashef 1990) if every open cover $\mathscr{U}$ of X has an open refinement that has a weak dissection, i.e. a σ-closure preserving closed refinement $\mathscr{C} = \bigcup_{n \in N} \mathscr{C}_n$ such that each member of $\mathscr{V}$ is a union of elements of $\mathscr{C}$ (such a refinement is called an N-refinement).

The class of weakly σ-spaces was introduced in (Al-Nashef 1990) as a generalization of both D-paracompact spaces and regular σ-spaces. Our aim in the present paper is to obtain characterizations of weakly σ-spaces similar to those known for D-paracompact spaces included in Theorem 1 of Brandenburg (1985). This is done in Section 2 which contains also the main definitions. In Section 3 we present more properties of weakly σ-spaces and prove that every normally situated subset of a weakly σ-space is a weakly σ-space. Finally, in Section 4, we pose three open questions.

2. CHARACTERIZATIONS OF WEAKLY σ-SPACES

We recall the following definitions from Al-Nashef (1990). Let $\mathscr{U}$ and $\mathscr{V}$ be two covers of a space X. $\mathscr{V}$ is called an N-refinement of $\mathscr{U}$ if $\mathscr{V}$ refines $\mathscr{U}$ and each member of $\mathscr{U}$ is union of members of $\mathscr{V}$. A weak dissection for a cover $\mathscr{U}$ is a closed σ-closure-preserving N-refinement $\mathscr{C} = \bigcup_{n\in N} \mathscr{C}_n$ of $\mathscr{U}$. A space X is called a weakly σ-space if every open cover of X has an open weakly dissectable refinement, i.e. an open refinement that has a weak dissection.

Remark. We agree to call a collection $\mathscr{A}$ of subsets of X (not necessarily a cover of X) weakly dissectable if we can find a σ-closure-preserving closed collection $\mathscr{C} = \bigcup_{n\in N} \mathscr{C}_n$ (we will call it a weak dissection for $\mathscr{A}$) such that each $C \in \mathscr{C}$ is contained in some $A \in \mathscr{A}$ and each member of $\mathscr{A}$ is a union of members of $\mathscr{C}$.

It is easy to see that regular σ-spaces as characterized by Heath & Hodel (1973) and also D-paracompact spaces as characterized by Brandenburg (1985) are weakly σ-spaces. On the other hand, every weakly σ-space is subparacompact, where X is subparacompact (Burke 1969) if each open cover of X has a σ-closure-preserving closed refinement.

To state the main theorem of this section which gives several characterizations of weakly σ-spaces, we introduce two definitions and recall several others.

Definition 2.1. A space X is called subexpandable if, given a discrete collection $\mathscr{F} = \{F_\alpha\colon \alpha \in I\}$ of closed subsets of X and given an open expansion $\mathscr{U} = \{U_\alpha\colon \alpha \in I\}$ of $\mathscr{F}$ with $F_\alpha \cap U_\beta = \phi$ for $\alpha, \beta \in I$ and $\alpha \neq \beta$, there exists a weakly dissectable collection $\mathscr{V} = \{V_\alpha\colon \alpha \in I\}$ of open subsets of X such that $F_\alpha \subset V_\alpha \subset U_\alpha$ for each $\alpha \in I$.

Definition 2.2. A space X is called strongly subparacompact if, for each open cover $\mathscr{U} = \{U_\alpha\colon \alpha \in I\}$ of X, there exists a sequence $(\mathscr{V}_n)_{n\in N}$ of open covers of X that satisfies (1) $\mathscr{V}_1 = \mathscr{U}$ and $\mathscr{V}_{n+1}$ is a precise refinement of $\mathscr{V}_n$, for each $n \in N$ and (2) for each $x \in X$ there exists $n(x) \in N$ such that $\operatorname{ord}(x, \mathscr{V}_n) = 1$ for all $n \geqslant n(x)$.

By Theorem 1.6 of Burke (1970), it follows that every strongly subparacompact space is subparacompact. Also, by Lemma 2.8 of Al-Nashef (1990), every weakly σ-space is strongly subparacompact.

