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On weighted Bergman spaces over star-shaped circular domains

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ABSTRACT

Let σ denote a circular probability measure on a star-shaped circular domain D in C^n. For $0 < p < \infty$ let $A^p(\sigma)$ and $A'^p(\sigma)$ denote, respectively, the spaces of holomorphic functions f on D which satisfy

$$\int_D |f|^p \, d\sigma < \infty \quad \text{and} \quad \sup_{0 \leqslant r < 1} \int_D |f(rz)|^p \, d\sigma(z) < \infty.$$

In this paper we prove that $A^p(\sigma) = A'^p(\sigma)$.

1. DEFINITIONS

In this paper D shall always denote a star-shaped circular domain in $C^n = R^{2n}$, i.e. $\lambda z \in D$ whenever $z \in D$ and $\lambda \in C$, $|\lambda| \leqslant 1$; and σ shall denote a circular probability Borel measure on D, i.e.

$$\int_D f(e^{i\theta} z) \, d\sigma(z) = \int_D f \, d\sigma$$

for all bounded Borel functions f on D.

For $0 < p < \infty$, let $A^p(\sigma)$ and $A'^p(\sigma)$ denote the spaces of holomorphic functions f on D which satisfy

$$\|f\|_{p,\sigma} = \left(\int_D |f|^p \, d\sigma \right)^{1/p} < \infty$$

and

$$\|f\|'_{p,\sigma} = \sup_{0 \leqslant r < 1} \|f_r\|_{p,\sigma} < \infty,$$

respectively. Note that the dilations

$$f_r(z) = f(rz) \quad (z \in \bar{D},\ 0 \leqslant r < 1)$$

are holomorphic on $\bar{D}$.

When $d\sigma \ll dV \equiv$ Lebesque measure on D, $A^p(\sigma)$ is referred to as a weighted Bergman space (Stoll 1978, Remark (ii)).

2. EQUIVALENCE OF $A^p(\sigma)$ AND $A'^p(\sigma)$

Theorem 1. For $0 < p < \infty$, $A^p(\sigma) = A'^p(\sigma)$ and $\|f\|_{p,\sigma} = \|f\|'_{p,\sigma}$.

Proof. Fix $p > 0$. For each $z \in D$ and each function f holomorphic in D, let f_z be the slice function

$$f_z(re^{i\theta}) = f(re^{i\theta}z) = f_r(e^{i\theta}z) \quad (0 \leqslant r < 1)$$

and set

$$M_p(f_z; r) = \int_{-\pi}^{\pi} |f_z(re^{i\theta})|^p \, d\theta.$$

It is clear that $f_r \to f$ (as $r \to 1$) uniformly on compact subsets of D; hence,

$$\lim_{r\to 1} M_p(f_z; r) = \int_{-\pi}^{\pi} |f(e^{i\theta}z)|^p \, d\theta.$$

Theorem (17.6) in Rudin (1974) states that $M_p(f_z; r)$ is a monotonically increasing function of r in $[0, 1)$. Thus, by the monotone convergence theorem,

$$\lim_{r\to 1} \int_D M_p(f_z; r) \, d\sigma(z) = \int_D d\sigma(z) \int_{-\pi}^{\pi} |f(e^{i\theta}z)|^p \, d\theta.$$

Finally, Fubini's theorem together with the circularity of the measure σ (applied to the integrals on both sides of the above equation) leads to,

$$\lim_{r\to 1} \int_D |f_r|^p \, d\sigma = \int_D |f|^p \, d\sigma.$$

Consequently, $f \in A^p(\sigma)$ if and only if $f \in A'^p(\sigma)$ with equality of "norms."

Remarks. (i) Theorem (1) in Marzuq (1984) corresponds to the special case where D is bounded and $d\sigma = [V(D)]^{-1} dV$.

