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The normal form theorem of groups acting on trees with inversions

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ABSTRACT

In this paper we extend the normal form theorem of groups acting on trees without inversions to the case in which the action with inversions is allowed. This result generalises the known ones for free groups, tree products, and HNN groups.

1. INTRODUCTION

Higgins (1976) introduced the concept of the fundamental groupoid π of a graph of groups and formulated the normal form theorem of π. In this paper we study the normal form theorem for groups acting on trees in general.

We begin by giving preliminary definitions. By a graph X we understand a pair of disjoint sets $V(X)$, $E(X)$ with $V(X)$ non-empty, together with a mapping $E(X) \to V(X) \times V(X)$, $y \to (o(y)), t(y))$, and a mapping $E(X) \to E(X)$, $y \to \bar{y}$ satisfying $\bar{\bar{y}} = y$ and $o(\bar{y}) = t(y)$, for all $y \in E(X)$. The case $\bar{y} = y$ is possible for some $y \in E(X)$. The elements of $V(X)$ are called vertices, and those of $E(X)$ are called edges. For $y \in E(X)$ we call $o(y)$ and $t(y)$ the ends of y.

There is an obvious definition of subgraphs, connected graphs, trees, action of groups on graphs and orbits.

We say that a group G acts without inversions on a graph X if $g(y) \neq \bar{y}$ for all $g \in G$ and all $y \in E(X)$, and acts with inversions on X if $g(y) = \bar{y}$ for some $g \in G$ and some $y \in E(X)$ where, for any $g \in G$ and $x \in X$, $g(x)$ means the image of x under g. Throughout this paper, G will be a group acting on a tree X in general, i.e. the action with inversions is allowed. T will be a subtree of X containing exactly one vertex from each G-orbit of vertices. Y will be a subgraph of X containing T such that each edge of Y has an end in T, and Y contains exactly one edge y (say) from each G-orbit of edges except in the case of an orbit containing pairs $x, \bar{x}$, in which case Y contains exactly one such pair $y, \bar{y}$. Thus, if an orbit contains edges x and $\bar{x}$ then Y contains exactly one such edge. For the existence of T and Y see Khanfar & Mahmud (1990).

2. NOTATION

(1) For each x, $y \in X$ (vertices or edges) let $G(x, y) = \{g \in G | g(y) = x\}$, and $G(x, x) = G_x$, the stabilizer of x. Thus, $G(x, y) \neq \phi$ if and only if x and y are in the same G-orbit.

(2) For each $v \in V(X)$ let v' be the unique vertex of T such that $G(v, v') \neq \phi$. In particular $v' = v$, if $v \in V(T)$, and in general $v'' = v'$. Also if $G(u, v) \neq \phi$, then $u' = v'$ for $u, v \in V(X)$.

(3) For each $y \in E(Y)$ such that $o(y) \in V(T)$ let $[y]$ be an element of $G(t(y), t(y)')$ subject to the following conditions:
(i) $[y] = 1$ if $y \in E(T)$
(ii) $[y](y) = \bar{y}$ if $G(\bar{y}, y) \neq \phi$
If $o(y) \notin V(T)$, then we choose $[y]$ as

$$[y] = \begin{cases} [\bar{y}]^{-1} & \text{if } G(\bar{y}, y) = \phi \\ [\bar{y}] & \text{if } G(\bar{y}, y) \neq \phi \end{cases}$$

(4) For each $y \in E(Y)$ let

$$e_y = \begin{cases} -1 & \text{if } o(y) \in V(T) \\ 0 & \text{if } o(y) \notin V(T) \end{cases}$$

$\dot{y} = [y]^{e_y}(y)$, and $\ddot{y} = [y]^{e_y+1}(y)$. It is clear that $t(\dot{y}) = t(y)'$, $o(\ddot{y}) = o(y)'$ and $\bar{\dot{y}} = \ddot{\bar{y}}$.

(5) For each $v \in V(T)$, let $\langle G_v | \text{rel } G_v \rangle$ stand for any presentation of G_v via a certain map, and $\tilde{G}_v$ be the set of generating symbols of this presentation. If w is a word in $\tilde{G}_v$, i.e. $w \in \langle \tilde{G}_v \rangle$ where $\langle \tilde{G}_v \rangle$ is the free group of base $\tilde{G}_v$, then $\bar{w}$ is the value of w under such map.