Recall that a space X is submetacompact (Junnila 1978) if, for each open cover $\mathscr{U}$ of X, there exists a sequence $(\mathscr{V}_n)_{n\in N}$ of open refinements of $\mathscr{U}$ such that if $x \in X$ then there exists $n(x) \in N$ for which $0 < \operatorname{ord}(x, \mathscr{V}_{n(x)}) < \omega$. It is well known that every subparacompact space is submetacompact (see Theorem 1.6 of Burke (1970)).

Theorem 2.3. The following statements are equivalent for a space X:

(a) X is a weakly σ-space.
(b) X is strongly subparacompact and subexpandable.
(c) X is subparacompact and subexpandable.
(d) X is submetacompact and subexpandable.

Proof. It is clear that (b) $\rightarrow$ (c) and (c) $\rightarrow$ (d). So we need only show that (a) $\rightarrow$ (b) and (d) $\rightarrow$ (a).

(a) → (b). Let X be a weakly σ-space. It is clear from Lemma 2.8 of Al-Nashef (1990) that X is strongly subparacompact. To prove that X is subexpandable we let $\mathscr{F} = \{F_\alpha: \alpha \in I\}$ be a discrete collection of closed subsets of X and let $\mathscr{U} = \{U_\alpha: \alpha \in I\}$ be a collection of open subsets of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in I$ and $F_\alpha \cap U_\beta = \phi$ when $\alpha, \beta \in I$ and $\alpha \neq \beta$. Now, $\{U_\alpha: \alpha \in I\} \cup \{X - \mathscr{F}^*\}$ is an open cover of the weakly σ-space X; so, by Lemma 2.6 of Al-Nashef (1990), it has a precise weakly dissectable refinement $\{V_\alpha: \alpha \in I\} \cup \{W\}$. It is clear that $\{V_\alpha: \alpha \in I\}$ is an open weakly dissectable collection that satisfies $F_\alpha \subset V_\alpha \subset U_\alpha$, for each $\alpha \in I$, and this proves that X is subexpandable.

(d) → (a). Let $\mathscr{U}$ be an open cover of X. Then it has a refining sequence $(\mathscr{V}_n)_{n\in N}$ of open covers of X where, for each $n \in N$, $\mathscr{V}_n = \{V_\alpha: \alpha \in I(n)\}$ and, for each $x \in X$, there exists $n(x)$ such that $0 < \operatorname{ord}(x, \mathscr{V}_{n(x)}) < \omega$.

For each $(n, k) \in N \times N$ we take $X(n, k) = \{x \in X: \operatorname{ord}(x, \mathscr{V}_n) \leqslant k\}$. Then it is clear that $X(n, k)$ is closed for each $(n, k) \in N \times N$.

Take $n \in N$. For each $k \in N$ we will define, by induction on k, a weakly dissectable collection $\mathscr{W}(n, k)$ of open subsets of X such that each $W \in \mathscr{W}(n, k)$ is contained in some $U \in \mathscr{U}$ and, if $x \in X$ with $\operatorname{ord}(x, \mathscr{V}_n) \leqslant k$ then x belongs to some element W of $\bigcup_{m \leqslant k} \mathscr{W}(n, m)$.

Let $k = 1$. For each $\alpha \in I(n)$ we let $F_\alpha = X(n, 1) \cap V_\alpha$ and put $\mathscr{F}(n, 1) = \{F_\alpha: \alpha \in I(n)\}$. Then $\mathscr{F}(n, 1)$ is a discrete collection of closed subsets of X. For each $\alpha \in I(n)$ we take $G_\alpha = V_\alpha - \bigcup\{F_\beta: \beta \in I(n), \beta \neq \alpha\}$. Then $\{G_\alpha: \alpha \in I(n)\}$ is an open expansion of $\mathscr{F}(n, 1)$ such that $F_\alpha \cap G_\beta = \phi$ for $\alpha, \beta \in I(n)$ with $\alpha \neq \beta$. Since X is subexpandable, there exists a weakly dissectable open collection $\mathscr{W}(n, 1) = \{W_\alpha: \alpha \in I(n)\}$, such that, for each $\alpha \in I(n)$, $F_\alpha \subset W_\alpha \subset G_\alpha \subset V_\alpha$ and $W_\alpha \subset U$ for some $U \in \mathscr{U}$. Also, if $x \in X$ and $\operatorname{ord}(x, \mathscr{V}_n) \leqslant 1$, then for some $\alpha \in I(n)$, $x \in F_\alpha \subset W_\alpha$. We take $H(n, 1) = \bigcup\{W_\alpha: \alpha \in I(n)\}$.

