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Submanifolds of a differentiable manifold with an $F(2q + 3, 1)$-structure

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ABSTRACT

In this paper some general relations between a manifold M and submanifold V of M are obtained. Investigation of invariant and non-invariant submanifold of an $F(2q + 3, 1)$-structure manifold is considered. Finally the necessary and sufficient condition for a non-invariant submanifold of an $F(2q + 3, 1)$-structure manifold to be an $f(2q + 3, 1)$-structure manifold is found.

INTRODUCTION

A C^∞, m-dimensional manifold M, on which there exists a C^∞, tensor field $f \neq 0$, of type (1, 1), such that

$$\text{(a)} \quad f^3 + f = 0, \qquad \text{(b)} \quad \operatorname{rank} f = r, \quad 1 \leqslant r \leqslant m \tag{1.1}$$

is called an f-structure manifold. Let

$$\text{(a)} \quad l = -f^2, \qquad \text{(b)} \quad m = f^2 + I, \tag{1.2}$$

then l and m are complementary projection operators acting on M_p, at each $p \in M$. If r is constant everywhere on M, then l and m define two C^∞ distributions P and Q on M, such that dimp $= r$, and dim $Q = m - r$ (Yano 1963).

Submanifolds of codimension $m - n$ in a manifold with an f-structure were studied by Mishra (1975). He showed that an invariant submanifold of a Riemannian f-structure manifold is also a Riemannian f-structure manifold. He gave the necessary and sufficient condition for a non-invariant submanifold of a Riemannian f-structure manifold to be an f-structure manifold.

Let M^m be a C^∞ manifold, let $f \neq 0$ be a tensor field of type (1, 1), and of a constant rank r, such that

$$f^{2q+3} + f = 0, \quad q \text{ is a positive integer,} \tag{1.3}$$

then M^m is called an $f(2q + 3, 1)$-structure manifold of rank r (Psomopoulou & Andreou 1985).

The operators given by

$$\text{(a)} \quad l = -f^{2q+2}, \qquad \text{(b)} \quad m = f^{2q+2} + I \tag{1.4}$$

applied to M_p, for each $p \in M^m$ are complementary projection operators. Since rank $f = r$ is constant everywhere on M^m, l and m define two complementary C^∞ distributions L and N on M^m of dimension r and $m - r$, respectively.

In an $f(2q + 3, 1)$-structure manifold M^m, there exists a positive Riemannian metric G, with respect to which L and N are orthogonal, and G is almost Hermitian with respect to L, i.e.

$$G(f\lambda, f\mu) = G(\lambda, \mu), \quad \text{for all } \lambda, \mu \in L. \tag{1.5}$$

Let V^n be a submanifold of a Riemannian manifold M^m, with dimension n, and immersion

$$b\colon V^n \to M^m, \tag{1.6}$$

This immersion induces a linear transformation B (called the Jacobian map), such that a vector field X in V^n at $p \in V^n$, $\mapsto BX$ in M^m at $b(p) \in M^m$.

Let g be the induced metric tensor in V^n, then for arbitrary vector fields X and Y in V^n, we have

$$G(BX, BY) \circ b = g(X, Y). \tag{1.7}$$

Let C_x, $x = 1, 2, \ldots, m - n$, be the field of normals to V^n, then

$$G(BX, C_x) \circ b = 0 \tag{1.8}$$

$$G(C_x, C_y) = \delta xy, \text{ where } \delta \text{ is the Kronecker delta.} \tag{1.9}$$

Let E be the Riemannian connection in M^m, decompose $E_{BX} BY$, where X, Y are vector fields in V^n, into components tangential and normal to V^n, i.e.

$$E_{BX} BY = BD_X Y + H^x(X, Y) C_x \quad \text{(Gauss formula)} \tag{1.10}$$

where D_X is a Riemannian connection in V^n.

$$E_{BX} C_x = -Bh_x(X) + r_x^y C_y \quad \text{(Weingarten formula)} \tag{1.11}$$

such that

$$H^x(X, Y) = g(h_x(X), Y) \tag{1.12}$$

and

$$r_x^y + r_y^x = 0 \text{ (Hicks 1965).} \tag{1.13}$$

In what follows, X, Y, X, . . . , will be taken as arbitrary vector fields in V^n.