(ii) Assume D is bounded. If we let B_D denote the Bergman kernel for D, then (Horowitz 1974; Kolaski 1982; Luecking 1985) the "classical" weighted Bergman spaces are obtained by setting $d\sigma(z) = c[B_D(z, z)]^\alpha dV(z)$, where c is a normalizing constant and α is greater than some negative constant K_D. That the measure σ is circular follows from the fact (Cartan 1931; Hahn 1972; Marzuq 1984) that

$$B_D(z, w) = \sum_{k=0}^{\infty} \sum_{v=1}^{m_k} c_{kv} \phi_{kv}(z) \bar{\phi}_{kv}(z)$$

where the functions ϕ_{kv} are homogeneous polynomials of degree k and the c_{kv} are positive normalizing constants.

(iii) A non-trivial example when D is unbounded is given by setting $D = \mathbf{C}^n$ and

$$d\sigma(z) = c|z|^{2(\alpha-1)} e^{-\beta|z|^2} dV(z)$$

where $\alpha > 0$, $\beta > 0$ and $c = [\beta^{(\alpha+n-1)}\Gamma(n)]/[\pi^n \Gamma(\alpha + n - 1)]$. In this setting $A^p(\sigma)$ is referred to as the generalized (α, β)-Fischer space (Burbea 1983a,b).

Having established Theorem (1), an obvious modification of the proof of Theorem (2) in Marzuq (1984) yields a proof of Theorem (2) below. For the benefit of the reader a brief sketch is provided.

Theorem 2. If $f \in A^p(\sigma)$ $(0 < p < \infty)$, then $\|f - f_r\|_{p,\sigma} \to 0$ as $r \to 1$. Moreover, if the domain D is bounded, then $A^p(\sigma)$ is separable and the analytic polynomials comprise a dense subset.

Proof. It was shown in the proof of Theorem (1) that

$$\lim_{r \to 1} \|f_r\|_{p,\sigma} = \|f\|_{p,\sigma}.$$

It now follows from Rudin (1974, p. 76) that $\|f - f_r\|_{p,\sigma} \to 0$ as $r \to 1$.

To complete the proof, let $\varepsilon > 0$ be given, choose $r_0 \in (0, 1)$ such that $\|f - f_{r_0}\|_{p,\sigma} < \varepsilon/2$. Recently Marzuq (1984) has shown that, if D is bounded, then f_{r_0} has a Fourier series expansion (see Remark (ii))

$$f_{r_0}(z) = \sum a_{kv} \phi_{kv}(z) \quad (z \in D)$$

and the nth partial sums S_{n,r_0} converge uniformly on $\bar{D}$ to f_{r_0}. It now follows easily that the analytic polynomials are dense and that $A^p(\sigma)$ is separable.

Remark. (iv) Let D be the unit disc in C^1, and let ψ be a positive measurable function on [0, 1) satisfying $\int_0^1 \psi(r)\, dr < \infty$. If we set $d\sigma\,(re^{i\theta}) = c\psi(r)\, d\theta\, dr$, then $A^p(\sigma)$ is the space A_ψ^p considered by Stoll (1978). He proved Theorem (2) in this setting for $0 < p \leqslant \infty$.

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حول فضاءات بيرجمان الموزونة في مجالات دائرية نجمية الأشكال

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خلاصة

لنفرض أن σ ترمز الى قياس احتمال دائري على مجال D دائري ونجمي الشكل في C^n . ولنفرض لكل $0 < p < \infty$ أن $A^p(\sigma)$ و $A'^p(\sigma)$ ترمز ، على التوالي ، إلى فضائين لتطابقات هولومورفية f على D بحيث تحقق :

$$\int_D |f|^p \, d\sigma < \infty \quad \text{and} \quad \sup_{0 \leqslant r \leqslant 1} \int_D |f(rz)|^p \, d\sigma(z) < \infty.$$

في هذا البحث أمكن اثبات أن :

$$A^p(\sigma) = A'^p(\sigma).$$