Next, we assume that for each $i < k$ we have already defined a weakly dissectable open collection $\mathscr{W}(n, i)$ such that each $W \in \mathscr{W}(n, i)$ is contained in some $U \in \mathscr{U}$ and that for $x \in X$ with $\operatorname{ord}(x, \mathscr{V}_n) \leqslant i$, there exists $W \in \bigcup_{m=1}^{i} \mathscr{W}(n, m)$ such that W contains x. Now, for each $A \in [I(n)]^k$, where $[I(n)]^k = \{A \subset I(n): |A| = k\}$, we define $F_A = X(n, k) \cap [\bigcap\{V_\alpha: \alpha \in A\}] \cap [X - \bigcup_{i=1}^{k-1} H(n, i)]$ and put $\mathscr{F}(n, k) = \{F_A: A \in [I(n)]^k\}$. Again $\mathscr{F}(n, k)$ is a discrete collection of closed subsets of X. For each $A \in [I(n)]^k$ we take $G_A = (\bigcap\{V_\alpha: \alpha \in A\}) \cap (X - \bigcup\{F_B: B \in [I(n)]^k, B \neq A\})$. Then $\{G_A: A \in [I(n)]^k\}$ is an open expansion of $\mathscr{F}(n, k)$ and $F_A \cap G_B = \phi$ for $A, B \in [I(n)]^k$ with $A \neq B$. Since X is subexpandable we find a weakly dissectable open collection $\mathscr{W}(n, k) = \{W_A: A \in [I(n)]^k\}$ such that $F_A \subset W_A \subset G_A \subset V_\alpha$ for some $\alpha \in A$. So W_A is contained in some $U \in \mathscr{U}$ for each $A \in [I(n)]^k$. Let $x \in X$ and $\operatorname{ord}(x, \mathscr{V}_n) = k$, say $V_{\alpha_1}, \ldots, V_{\alpha_k} \in \mathscr{V}_n$ and $x \in \bigcap_{i=1}^{k} V_{\alpha_i}$. Assume that x does not appear in any $W \in \bigcup_{i=1}^{k-1} \mathscr{W}(n, i)$. Then $x \in F_A \subset W_A \in \mathscr{W}(n, k)$ where $A = \{\alpha_1, \ldots, \alpha_k\} \in [I(n)]^k$.

Now we take $\mathscr{W}(n) = \bigcup_{k\in N} \mathscr{W}(n, k)$ and $\mathscr{W} = \bigcup_{n\in N} \mathscr{W}(n) = \bigcup_{n\in N} \bigcup_{k\in N} \mathscr{W}(n, k)$. Then $\mathscr{W}$ is σ-weakly dissectable and hence, by Lemma 2.5 of Al-Nashef (1990), a weakly dissectable collection of open subsets of X such that each $W \in \mathscr{W}$ is contained in some $U \in \mathscr{U}$. To show that $\mathscr{W}$ covers X, we let $x \in X$. There exists $n(x) \in N$ such that $0 < \operatorname{ord}(x, \mathscr{V}_{n(x)}) < \omega$, say, $\operatorname{ord}(x, \mathscr{V}_{n(x)}) = k$. Then x belongs to some element of $\bigcup_{i=1}^{k} \mathscr{W}(n(x), i) \subset \mathscr{W}$. So, we have proved that $\mathscr{U}$ has a weakly dissectable open refinement $\mathscr{W}$ and this proves that X is a weakly σ-space.