(6) For each $y \in E(Y)$ let E_y be a set of generators of $G_{\dot{y}}$ such that E_y and $E_{\bar{y}}$ are in one-to-one correspondence under the map $g \leftrightarrow [y]g[y]^{-1}$; for $g \in G_{\dot{y}}$, let $\tilde{G}_y$ be a set of words in $\tilde{G}_{t(y)'}$ mapping onto E_y; let A_y be a right transversal for $G_{\dot{y}}$ in $G_{t(y)'}$ subject only to the condition that 1 is the representative for the coset $G_{\dot{y}}$.

(7) Define the map $\phi_y: \tilde{G}_y \to \tilde{G}_{\bar{y}}$ by taking the word which represents the element g in E_y to the word which represents the element $[y]g[y]^{-1}$ in $E_{\bar{y}}$, and extend it to $\langle \tilde{G}_y \rangle$. If $g \in G_{\dot{y}}$, then $\bar{\phi}_y(g)$ is defined to be the element $[y]g[y]^{-1}$ which is in $G_{\dot{\bar{y}}}$.

(8) Let $yG_y y^{-1} = G_{\bar{y}}$ stand for the set of relations $ygy^{-1} = \phi_y(g)$, $g \in \tilde{G}_y$.

(9) Let S_y be a word in $\tilde{G}_{o(y)'}$ of value $[y][\bar{y}]$, i.e. $\bar{S}_y = [y][\bar{y}]$.

3. THE NORMAL FORM THEOREM FOR *G*

Lemma (Mahmud 1990). G is generated by the set of elements of G_v, and the elements $[y]$ where $v \in V(T)$ and $y \in E(Y)$, and G has the presentation $\langle P(Y) \backslash R(Y) \rangle$ where $P(Y)$ is the set of generating symbols

(1) $\tilde{G}_v$, for $v \in V(T)$
(2) y, for $y \in E(Y)$, and

$R(Y)$ is the set of relations

(1) rel G_v, for all $v \in V(T)$

(2) $yG_y y^{-1} = G_{\bar{y}}$, for all $y \in E(Y)$
(3) $y = 1$, for all $y \in E(T)$
(4) $\bar{y} = S_y y^{-1}$, for all $y \in E(Y)$
(5) $y^2 = S_y$, for all $y \in E(Y)$ such that $G(\bar{y}, y) \neq \phi$

via the map $\tilde{G}_v \mapsto GvHy \mapsto [y]$.

Definition. By a *word* of G we mean an expression of the form $w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$, $n \geqslant 0$, where $y_i \in E(Y)$, $i = 1, 2, \ldots, n$ such that

(1) g_0 is an element of $G_{o(y_1)'}$
(2) g_i is an element of $G_{t(y_i)'}$, for $j = 1, 2, \ldots, n$
(3) $t(y_i)' = o(y_{i+1})'$, for $i = 1, \ldots, (n-1)$.

The inverse w^{-1} of w is defined to be the word

$$w^{-1} = g_n^{-1} \cdot \bar{y}_n \cdot \ldots \cdot g_1^{-1} \cdot \bar{y}_1 \cdot g_0^{-1}.$$

w is called a *reduced* word of G if w contains no subword of the forms

$$1 \cdot y_i \cdot g_i \cdot \bar{y}_1 \cdot 1, \quad \text{if } g_i \in G_{\bar{y}_i}, \quad \text{for } i = 1, \ldots, n$$

or

$$1 \cdot \bar{y}_i \cdot g_i \cdot y_i \cdot 1, \quad \text{if } g_i \in G_{\bar{y}_i}, \quad \text{for } i = 1, \ldots, n$$

or

$$1 \cdot y_i \cdot g_i \cdot y_i \cdot 1, \quad \text{if } g_i \in G_{\bar{y}_i} \text{ such that } G(\bar{y}_i, y_i) \neq \phi, \quad \text{for } i = 1, \ldots, n.$$

w is called a *normal* word of G if it is reduced and satisfies the following:

(1) g_0 is any element of $G_{o(y_1)'}$
(2) $g_i \in A_{y_i}$, for $i = 1, \ldots, n$
(3) If $y_{i+1} = \bar{y}_i$ for some i, $1 \leqslant i \leqslant n-1$, then $g_i \neq 1$
(4) If $y_{i+1} = y_i$ for some i, $1 \leqslant i \leqslant n-1$ and $G(\bar{y}_i, y_i) \neq 0$, then $g_i \neq 1$.

