J. Univ. Kuwait (Sci.) **18,** 1991

Multipliers for $|C, \alpha, \gamma|$ summability of Fourier series

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ABSTRACT

In 1978, Pandey proved a result concerning $|C, 1|$ summability of a factored Fourier series and in 1985, Tripathi and Tripathi proved the result for the same series but concerning $|C, \alpha|$, $\alpha > 1$. These two results are special cases of our theorem we are giving here.

1. INTRODUCTION

Let Σu_n be an infinite series with sequence of partial sums $\{s_n\}$, and let $t_n = t_n^0 = nu_n$. By $\{s_n^\alpha\}$ and $\{t_n^\alpha\}$ we denote the nth Cesàro means of order $\alpha > -1$ of the sequences $\{s_n\}$ and $\{t_n\}$ respectively. Then

$$s_n^\alpha = (A_n^\alpha)^{-1} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} s_v,$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}.$$

The series Σu_n is said to be summable $|C, \alpha, \gamma|_k$, $k \geqslant 1$, $\alpha > -1$, and $\gamma \geqslant 0$, if

$$\sum n^{k\gamma+k-1} |s_n^\alpha - s_{n-1}^\alpha|^k < \infty \qquad \text{(Flett 1958)} \tag{1.1}$$

or equivalently

$$\sum n^{k\gamma-1} |t_n^\alpha|^k < \infty \tag{1.2}$$

For $\gamma = 0$, the summability $|C, \alpha, \gamma|_k$ reduces to summability $|C, \alpha|_k$ (Flett 1957) and for $\gamma = 0$, and $k = 1$, it reduces to summability $|C, \alpha|$.

Let the formal expansion of a function $f(x)$, periodic with period 2π and integrable in the sense of Lebesgue over $[-\pi, \pi]$, in a Fourier-trigonometric series be given by

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1.3}$$

We write

$$\phi(u) = f(x+t) + f(x-t) - 2f(x)$$

$$g_n = \left(\prod_{v=1}^{l-1} \log^v n\right) (\log^l n)^{1+\varepsilon},\ \log^l n_0 > 0,\ \varepsilon > 0,\ n \geqslant n_0$$

where

$$\log^l n = \log(\log^{l-1} n), \ldots, \log^2 n = \log\log n.$$

The following results are known:

Theorem A. (Pandey 1978). If

$$\varphi(t) = \int_t^\delta \frac{|\phi(u)|}{u}\, du = 0\,\{(\log^l(1/t))^\eta\} \quad \text{as } t \to +0 \tag{1.4}$$

$0 < \delta \leqslant \pi$, then the series

$$\sum_{n=n_0}^{\infty} A_n(x)/g_n \tag{1.5}$$

is summable $|C, 1|$ for $0 < \eta < \varepsilon$.

Theorem B. (Tripathi & Tripathi 1985). If (1.4) holds, then the series (1.5) is summable $|C, \alpha|$, $\alpha > 1$, for $0 < \eta < \varepsilon$.

We prove the following:

Theorem C. If (1.4) holds, then for $0 < \eta < \varepsilon$, $0 \leqslant \gamma < 1$,
(1) the series $\Sigma\, A_n(x)/n^\gamma g_n$ is summable $|C, \alpha, \gamma|$, $\alpha > 1$;
(2) the series $\Sigma\, A_n(x)/n^{1+\gamma-\alpha} g_n$ is summable $|C, \alpha, \gamma|$, $0 < \alpha \leqslant 1$.

2. NEEDED RESULTS

Lemma. If $\sigma > \delta > 0$, then

$$\sum_{n=v+1}^{\infty} \frac{(n-v)^{\delta-1}}{n^\sigma} = 0(v^{\delta-\sigma}).$$

Proof.

$$\begin{aligned}\sum_{n=v+1}^{\infty} \frac{(n-v)^{\delta-1}}{n^\sigma} &= 0(1) \int_{v+1}^{\infty} x^{-\sigma}(x-v)^{\delta-1}\, dx \\ &= 0(v^{\delta-\sigma}) \int_0^{v/(v+1)} u^{\sigma-\delta-1}(1-u)^{\delta-1}\, du \\ &= 0(v^{\delta-\sigma}),\end{aligned}$$

as

$$\int_0^{v/(v+1)} u^{\sigma-\delta-1}(1-u)^{\delta-1}\, du \leqslant \int_0^1 u^{\sigma-\delta-1}(1-u)^{\delta-1}\, du = \beta(\sigma-\delta, \delta).$$