Theorem 2.4. A space X is a weakly σ-space if and only if for each open cover $\mathscr{U}$ of X we can find an open refinement $\mathscr{V} = \{V_\alpha : \alpha \in I\}$ of $\mathscr{U}$ and a countable closed cover $\mathscr{E}$ of X such that:

(i) For each $E \in \mathscr{E}$, $\mathscr{V} \mid E = \{V_\alpha \cap E: \alpha \in I\}$ has a closure-preserving closed (in E) refinement $\mathscr{C}(E)$ that covers E.

(ii) If we put $\mathscr{C} = \bigcup\{\mathscr{C}(E): E \in \mathscr{E}\}$ then $V_\alpha = \bigcup\{C \in \mathscr{C}: C \subseteq V_\alpha\}$, for each $\alpha \in I$.

Proof. Let $\mathscr{U}$ be an open cover of a weakly σ-space X. Let $\mathscr{V} = \{V_\alpha : \alpha \in I\}$ be a weakly dissectable open refinement of $\mathscr{U}$ that has a weak dissection $\mathscr{C} = \bigcup_{n \in N} \mathscr{C}_n$. We take $\mathscr{E} = \{\mathscr{C}_n^* : n \in N\}$. Then $\mathscr{E}$ is a countable closed cover of X. If $E \in \mathscr{E}$, say $E = \mathscr{C}_n^*$ for some $n \in N$, then $\mathscr{V} \mid E$ has the collection $\mathscr{C}_n$ as a closure-preserving closed (in E) refinement that covers E. Moreover, since $\mathscr{C} = \bigcup_{n \in N} \mathscr{C}_n$ is an N-refinement of $\mathscr{V}$, then it is clear that $V_\alpha = \bigcup\{C \in \mathscr{C}: C \subseteq V_\alpha\}$ for each $\alpha \in I$.

For the converse, let $\mathscr{U}$ be an open cover of X. By assumption, there exist an open refinement $\mathscr{V} = \{V_\alpha : \alpha \in I\}$ of $\mathscr{U}$ and a countable closed cover $\mathscr{E}$ of X that satisfy conditions (i) and (ii) in the theorem. To show that the collection $\mathscr{C} = \bigcup\{\mathscr{C}(E): E \in \mathscr{E}\}$ is a weak dissection for $\mathscr{V}$, we first notice that, by condition (ii), $\mathscr{C}$ is an N-refinement of $\mathscr{V}$. Also, for each $E \in \mathscr{E}$, $\mathscr{C}(E)$ is a closure-preserving collection of closed subsets of X.

In this way we have shown that X is a weakly σ-space and the proof is complete.

3. PROPERTIES OF WEAKLY σ-SPACES

In this section we give more properties of weakly σ-spaces. It is clear that the product of two weakly σ-spaces need not be weakly σ-space, since there exists an example, constructed by Alster & Enkelking (1972), of a paracompact space (hence, a weakly σ-space) X, such that $X \times X$ is not even subparacompact (and thus not a weakly σ-space).

Next we study subspaces of weakly σ-spaces. First we show that a subspace of a weakly σ-space need not be a weakly σ-space.

A space X is called by Brandenburg (1981) D-normal if for each pair A, B of disjoint closed subsets of X there exists a pair G, H of disjoint closed G_δ-subsets of X such that $A \subset G$ and $B \subset H$. We introduce the following definition.

Definition 3.1. A space X is called collectionwise D-normal if for each discrete collection $\mathscr{F}$ of closed subsets of X there exists a disjoint collection $\mathscr{H}$ of closed G_δ-subsets such that each $F \in \mathscr{F}$ is contained in some $H \in \mathscr{H}$.

It is clear that every perfect space is collectionwise D-normal and every collectionwise D-normal space is D-normal. We now prove the following property of weakly σ-spaces.

Theorem 3.2. Every weakly σ-space is collectionwise D-normal.