INVARIANT SUBMANIFOLDS

Theorem 1. Suppose that F is a C^∞ tensor field of type (1.1) on M^m. Write $F(BX)$ as the sum of tangential and normal parts

$$F(BX) = B(fX) + p^x(X) C_x \tag{2.1}$$

where f is a C^∞ tensor field of type (1.1) on V^n. Write $F(C_x)$ as the sum of tangential and normal parts

$$F(C_x) = -BP_x + a_x^y C_x \tag{2.2}$$

$$\text{(a)}\quad J = -F^t, \qquad \text{(b)}\quad K = F^t + I; \quad t \text{ is a positive integer.} \tag{2.3}$$

Then

$$\begin{aligned} F^t(BX) &= B(f^tX) + p^x(f^{t-1}X)C_x + p^x(f^{t-2}X)F(C_x) + \dots \\ &\quad + p^x(fX)F^{t-2}(C_x) + p^x(X)F^{t-1}(C_x) \\ &= B(f^tX) + BH_t(X) + R_t(X)C_y \end{aligned} \tag{2.4}$$

where BH_t and $R_t(X)\,C_y$ are the tangential part and the normal part to V_n respectively.

$$\begin{aligned} F^t(C_x) &= -B(f^{t-1}P_x) - p^x(f^{t-2}p_x)C_x - p^x(f^{t-3}p_x)F(C_x) - \dots \\ &\quad - p^x(fP_x)F^{t-3}(C_x) - p^x(P_x)F^{t-2}(C_x) \\ &\quad + a_x^y F^{t-1}(C_y) = BS_t(X) + T_t(X)C_y \end{aligned} \tag{2.5}$$

where $BS_t(X)$ and $T_t(X)C_y$ are the tangential part and normal part to V_n respectively.

$$\begin{aligned} G(J(BX), K(BY))\circ b &= -[g(f^tX, f^ty) + g(f^tX, y) \\ &\quad + g(f^tX, H_t(Y) + g(H_t(X), f^tY) \\ &\quad + g(H_t(X), H_t(Y)) + g(H_t(X), Y) \\ &\quad + R_t(X)R_t(Y)] \end{aligned} \tag{2.6}$$

$$G(F(BX), F(BY))\circ b = g(fX, gY) + p^x(X)p^y(Y)\,\delta xy \tag{2.7}$$

$$G(F(BX), C_x)\circ b = p^x(X) \tag{2.8}$$

$$G(F(C_x), BY)\circ b = -g(P_x, Y) \tag{2.9}$$

Proof.

$$F^2(BX) = B(f^2X) + p^x(fX)C_x + p^x(X)F(C_x)$$

Hence, it is true for $t = 2$. Suppose that it is true for $t = q$, then

$$F^q(BX) = B(f^qX) + p^x(f^{q-1}X)C_x + \cdots + p^x(X)F^{q-1}(C_x).$$

Multiply by F, we get

$$\begin{aligned} F^{q+1}(BX) &= F[B(f^qX] + p^x(f^{q-1}X)F(C_x) + \cdots + p^x(X)F^q(C_x) \\ &= B(f^{q+1}X) + p^x(f^qX)C_x + p^x(f^{q-1}X)F(C_x) \\ &\quad + \dots + p^x(X)F^{q+1}(C_x). \end{aligned}$$

Hence, it is true for $q + 1$. That $F^t(BX) = B(f^tX) + BH_t(X) + R_t(X)C_y$ is due to the fact that the $F(C_x)$, $F^2(C_x)$, $\dots$, $F^{t-1}(C_x)$, can be resolved tangentially and normally to V^n. This proves (2.4).

$$\begin{aligned} G(J(BX), K(BX))\circ b &= -G(F^t(BX), F^t(BY))\circ b - G(F^t(BX), BY)\circ b \\ &= -G(B(f^tX), B(f^tY))\circ b - G(B(f^tX), BH_t(Y))\circ b \\ &\quad - G(BH_t(X), B(f^tY))\circ b - G(BH_t(X), BH_t(Y))\circ b \\ &\quad - G(R_t(X)C_y, R_t(Y)C_y)\circ b - G(B(f^tX), BY)\circ b \\ &\quad - G(BH_t(X), BY)\circ b \end{aligned}$$

$$
\begin{aligned}
&= -g(f^tX, f^tY) - g(f^tX, H_t(Y)) - g(H_t(X), H_t(Y)) \\
&\quad - g(H_t(X), f^tY) - g(f^tX, Y) - g(H_t(X), Y) \\
&\quad - R_t(X)R_t(Y),
\end{aligned}
$$

which is (2.6).

$$
\begin{aligned}
G(F(BX), F(BY))\circ b &= G(B(fX) + p^x(X)C_x, B(fY) + p^y(Y)C_y)\circ b \\
&= g(fX, fY) + p^x(X)p^y(Y)\,\delta xy,
\end{aligned}
$$

which is (2.7).

$$G(F(BX), C_x)\circ b = G(B(fX) + p^x(X)C_x, C_x)\circ b = p^x(X),$$

which is (2.8).

$$G(F(C_x), BY)\circ b = G(-BP_x + a_x^y C_y, BY)\circ b = -g(P_x, Y),$$

which is (2.9).‖

Suppose that the tangent space at every point $x \in V^n$ is invariant under the action of F, i.e. $F(BH) = B(fX)$, then V^n is called an invariant submanifold of M^m.