We define $o(w) = o(y_1)'$ and $t(w) = t(y_n)'$.

If $o(w) = t(w)$, then w is called a *closed* word of *type* $o(w)$.

The value $[w]$ of w is defined to be the element

$$[w] = g_o[y_1]g_1 \ldots [y_n]g_n$$

We define n to be the *length* of w, and denote $|w| = n$. If $w_1 = h_0 \cdot x_1 \cdot h_1 \ldots x_m \cdot h_m$ is a word of G, then we define the equality of w and w_1, i.e. $w = w_1$ if $n = m$, $y_i = x_i$ for $i = 1, \ldots, n$, and $g_i = h_i$ for $i = 0, 1, \ldots, n$.

If $t(w) = o(w_1)$ then $w \cdot w_1$ is defined to be the word $w \cdot w_1 = g_0 \cdot y_1 \cdot g_1 \ldots y_n \cdot g_n ho \cdot x_1 \cdot h_1 \ldots x_m \cdot h_m$.

Theorem. Every element of G is the value of a unique normal word of G of type v_0 for an arbitrary vertex v_0 of $V(T)$.

Proof. By the lemma above, the set $\{[y]/y \in E(Y)\} \cup \{g/g \in G_v, v \in V(T)\}$ generates G. So every element g of G can be expressed as a product $g_0[y_1]g_1 \ldots [y_n]g_n$, where $g_i \in G_{u_i}$ for some vertices $u_0, u_1, \ldots, u_n$ in T and edges $y_1, \ldots, y_n$ in Y. By taking the unique reduced paths in T between v_0 and v_i, between v_0 and $o(y_i)'$, and between $t(y_i)'$ and v_0 and the identities of $G_{t(y_i)'}$ we may choose this product so that $w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$ is a word of type v_0.

Thus every element of G is the value of a word of G of type v_0 (not necessarily unique).

Now we introduce a specific process * for reducing a word $w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$ of G (not necessarily closed or reduced) to a normal word w^* of G. We define * as follows: Let $w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$ be a word of G (not necessarily closed or reduced).

*The reduction process * on w* is obtained inductively as follows: If $n = 0$, i.e. $w = g_0$, we take $w^* = g_0$ which by definition is a normal word of G. Suppose that $n \geqslant 1$ and $g_1 \cdot y_2 \cdot g_2 \ldots y_n$ is a normal word of G, then

$$(g_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \ldots y_n \cdot g_n)^* = \begin{cases} g_0 \bar{\Phi}_{y_1}(g_1) g_2 \cdot y_3 \cdot g_3 \cdot \ldots \cdot y_n \cdot g_n, & \text{if } \bar{y}_1 = y_2 \text{ and } g_1 \in G_{\dot{y}_1} \\ g_0 \bar{\Phi}_{y_1}(g_1 \bar{S}_{y_1}) g_2 \cdot y_3 \cdot g_3 \cdot \ldots \cdot y_n \cdot g_n, & \text{if } y_1 = y_2 \text{ such that } G(\bar{y}_1, y_1) \neq \phi \text{ and } g_1 \in G_{\dot{y}_1} \\ g_0 \bar{\Phi}_{y_1}(h) \cdot y_1 \cdot a \cdot y_2 \cdot g_2 \cdot \ldots y_n \cdot g_n, & \text{otherwise, where } h \in G_{\dot{y}_1} \text{ and } a \in A_{y_1} \text{ are such that } g_1 = ha \end{cases}$$

The theorem will follow easily once we establish the following properties of *.