Theorem 1. The series $\Sigma\, \mu_n a_n$ is summable $|C, \alpha, \gamma|_k$, $k \geqslant 1$, $\alpha > 0$, $0 \leqslant \gamma < 1/k$ if the following holds:

$$\sum n^{k(1+\gamma-\alpha)-1}\, |\mu_n|^k\, |s_n|^k < \infty \quad (\alpha \leqslant 1),$$

$$\sum n^{k(1+\gamma)-1}\, |\Delta\mu_n|^k\, |s_n|^k < \infty.$$

Proof. Let T_n^α be the nth Cesàro mean of order α of the sequence $\{n\mu_n a_n\}$. Then

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1}\, v\mu_v a_v,$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)}.$$

$$T_n^\alpha = \frac{1}{A_n^\alpha}\left[\sum_{v=1}^{n-1} \{-A_{n-v}^{\alpha-1}\mu_v s_v + (v+1)\Delta_v A_{n-v}^{\alpha-1}\mu_v s_v + (v+1)\, A_{n-v-1}^{\alpha-1}\Delta\mu_v s_v\} + n\mu_n s_n\right]$$

$$= T_{n,1}^\alpha + T_{n,2}^\alpha + T_{n,3}^\alpha + T_{n,4}^\alpha, \text{ say.}$$

In order to prove the theorem, it is sufficient, by Minkowski's inequality to show that

$$\sum_{n=1}^{\infty} n^{k\gamma-1}\, |T_{n,j}^\alpha|^k < \infty, \qquad j = 1, 2, 3, 4.$$

Let us first consider the case $0 < \alpha \leqslant 1$. Applying Hölder's inequality,

$$\sum_{n=2}^{m+1} n^{k\gamma-1}\, |T_{n,1}^\alpha|^k \leqslant \sum_{n=2}^{m+1} n^{k\gamma-1} \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_{n-v}^{\alpha-1}\, |\mu_v|^k\, |s_v|^k \left\{\frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_{n-v}^{\alpha-1}\right\}^{k-1}$$

$$\leqslant \sum_{v=1}^{m} |\mu_v|^k\, |s_v|^k \sum_{n=v+1}^{m+1} \frac{A_{n-v}^{\alpha-1}}{n^{1-k\gamma} A_n^\alpha}$$

$$= 0(1) \sum_{v=1}^{m} |\mu_v|^k\, |s_v|^k \sum_{n=v+1}^{m+1} \frac{(n-v)^{\alpha-1}}{n^{1-k\gamma+\alpha}}$$

$$= 0(1) \sum_{v=1}^{m} v^{k\gamma-1}\, |\mu_v|^k\, |s_v|^k$$

$$= 0(1).$$

$$\sum_{n=2}^{m+1} n^{k\gamma-1}\, |T_{n,2}^\alpha|^k \leqslant \sum_{n=2}^{m+1} n^{k\gamma-1} \frac{1}{(A_n^\alpha)^k} \sum_{v=1}^{n-1} (v+1)^k\, |\Delta_v A_{n-v}^{\alpha-1}|\, |\mu_v|^k\, |s_v|^k$$

$$\times \left\{\sum_{v=1}^{n-1} |\Delta_v A_{n-v}^{\alpha-1}|\right\}^{k-1}$$

(when $\alpha = 1$, $T_{n,2}^\alpha = 0$ as $\Delta_v A_{n-v}^{\alpha-1} = 0$)

$$= 0(1) \sum_{n=2}^{m+1} n^{k\gamma-1} \frac{1}{(A_n^\alpha)^k} \sum_{v=1}^{n-1} v^k\, |\Delta_v A_{n-v}^{\alpha-1}|\, |\mu_v|^k\, |s_v|^k$$

(as $\Sigma_{v=1}^{n-1} |\Delta_v A_{n-v}^{\alpha-1}| = 0\ \Sigma_{v=1}^{n-1} (n-v)^{\alpha-2} = 0(1)$, $0 < \alpha < 1$)

$$= 0(1) \sum_{v=1}^{m} v^k\, |\mu_v|^k\, |s_v|^k \sum_{n=v+1}^{m+1} \frac{|\Delta_v A_{n-v}^{\alpha-1}|}{n^{1-k\gamma}(A_n^\alpha)^k}$$

$$= 0(1) \sum_{v=1}^{m} v^k |\mu_v|^k |s_v|^k \sum_{n=v+1}^{m+1} \frac{(n-v)^{\alpha-2}}{n^{1-k\gamma+k\alpha}}$$

$$= 0(1) \sum_{v=1}^{m} v^{k(1+\gamma-\alpha)-1} |\mu_v|^k |s_v|^k$$

$$= 0(1).$$

$$\sum_{n=2}^{m+1} n^{k\gamma-1} |T_{n,3}^{\alpha}|^k \leqslant \sum_{n=2}^{m+1} n^{k\gamma-1} \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} (v+1)^k A_{n-v}^{\alpha-1} |\Delta\mu_v|^k |s_v|^k \left\{ \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} A_{n-v}^{\alpha-1} \right\}^{k-1}$$