Theorem 2. Let V^n be an invariant submanifold of M^m. Then

$$F^t(BX) = B(f^tX) \tag{2.10}$$

$$
\begin{aligned}
F^t(C_x) &= -B(f^{t-1}P_x) - a_x^y B(f^{t-2}P_y) - a_x^y a_y^z B(f^{t-3}P_z) - \dots \\
&\quad - a_x^y a_y^z \dots a_g^e B(fP_e) + a_x^y a_y^z \dots a_g^e a_e^k C_k
\end{aligned} \tag{2.11}
$$

$$G(J(BX), K(BY))\circ b = -g(f^tX, f^tY + Y) \tag{2.12}$$

$$G(F(BX), F(BY))\circ b = g(fX, fY) \tag{2.13}$$

$$G(F(BX), C_x)\circ b = 0 \tag{2.14}$$

Proof. V^n is an invariant submanifold of M^m means that $p^x = 0$. (2.10) follows from (2.4) by putting $p^x = 0$.

Putting $p^x = 0$ in (2.5), we have

$$
\begin{aligned}
F^t(C_x) &= -B(f^{t-1}P_x) + a_x^y F^{t-1}(C_y) \\
&= -B(f^{t-1}P_x) + a_x^y[-B(f^{t-2}P_y) + a_y^z F^{t-2}(C_z)] \\
&= -B(f^{t-1}P_x) - a_x^y B(f^{t-2}P_y) + a_x^y a_y^z F^{t-2}(C_z) \\
&= \dots = -B(f^{t-1}P_x) - a_x^y B(f^{t-2}P_y) - a_x^y a_y^z B(f^{t-3}P_z) \\
&\quad - \dots - a_x^y a_y^z \dots a_g^e B(fP_e) + a_x^y a_y^z \dots a_g^e a_e^k C_k,
\end{aligned}
$$

which is (2.11).

$p^x = 0 \Rightarrow H_t(X) = 0$ and $R_t(X) = 0$. This is in (2.6) gives (2.12). (2.13) and (2.14) follow from (2.7) and (2.8) by putting $p^x = 0$.‖

Theorem 3. Let M^m be an $F(2q + 3, 1)$-structure manifold. Then

$$f^{2q+3}X + fX + H_{2q+3}(X) = 0 \tag{2.15}$$

$$R_{2q+3}(X) + p^y(X) = 0 \tag{2.16}$$

$$g(f^{2q+2}X, f^{2q+2}Y + Y) + g(f^{2q+2}X + H_{2q+2}(X), H_{2q+2}(Y))$$
$$+ g(H_{2q+2}(X), f^{2q+2}Y + H_{2q+2}(Y)) + R_{2q+2}(X)R_{2q+2}(Y) = 0 \quad (2.17)$$
$$g(z, w) = g(fZ, fW) + p^x(Z)p^y(W)\,\delta xy;\ BZ, BW \in L \quad (2.18)$$

Proof. We have, since M^m is an $F(2q + 3, 1)$-structure manifold

$$F^{2q+3}(BX) + F(BX) = 0, \quad (2.19)$$
$$G(lBX, mBY) = 0, \quad \text{where } l = -F^{2q+2}, \quad m = F^{2q+2} + I \quad (2.20)$$
$$G(G\lambda, F\mu) = G(\lambda, \mu); \quad \lambda, \mu \in L. \quad (2.21)$$

From (2.4), (2.1) and (2.19) we have

$$B(f^{2q+3}X) + BH_{2q+3}(X) + R_{2q+3}(X)C_y + B(fX) + p^x(X)C_x = 0$$
$$B[f^{2q+3}X) + fX + H_{2q+3}(X)] + [R_{2q+3}(X) + p^y(X)]C_y = 0,$$

and this is (2.15) and (2.16).

From (2.20), we have

$$G(F^{2q+2}(BX), F^{2q+2}(BY))\circ b + G(F^{2q+2}(BX), BY)\circ b = 0$$

Then from (2.6) we have (2.17).

$$g(Z, W) = G(BZ, BW)\circ b = G(F(BZ), F(BW))\circ b,$$
$$= g(fZ, fW) + p^x(Z)p^y(W)\,\delta xy;\ BZ, BW \in L$$

which is (2.18).‖

Theorem 4. Let V^n be an invariant submanifold of an $F(2q + 3, 1)$-structure manifold M^m. Then

$$f^{2q+3}X + fX = 0 \quad (2.22)$$
$$g(f^{2q+2}X, f^{2q+2}Y + Y) = 0 \quad (2.23)$$
$$g(Z, W) = g(fZ, fW); \quad BZ, BW \in L \quad (2.24)$$

i.e. V^n itself is an $f(2q + 3, 1)$-structure manifold.