Proof. Let X be a weakly σ-space. Let $\mathscr{F} = \{F_\alpha : \alpha \in I\}$ be a discrete collection of closed subsets of X. For $\alpha \in I$ we take $V_\alpha = X - \bigcup\{F_\beta : \beta \in I, \beta \neq \alpha\}$. Then $\mathscr{V} = \{V_\alpha : \alpha \in I\}$ is an open cover of X and so it has a precise weakly dissectable open refinement $\mathscr{W} = \{W_\alpha : \alpha \in I\}$, with $W_\alpha \subset V_\alpha$ for each $\alpha \in I$. Let $\mathscr{C} = \bigcup_{n \in N} \mathscr{C}_n$ be a weak dissection for $\mathscr{W}$ where, for $n \in N$, $\mathscr{C}_n = \{C(\alpha, n): \alpha \in I\}$ and $C(\alpha, n) \subset W_\alpha$ for each $\alpha \in I$. Also, if $\alpha \in I$ then $W_\alpha = \bigcup_{n \in N} C(\alpha, n)$. For each $\alpha \in I$ we define $H_\alpha = X - \bigcup\{W_\beta: \beta \in I, \alpha \neq \beta\}$. Then H_α is a closed G_δ-subset of X, because we can write $\bigcup\{W_\beta: \beta \in I, \beta \neq \alpha\} = \bigcup_{n \in N} (\bigcup\{C(\beta, n): \beta \in I, \alpha \neq \beta\})$ as a countable union of closed sets (since $\mathscr{C}_n$ is closure preserving for each $n \in N$). For each $\alpha \in I$, $F_\alpha \subset H_\alpha$ and for $\alpha, \beta \in I$ with $\alpha \neq \beta$ we get that $H_\alpha \cap H_\beta = [X - \bigcup\{W_\delta: \delta \in I, \delta \neq \alpha\}] \cap [X - \bigcup\{W_\delta: \delta \in I, \delta \neq \beta\}] = X - \bigcup\mathscr{W} = \phi$.

Example 3.3. Let ω_0 and ω_1 denote the first infinite ordinal and the first uncountable ordinal, respectively. The product space $[o, \omega_0] \times [o, \omega_1]$ is a compact Hausdorff space, but its subspace $[o, \omega_0] \times [o, \omega_1] - \{(\omega_0, \omega_1)\}$ (known as the Tychonoff plank) is not even D-normal; therefore, it is not a weakly σ-space.

Example 3.4. The space constructed in Example 2 of Brandenburg (1985) is a strongly subparacompact space, but it is not a weakly σ-space since it is not D-normal.

Lemma 3.5. Every F_σ-subset of a weakly σ-space is a weakly σ-space.

Proof. Let X be a weakly σ-space and let $Y = \bigcup_{n \in N} Y_n$ be an F_σ-subset of X, where Y_n is a closed subset of X for each $n \in N$. Let $\mathscr{U}$ be an open cover of the subspace Y. For each $U \in \mathscr{U}$ choose an open subset $V(U)$ of X such that $U = V(U) \cap Y$. Then, for each $n \in N$, we obtain an open cover $\mathscr{V}_n = \{V(U): U \in \mathscr{U}\} \cup \{X - Y_n\}$ of the space X. Let $\mathscr{W}_n$ be a weakly dissectable precise open refinement of $\mathscr{V}_n$ and let $\mathscr{A}_n = \{W \cap Y: W \in \mathscr{W}_n$ and $W \cap Y_n \neq \phi\}$. Let $\mathscr{A} = \bigcup_{n \in N} \mathscr{A}_n$. Then $\mathscr{A}$ is a σ-weakly dissectable (hence, weakly dissectable) open refinement of $\mathscr{U}$. So, Y is a weakly σ-space.

Lemma 3.6. Let A be a subset of a space X such that if $A \subset U \subset X$, with U open, then there exists a weakly σ-space $B \subset X$ for which $A \subset B \subset U$. Then A is a weakly σ-space.