(a) w^* is a normal word of G, for all words w of G.
(b) If w is a normal word of G then $w^* = w$. In particular $(w^*)^* = w^*$.
(c) For any word w of G, $[w^*] = [w]$.
(d) If w is a closed word of G of type v for $v \in V(T)$, then w^* is a closed word of type v.
(e) If $w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$ is any normal word of G and $y \in E(Y)$ such that $t(y)' = o(y_1)'$, then

$$(1 \cdot y \cdot w)^* = \begin{cases} \bar{\Phi}_y(g_0) g_1 \cdot y_2 \cdot g_2 \cdot \ldots \cdot y_n \cdot g_n, \text{ if } \bar{y} = y_1 \text{ and } g_0 \in G_{\dot{y}} \\ \bar{\Phi}_y(g_0 \bar{S}_y) g_1 \cdot y_2 \cdot g_2 \cdot \ldots \cdot y_n \cdot g_n, \text{ if } y = y_1 \text{ such that } G(\bar{y}, y) \neq \phi \text{ and } g_0 \in G_{\dot{y}} \\ \bar{\Phi}_y(h) \cdot y \cdot a \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdot \ldots \cdot y_n \cdot g_n, \text{ if } \bar{y} \neq y_1, \text{ (or } y \neq y_1 \text{ if } G(\bar{y}, y) \neq \phi), \text{ or } g_0 \notin G_{\dot{y}}, \text{ where } h \in G_{\dot{y}} \text{ and } a \in A_y \text{ such that } g_0 = ha. \end{cases}$$

(f) If $w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$ is a normal word of G and $y \in E(Y)$ such that $o(y)' = o(y_1)'$, then $(1 \cdot y \cdot g \cdot \bar{y} \cdot w)^* = \bar{\Phi}_y(g\bar{S}_y) g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$ for all $g \in G_{\dot{y}}$.
(g) For any words w_1 and w_2 of G such that $t(w_1) = o(w_2)$, $(w_1 \cdot w_2)^* = (w_1 \cdot w_2^*)^*$.

Proof. Properties (a), (b), (c) and (d) follow immediately from the definition of *. Properties (e) and (f) require an argument which we now give. Property (g) follows from the definition of * and $w_1 \cdot w_2$.

Proof of Property (*e*)

Case (*i*) Suppose that $\bar{y} = y_1$, (or $y = y_1$ if $G(\bar{y}, y) \neq \phi$) and $g_0 \in G_{\dot{y}}$. Then

$$(1 \cdot y \cdot w)^* = \bar{\phi}_y(g_0 \bar{S}_y)g_1 \cdot y_2 \cdot g_2 \cdot \ldots \cdot y_n \cdot g_n, \text{ (property (b))}$$

Case (*ii*) Suppose that $\bar{y} \neq y_1$, (or $y \neq y_1$ if $G(\bar{y}, y) \neq \phi$), or $g_0 \in G_{\dot{y}}$. Then g_0 can be written as $g_0 = ha$ where $h \in G_{\dot{y}}$ and $a \in A_y$. Hence $(1 \cdot y \cdot w)^* = \bar{\phi}_y(h\bar{S}_y) \cdot y \cdot a \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$, (by the definition of *)

N.B. $\bar{S}_y = 1$ if $G(\bar{y}, y) = \phi$. The case when $w = g_0$, i.e. $n = 0$ is subsumed under case (ii) and this gives $(1 \cdot y \cdot w)^* = \bar{\phi}_y(h) \cdot y \cdot a$.

Proof of Property (*f*):

We have the following cases:

Case (*i*) Suppose that $y = y_1$ and $g_0 \in G_{\dot{y}}$. Then

$$\begin{aligned}(g \cdot \bar{y} \cdot w)^* &= g\bar{\phi}_y^{-1}(g_0)g_1 \cdot y_2 \cdot g_2 \cdot \ldots \cdot y_n \cdot g_n \text{ (by the definition of *)} \\ (1 \cdot y \cdot g \cdot \bar{y} \cdot w)^* &= (1 \cdot y \cdot g\bar{\phi}_y^{-1}(g_0)g_1 \cdot y_2 \cdot g_2 \cdot \ldots \cdot y_n \cdot g_n)^* \\ &= \bar{\phi}_y(g)g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n \text{ (by the definition of *),} \\ &\quad \text{(note that } \bar{S}_y = 1).\end{aligned}$$

Suppose that $y = \bar{y}_1$ if $G(\bar{y}, y) \neq \phi$, and $g_0 \in G_{\dot{y}}$. Then as above we have $(1 \cdot y \cdot g \cdot \bar{y} \cdot w)^* = \bar{\phi}_y(g\bar{S}_y)g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$, where $\bar{S}_y \neq 1$. (Note that $g_1 \in A_{y_1}$ and if $\bar{y} = y_2$, (or $y = y_2$ if $G(\bar{y}, y) \neq \phi$), then $g_1 \neq 1$).