$$= 0(1) \sum_{n=2}^{m+1} n^{k\gamma-1} \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} v^k A_{n-v}^{\alpha-1} |\Delta\mu_v|^k |s_v|^k$$

$$= 0(1) \sum_{v=1}^{m} v^k |\Delta\mu_v|^k |s_v|^k \sum_{n=v+1}^{m+1} \frac{(n-v)^{\alpha-1}}{n^{1-k\gamma+\alpha}}$$

$$= 0(1) \sum_{v=1}^{m} v^{k(1+\gamma)-1} |\Delta\mu_v|^k |s_v|^k$$

$$= 0(1).$$

$$\sum_{n=2}^{m+1} n^{k\gamma-1} |T_{n,4}^{\alpha}|^k \leqslant \sum_{n=2}^{m+1} n^{k\gamma-1} \frac{1}{(A_n^{\alpha})^k} n^k |\mu_n|^k |s_n|^k$$

$$= 0(1) \sum_{n=2}^{m+1} n^{k(1+\gamma-\alpha)-1} |\mu_n|^k |s_n|^k$$

$$= 0(1).$$

The case $\alpha > 1$ follows from the case $\alpha = 1$ and the known result (Flett 1958, Theorem 1) that summability $|C, 1, \gamma)$ implies summability $|C, \alpha, \gamma|$ when $\alpha > 1 \geqslant \gamma > 0$.

3. PROOF OF THEOREM C

Write

$$S_n(x) = \sum_{v=0}^{n} A_v(x),$$

then, we have

$$S_n(x) - f(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \frac{\sin nt}{t} \, dt + o(1)$$

$$= \frac{2}{\pi} \left\{ \int_0^{1/n} + \int_{1/n}^{\pi} \right\} \phi(t) \frac{\sin nt}{t} \, dt + o(1)$$

$$= \frac{2}{\pi} \{I_1 + I_2\} + o(1), \text{ say.}$$

Since

$$\int_0^t |\phi(u)| \, du = \int_0^t - u\phi'(u) \, du = -[u\phi(u)]_0^t + \int_0^t \phi(u) \, du$$

$$= 0\,\{t(\log^{l}(1/t))^{\eta}\} + 0\{t^{\beta}\,(\log^{l}(1/t))^{\eta}\int_0^t u^{-\beta}\,\mathrm{d}u\}, \quad 0 < \beta < 1$$

$$= 0\,\{t(\log^{l}(1/t))^{\eta}\},$$

therefore

$$|I_1| \leqslant n\int_0^{1/n} |\phi(t)|\,\mathrm{d}t = 0\,\{(\log^{l} n)^{\eta}\}.$$

$$|I_2| = 0(1)\int_{1/n}^{\pi} \frac{|\phi(t)|}{t}\,\mathrm{d}t = 0\,\{(\log^{l} n)^{\eta}\}.$$

Hence

$$|S_n(x) - f(x)| = 0\,\{(\log^{l} n)^{\eta}\},$$

$$|S_n(x)| \leqslant |S_n(x) - f(x)| + |f(x)|$$

$$= 0\,\{\log^{l} n)^{\eta}\}.$$

Since

$$\Delta(1/n^{\gamma} g_n) = 0(1/n^{1+\gamma} g_n),\ \Delta(1/n^{1+\gamma-\alpha} g_n) = 0(1/n^{2+\gamma-\alpha} g_n),$$

the result follows by an application of Theorem 1 with $k = 1$.

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(*Received 10 October 1988, revised 25 December 1989*)

مضروبات حول تجميع $(C, \propto, \gamma)$ التابعة لمسلسلة فورييه

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قسم العلوم التطبيقية بكلية الدراسات التكنولوجية ،
ص . ب . ٤٢٣٢٥ ، الكويت ٧٠٦٥٤

خلاصـــة

في سنة ١٩٧٨ برهن پاندى نظرية تتعلق بتجميع $(C, 1)$ لمعاملات مسلسلات فورييه ، وفي سنة ١٩٨٥ برهن تريپاثى وتريپاثى ما يكافىء النظرية ولكن فيها يتعلق بتجميع $(C, \propto)$ عندما تكون $1 > \propto$. الغاية من البحث الحالي هو برهان نظرية جديدة تتعلق بتجميع $(C, \propto, \gamma)$ بحيث تكون النظريتان السابقتان حالة خاصة من هذه النظرية .