Proof. Let $\{W_\alpha: \alpha \in I\}$ be an open cover of the subspace A. For each $\alpha \in I$, let U_α be an open subset of X such that $W_\alpha = U_\alpha \cap A$. Let $U = \bigcup\{U_\alpha: \alpha \in I\}$. Then U is open and $A \subset U$; so, by assumption, there exists a weakly σ-space $B \subset X$ such that $A \subset B \subset U$. Since $\{U_\alpha \cap B: \alpha \in I\}$ is an open cover of B, it has a weakly dissectable open precise refinement $\{V_\alpha: \alpha \in I\}$ with weak dissection $\mathscr{C} = \bigcup_{n \in N} \mathscr{C}_n$ in B. It is clear that $\{V_\alpha \cap A: \alpha \in I\}$ is an open cover of A that refines $\{W_\alpha: \alpha \in I\}$. If we let $\mathscr{D}_n = \{C \cap A: C \in \mathscr{C}_n\}$, for each $n \in N$, then it is easy to see that $\mathscr{D}_n$ is a closure-preserving collection of closed subsets of A, for each $n \in N$, and $\mathscr{D} = \bigcup_{n \in N} \mathscr{D}_n$ is an N-refinement (therefore, a weak dissection) for $\{V_\alpha \cap A: \alpha \in I\}$. So, A is a weakly σ-space.

A subset A of a space X is called by Singal & Jain (1971) a generalized F_σ-subset if, when $A \subset U$ with U open, there exists an F_σ-subset B of X such that $A \subset B \subset U$. Therefore from Lemmas 3.5 and 3.6 we get the following result.

Corollary 3.7. Every generalized F_σ-subset of a weakly σ-space is a weakly σ-space.

Lemma 3.8. If a space X has a locally finite cover $\{G_\alpha: \alpha \in \Gamma\}$ by open F_σ-subsets such that each G_α is a weakly σ-space, then X is a weakly σ-space.

Proof. For each $\alpha \in \Gamma$, let $G_\alpha = \bigcup_{n \in N} H(\alpha, n)$, where $H(\alpha, n)$ is a closed subset of X for each $n \in N$. Let $\mathcal{U} = \{U_i: i \in I\}$ be an open cover of X. For each $\alpha \in \Gamma$, the collection $\mathcal{U}_\alpha = \{U_i \cap G_\alpha: i \in I\}$ is an open cover of the weekly σ-space G_α. Let $\mathcal{V}_\alpha$ be a weakly dissectable open refinement of $\mathcal{U}_\alpha$ and assume that $\mathcal{C}_\alpha = \bigcup_{k \in N} \mathcal{C}(\alpha, k)$ is a weak dissection for $\mathcal{V}_\alpha$. For each $(k, n) \in N \times N$, we define $\mathcal{C}(\alpha, k, n) = \{C \cap H(\alpha, n): C \in \mathcal{C}(\alpha, k)\}$. Then, it is easy to see that $\mathcal{C}(\alpha, k, n)$ is a closure-preserving collection of closed subsets of X, for each $\alpha \in \Gamma$ and for each $(k, n) \in N \times N$. If, for each $(k, n) \in N \times N$, we let $\mathcal{D}(k, n) = \bigcup\{\mathcal{C}(\alpha, k, n): \alpha \in \Gamma\}$ then $\mathcal{D}(k, n)$ is still a closure-preserving collection of closed subsets of X, because $\{G_\alpha: \alpha \in \Gamma\}$ is locally finite. Therefore, the collection $\mathcal{D} = \bigcup\{\mathcal{D}(k, n): (k, n) \in N \times N\}$ is a weak dissection of the open cover $\mathcal{V} = \bigcup\{\mathcal{V}_\alpha: \alpha \in \Gamma\}$ of X and is a refinement of $\mathcal{U}$. So, the space X is a weakly σ-space.