Case (*ii*) Suppose that $y \neq y_1$, (or $y \neq \bar{y}_1$ if $G(\bar{y}, y) \neq \phi$) or $g_0 \notin G_{\dot{y}}$. Then g_0 can be written as $g_0 = ha$ where $h \in G_{\bar{y}}$ and $a \in A_{\bar{y}}$, ($a \neq 1$ if $y = \bar{y}_1$). Then

$$\begin{aligned}(g \cdot \bar{y} \cdot w)^* &= g\bar{\phi}_y^{-1}(h) \cdot \bar{y} \cdot a \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n \text{ (by the definition of *)} \\ (1 \cdot y \cdot g \cdot \bar{y} \cdot w)^* &= \bar{\phi}_y(g)ha \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n \text{ (by the definition of *)} \\ &= \bar{\phi}_y(g)g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n\end{aligned}$$

Now every element of G is the value of a normal word of G of type v_0. It remains to show the uniqueness. We use the well-known argument of Artin–van der Waerden, that is, we represent G as a permutation group on the set of normal words of type v_0.

Let $S(v_0)$ be the set of all normal words of G of type v_0 and $S^*(v_0)$ denote the group of all permutations of $S(v_0)$. We shall construct a homomorphism ψ: $G \to S^*(v_0)$. It is sufficient to define for each g in $\tilde{G}_v$, $v \in V(T)$ and each y in $E(Y)$, bijections ψ_g: $S(v_0) \to S(v_0)$ and ψ_y: $S(v_0) \to S(v_0)$ satisfying:

(1) $\psi_{r_1} = \psi_{r_2}$, for all relations $r_1 = r_2$ in rel G_v
(2) $\psi_y \psi_g \psi_y^{-1} = \psi_{\phi_y}(g)$, for all $g \in \tilde{G}_y$
(3) $\psi_y = 1$ if $y \in E(T)$
(4) $\psi_y \psi_{\bar{y}} = 1$ if $G(\bar{y}, y) = \phi$
(5) $\psi_y \psi_{\bar{y}} = \psi_{S_y}$ if $G(\bar{y}, y) \neq \phi$
(6) $\psi_y^2 = \psi_{S_y}$ if $G(\bar{y}, y) \neq \phi$.

For each $v \in V(T)$ let γ_v be the word $1 \cdot y_1 \cdot 1 \cdot y_2 \cdot 1 \cdot \ldots \cdot y_n \cdot 1$, where $y_1, y_2, \ldots, y_n$ is the unique reduced path in T from v_0 to v. We define γ_{v_0} to be the identity element of G_{v_0}.

Let $w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$ be a normal word of G of type v_0. For each $v \in V(T)$ and each $g \in \tilde{G}_v$ define the map ψ_g: $S(v_0) \to S(v_0)$ by ψ_g: $w \to (\gamma_v \cdot \bar{g} \cdot \gamma_v^{-1} \cdot w)^*$. Then for each relation $r_1 = r_2$ in rel G_v we have

(1) $$\begin{aligned}\psi_{r_1}\psi_{r_2^{-1}}(w) &= \psi_{r_1}((\gamma_v \cdot \bar{r}_2^{-1} \cdot \gamma_v^{-1} \cdot w^*))^* \\ &= (\gamma_v \cdot \bar{r}_1 \cdot \gamma_v^{-1} \cdot (\gamma_v \cdot \bar{r}_2^{-1} \cdot \gamma_v^{-1} \cdot w)^*)^* \\ &= (\gamma_v \cdot \bar{r}_1 \cdot (\gamma_v^{-1} \cdot \gamma_v \cdot \bar{r}_2^{-1} \cdot \gamma_v^{-1} \cdot w)^*)^* \text{ (Property (g))} \\ &= ((\gamma_v \cdot \bar{r}_1\bar{r}_2^{-1} \cdot \gamma_v^{-1} \cdot w)^*)^* \text{ (Properties (e) and (g))} \\ &= ((\gamma_v \cdot 1 \cdot \gamma_v^{-1} \cdot w)^*)^* \text{ (by the relation } r_1 = r_2 \text{ in rel } G_v) \\ &= ((w)^*)^* \text{ (Property (f))} \\ &= w^* \quad \text{(Property (b))} \\ &= w \quad \text{(Property (b))}.\end{aligned}$$

Thus $\psi_{r_1} = \psi_{r_2}$ for all relations $r_1 = r_2$ in rel G_v and all $v \in V(T)$. Moreover ψ_g is a permutation of $S(v_0)$.