Proof.

$$p^x = 0 \Rightarrow H_{2q+2}(X) = H_{2q+3} = 0, \text{ and } R_{2q+3}(X) = R_{2q+2}(X) = 0.$$

(2.22), (2.23) and (2.24) are consequences of (2.15), (2.17) and (2.18) respectively.

| That V^n is an $f(2q + 3, 1)$-structure manifold follows from (2.22).‖

Theorem 5. Let V^n be an invariant submanifold of an $F(2q + 3, 1)$-structure manifold M^m, then we have

$$(E_{BX}F)(BY) = B(D_Xf)Y + H^x(X, Y)BP_x + [H^x(X, fY) - H^y(X, Y)a_y^x]C_x \quad (2.25)$$

$$(E_{BX}F)(C_x) = B[fh_x(X) + r_x^y(X)P_y - D_XP_x - a_x^y h_y(X)]$$
$$+ [X(a_x^z) + a_x^y r_y^z(X) - r_x^y(X)a_y^z - H^z(X, P_x)]C_z \quad (2.26)$$

Proof.

$$\begin{aligned}(E_{BX}F)(BY) &= E_{BX}F(BY) - F(E_{BX}BY)\\ &= E_{BX}(B(fX) + p^x(Y)C_x) - F[BD_XY + H^x(X, Y)C_x]\\ &= BD_X(fY) + H^x(X, fY)C_x + (D_Xp^x)(Y)C_x\\ &\quad + p^x(Y)[-Bh_x(X)\, r_x^y(X)C_y] - B[f(D_XY)]\\ &\quad - H^x(X, Y)[-BP_x + a_x^yC_y].\end{aligned}$$

Putting $p^x = 0$, we have (2.25).
Similarly, we have

$$\begin{aligned}(E_{BX}F)(C_x) &= B[fh_x(X) + r_x^y(X)P_y - D_XP_x - a_x^yh_y(X)]\\ &\quad + [p^x(h_x(X)) + X(a_x^z) + a_x^yr_y^z(X) - r_x^y(X)a_y^z - H^z(X, P_x)]C_z.\end{aligned}$$

Putting $p^x = 0$, we have (2.26).‖

$F(2q + 3, 1)$-STRUCTURE MANIFOLD WITH NON-INVARIANT SUBMANIFOLD

Theorem 6. Let M^m be an $F(2q + 3, 1)$-structure manifold. Let V^n be a non-invariant submanifold of M^m. Then

$$f^{2q+3}X + fX + H_{2q+3}(X) = 0 \tag{3.1}$$

$$R_{2q+3}(X) + p^y(X) = 0 \tag{3.2}$$

$$S_{2q+3}(X) - P_x = 0 \tag{3.3}$$

$$T_{2q+3}(X) + a_x^y = 0 \tag{3.4}$$

Proof. We have $F^{2q+3}(BX) + F(BX) = 0$, and $F^{2q+3}(C_x) + F(C_x) = 0$. From (2.1) and (2.4) by putting $t = 2q + 3$ we get (3.1) and (3.2). From (2.2) and (2.5) by putting $t = 2q + 3$ we get (3.3) and (3.4).‖

Theorem 7. The necessary and sufficient condition for a non-invariant submanifold V^n of an $F(2q + 3, 1)$-structure manifold M^m, to be also an $f(2q + 3, 1)$-structure manifold is

$$H_{2q+3}(X) = 0 \tag{3.5}$$

Proof.
(i) Suppose that $H_{2q+3}(X) = 0$, then from (3.1) we have $f^{2q+3} + f = 0$.
(ii) Suppose that $f^{2q+3} + f = 0$, then from (3.1) we have $H_{2q+3}(X) = 0$.‖

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ذوات طيات تحتانية لذي طيات تفاضلي
يتصف بتركيب $F(2q+3,1)$

عدنان العقيل
قسم الرياضيات بجامعة الكويت ،
ص . ب . ٥٩٦٩ ، الصفاة ١٣٠٦٠ ، الكويت

خلاصة

لقد أمكن الحصول ، في هذا البحث ، على بعض العلاقات العامة بين ذي طيات M وذي طيات تحتاني V لذي الطيات M . وتم اجراء استكشاف ذي طيات تحتاني صامد وآخر غير صامد يتصف كل منها بتركيب $F(2q+3,1)$. وأخيرا ، أمكن الحصول على شرط ضروري وكاف لذي طيات تحتاني غير صامد في ذي طيات يتصف بتركيب $F(2q+3,1)$ ليكون ذا طيات يتصف بتركيب $F(2q+3,1)$.