Definition 3.9. A subset A of a space X is called by Pears (1975) normally situated in X if, for each open subset U of X with $A \subset U$, there exists an open subset V of X such that $A \subset V \subset U$ and $V = \bigcup\{V_\alpha: \alpha \in I\}$, where $\{V_\alpha: \alpha \in I\}$ is an open locally finite (in V) collection of F_σ-subsets of X.

Theorem 3.10. Every normally situated subset of a weakly σ-space is a weakly σ-space.

Proof. Let A be a normally situated subset of a weakly σ-space X. Let $A \subset U$, where U is an open subset of X. There exists an open set $V \subset X$ such that $A \subset V \subset U$ and $V = \bigcup\{V_\alpha: \alpha \in I\}$, where $\{V_\alpha: \alpha \in I\}$ is a locally finite (in V) collection of open F_σ-subsets of X. By Lemma 3.5, V_α is a weakly σ-space, for each $\alpha \in I$. So, by Lemma 3.8, V is a weakly σ-space. Finally, we apply Lemma 3.6 to conclude that A is a weakly σ-space.

4. OPEN PROBLEMS

Chaber (1984) called a space X collectionwise d-normal (Definition 1.6) if for any discrete collection $\mathcal{F}$ of closed subsets of X and an open expansion $\mathcal{U} = \{U(F): F \in \mathcal{F}\}$ of $\mathcal{F}$ there exist a sequence $\beta = (\mathcal{V}_n)_{n \in N}$ of open covers of X and an open expansion $\mathcal{W} = \{W(F): F \in \mathcal{F}\}$ of $\mathcal{F}$ such that $F \subset W(F) \subset \text{int}_\beta\, U(F)$, for each $F \in \mathcal{F}$. Recall that, for a subset U of X, $\text{int}_\beta U = \{x \in X: \text{st}(x, \mathcal{V}_n) \subset U, \text{ for some } n \in N\}$. It is shown in Proposition 1.8 of Chaber (1984) that every submetacompact collectionwise d-normal space is D-paracompact.

Since the Sorgenfrey plane $S \times S$ is a submetacompact perfect space (and hence collectionwise D-normal space) which is not D-paracompact as demonstrated in Example 1 of Brandenburg (1985), it follows that our concept of collectionwise D-normality differs from that of Chaber's mentioned above.

On the other hand, a well known result, which appears as Theorem 5.3.3 in Engelking (1977), states that every metacompact collectionwise normal space is paracompact. So, we ask the following question.

Question 4.1. Is every metacompact collectionwise D-normal space a D-paracompact space?

If $\mathcal{U}$ is an open cover of a space X, then a $\mathcal{U}$-mapping of X onto a space Y is a continuous map f for which there exists an open cover $\mathcal{V}$ of Y such that $f^{-1}(\mathcal{V})$

refines $\mathscr{U}$. Paracompact (D-paracompact) spaces are characterized by admitting, for each open cover $\mathscr{U}$, a $\mathscr{U}$-mapping onto a metrizable space (developable T_1-space). So we ask the following:

Question 4.2. Can weakly σ-spaces be characterized by admitting, for each open cover $\mathscr{U}$, a $\mathscr{U}$-mapping onto some regular σ-space?

Question 4.3. Is the property of being a weakly σ-space invariant under closed mappings? Or under perfect mappings?

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الفضاءات القريبة من النوع σ

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خلاصـــة

نقوم في هذا البحث بتقديم ودراسة الفضاءات التبولوجية التي يميزها أن كل غطاء مفتوح $\mathscr{U}$ لها يملك تنقيحا مفتوحا $\mathscr{V}$ ، وبحيث $\mathscr{V}$ له تنقيح مغلق $\mathscr{C} = \bigcup_{n \in N} \mathscr{C}_n$ ، وهنا $\mathscr{C}_n$ تحافظ على الانغلاق ، لكل $n \in N$. بينما كل عنصر في $\mathscr{V}$ هو اتحاد عناصر في $\mathscr{C}$. تتناول هذه الدراسات اعطاء عدة تمييزات لهذه الفضاءات كما أنها تدرس بعض الخواص المتعلقة بفضاءاتها الجزئية .