For each $y \in E(Y)$ define the map ψ_y: $S(v_0) \to S(v_0)$ by ψ_y: $w \to (\gamma_{o(y)'} \cdot y \cdot \gamma_{t(y)'}^{-1} \cdot w)^*$.

Then we have

(2) $$\begin{aligned}\psi_y\psi_g(w) &= \psi_y((\gamma_{t(y)'} \cdot \bar{g} \cdot \gamma_{t(y)'}^{-1} \cdot w)^*)^*, \text{ where } g \in \tilde{G}_y \\ &= (\gamma_{o(y)'} \cdot y \cdot \gamma_{t(y)'}^{-1} \cdot (\gamma_{t(y)'} \cdot \bar{g} \cdot \gamma_{t(y)'}^{-1} \cdot w)^*)^* \\ &= (\gamma_{o(y)'} \cdot y \cdot 1 \cdot (\gamma_{t(y)'}^{-1} \cdot \gamma_{t(y)'} \cdot \bar{g} \cdot \gamma_{t(y)'}^{-1} \cdot w)^*)^* \text{ (Property (g))} \\ &= ((\gamma_{o(y)'} \cdot y \cdot 1) \cdot (\bar{g} \cdot \gamma_{t(y)'}^{-1} \cdot w)^*)^* \text{ (Property (f))} \\ &= (\gamma_{o(y)'} \cdot (1 \cdot y \cdot \bar{g} \cdot \gamma_{t(y)'}^{-1} \cdot w)^*)^* \text{ (Property (g))} \\ &= (\gamma_{o(y)'} \cdot (\bar{\phi}_y(\bar{g}) \cdot y \cdot \gamma_{t(y)'}^{-1} \cdot w)^*)^* \text{ (Property (f) and the definition of *)} \\ &= (\gamma_{o(y)'} \cdot \bar{\phi}_y(\bar{g}) \cdot (\gamma_{o(y)'}^{-1} \cdot \gamma_{o(y)'} \cdot y \cdot \gamma_{t(y)'}^{-1} \cdot w)^*)^* \text{ (Property (f))} \\ &= \psi_{\phi_y(g)}((\gamma_{o(y)'} \cdot y \cdot \gamma_{t(y)'}^{-1} \cdot w)^*) \\ &= \psi_{\phi_y(g)}\psi_y(w).\end{aligned}$$

Thus $\psi_y\psi_g\psi_y^{-1} = \psi_{\phi_y(g)}$, for all $g \in \tilde{G}_y$.

(3) If $y \in E(T)$, then

$$\begin{aligned}\psi_y(w) &= (\gamma_{o(y)} \cdot y \cdot \gamma_{t(y)}^{-1} \cdot w)^*, \text{ since } o(y)' = o(y) \text{ and } t(y)' = t(y) \\ &= w \text{ (Properties (f) and (g))}.\end{aligned}$$

Thus $\psi_y = 1$ for all $y \in E(T)$. Moreover, ψ_y is a permutation of $S(v_0)$.

(4) If $G(\bar{y}, y) = \phi$, then

$$\begin{aligned}\psi_y\psi_{\bar{y}}(w) &= \psi_y(\gamma_{t(y)'} \cdot \bar{y} \cdot \gamma_{o(y)'}^{-1} \cdot w)^*)^* \\ &= (\gamma_{o(y)'} \cdot y \cdot \gamma_{t(y)'}^{-1} \cdot (\gamma_{t(y)'} \cdot \bar{y} \cdot \gamma_{o(y)'}^{-1} w)^*)^* \\ &= (\gamma_{o(y)'} \cdot y \cdot 1 \cdot (\gamma_{t(y)'}^{-1} \cdot \gamma_{t(y)'} \cdot \bar{y} \cdot \gamma_{t(y)'}^{-1} \cdot w)^*)^* \text{ (Property (g))}\end{aligned}$$

$$= (\gamma_{o(y)'} \cdot (1 \cdot y \cdot 1 \cdot \bar{y} \cdot \gamma^{-1}_{o(y)'} \cdot w)^*)^* \text{ (Properties (f) and (g))}$$
$$= (w)^* \text{ (Property (f))}$$
$$= w \quad \text{(Property (b))}.$$

Thus, $\psi_y \psi_{\bar{y}} = 1$, for all $y \in E(Y)$ such that $G(\bar{y}, y) = \phi$.
Similarly $\psi_{\bar{y}} \psi_y = 1$, for all $y \in E(Y)$ such that $G(\bar{y}, y) = \phi$.
Then ψ_y is a permutation of $S(v_0)$.

(5) If $G(\bar{y}, y) \neq \phi$, then

$$\begin{aligned}\psi_y \psi_{\bar{y}}(w) &= \psi_y(\gamma_{t(y)'} \cdot \bar{y} \cdot \gamma^{-1}_{o(y)'} \cdot w)^*)^* \\ &= (\gamma_{o(y)'} \cdot y \cdot \gamma^{-1}_{t(y)'} \cdot \gamma_{t(y)'} \cdot \bar{y} \cdot \gamma^{-1}_{o(y)'} \cdot w)^* \text{ (Property (g))} \\ &= (\gamma_{o(y)'} \cdot y \cdot 1 \cdot \bar{y} \cdot \gamma^{-1}_{o(y)'} \cdot w)^* \text{ (Property (g))} \\ &= (\gamma_{o(y)'} \cdot \bar{S}_y \cdot \gamma^{-1}_{o(y)'} \cdot w)^* \text{ (Property (f))} \\ &= \psi_{S_y}(w), \ (S_y \in \langle \tilde{G}_{o(y)'} \rangle).\end{aligned}$$

Thus $\psi_y \psi_{\bar{y}} = \psi_{S_y}$, for all $y \in E(Y)$ such that $G(\bar{y}, y) \neq \phi$.
Then ψ_y is a permutation of $S(v_0)$.

$$\begin{aligned}(6)\ \psi_y^2(w) &= \psi_y((\gamma_{o(y)'} \cdot y \cdot \gamma^{-1}_{t(y)'} \cdot w)^*)^* \\ &= (\gamma_{o(y)'} \cdot y \cdot \gamma^{-1}_{t(y)'} \cdot (\gamma_{o(y)'} \cdot y \cdot \gamma^{-1}_{t(y)'} \cdot w)^*)^* \\ &= (\gamma_{o(y)'} \cdot y \cdot 1 \cdot (\gamma^{-1}_{t(y)'} \cdot \gamma_{o(y)'} \cdot y \cdot \gamma^{-1}_{t(y)'} \cdot w)^*)^* \text{ (Property (g))} \\ &= (\gamma_{o(y)'} \cdot y \cdot 1 \cdot y \cdot \gamma^{-1}_{t(y)'} \cdot w)^* \text{ (Property (f) and } o(y)' = t(y)') \\ &= (\gamma_{o(y)'} \cdot \bar{S}_y \cdot \gamma^{-1}_{o(y)'} \cdot w)^*, \text{ since } t(y)' = o(y)' \\ &= \psi_{S_y}(w).\end{aligned}$$

Thus $\psi_y^2 = \psi_{S_y}$ for all $y \in E(Y)$ such that $G(\bar{y}, y) \neq \phi$.

From above we deduce that ψ_g and ψ_y induce a unique homomorphism ψ: $G \to S^*(v_0)$. We need to show that ψ is injective, that is, $\psi_w \neq w_{w'}$, if $w \neq w'$ where $w, w' \in S(v_0)$. Let $w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$, $h_0 \in \tilde{G}_{o(y_1)'}$ and $h_i \in \tilde{G}_{t(y_i)'}$, such that $\bar{h}_0 = g_0$ and $\bar{h}_i = g_i$, for $i = 1, \ldots, n$. If we apply the map ψ_w to the word 1 in $S(v_0)$ we obtain

$$\begin{aligned}\psi_w(1) &= \psi_{h_0} \psi_{y_1} \psi_{h_1} \ldots \psi_{y_n} \psi_{h_n}(1) \\ &= (g_0 \cdot y_1 \cdot \ldots \cdot (\gamma^{-1}_{t(y)'} \cdot y_n \cdot \gamma^{-1}_{t(y)'} \cdot (\gamma_{t(y)'} \cdot g_n \cdot \gamma^{-1}_{t(y)'})^*)^*)^*.\end{aligned}$$

Since $w \in S(v_0)$, it is easy to see that this computation gives precisely the word w. Similarly $\psi_{w'}(1) = w' \neq w$, and hence $\psi_w \neq \psi_{w'}$ as required and therefore distinct elements in $S(v_0)$ represent distinct elements of G. This completes the proof of the main theorem.

Corollary 1. Let w be a non-trivial reduced word of G of type v_0 for an arbitrary v_0 in $V(T)$, then $[w]$ is not the identity element of G.

Proof. Let $w = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$
The hypotheses on w are
(i) if $n = 0$, then $g_0 \neq 1$, and

(ii) if $n > 0$ and $y_{i+1} = \bar{y}_i$, (or $y_{i+1} = y_i$ if $G(\bar{y}_i, y_i) \neq \phi$) for some i, then $g_i \notin G_{\dot{y}_i}$.

By induction on n we show that $[w^*] \neq 1$, where w^* is the normal work of G of type y_0 such that $[w^*] = [w]$.

The case $n = 0$ is trivial, so suppose that $n \geqslant 1$ and define $h_1, \ldots, h_n, a_1, \ldots, a_n$ inductively by the equations $g_n = h_n a_n$, where $h_n \in G_{\dot{y}_n}$ and $a_n \in A_{\dot{y}_n}$. $g_{n-i}\bar{\phi}_{y_{n-i+1}} \times (h_{n-i+1}) = h_{n-i}a_{n-i}$ for $i = 1, \ldots, (n-1)$, where $h_{n-i} \in G_{\dot{y}_{n-i}}$ and $a_{n-i} \in A_{y_{n-i}}$. Then define the word $w_0 = g \cdot y_1 \cdot a_1 \cdot y_2 \cdot a_2 \cdot \ldots \cdot y_n \cdot a_n$, where $a_i \in A_{y_i}$, $g \in G_{o(y)'} = G_{v_0}$, and $g = g_0\bar{\phi}_{y_1}(h_1)$. If $y_{i+1} = \bar{y}_i$, (or $y_{i+1} = y_i$ if $G(\bar{y}_i, y_i) \neq \phi$) for some $i = 1, \ldots, (n-1)$, then h_i and $\bar{\phi}_{y_i}(h_{i+1})$ are in $G_{\dot{y}_{i+1}}$, and $g_i\bar{\phi}_{y_{i+1}}(h_{i+1}) = h_i a_i$.

Since h_i and $\bar{\phi}_{y_{i+1}}(h_{i+1})$ lie in $G_{\dot{y}_i}$ but g_i does not, we have $a_i \notin G_{\dot{y}_i}$, so $a_i \neq 1$. This shows that $w_0 = g \cdot y_1 \cdot a_1 \cdot y_2 \cdot a_2 \cdot \ldots \cdot y_n \cdot a_n$ is a normal word of G of type v_0. So $w_0 = w^*$. By the theorem $[w] \neq 1$ in G. This completes the proof of Corollary 1.

Corollary 2. Let $w_1 = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_n \cdot g_n$, and $w_2 = h_0 \cdot X_I \cdot h_1 \cdot \ldots \cdot X_M \cdot h_m$ be two reduced words of G of the same type such that $[w_1] = [w_2]$. Then $n = m$, $y_i = X_I$, or ($\bar{X}_I$ if $G(\bar{X}_I, X_I) \neq \phi$), for $i = 1, \ldots, n$, and there is a unique sequence $\pi_1, \pi_2, \ldots, \pi_{2n}$, where $\pi_{2i} \in G_{\dot{y}_i}$ and $\pi_{2i-1} \in G_{\dot{y}_i}$ for $i = 1, \ldots, n$ such that $g_0 = h_0\pi_1, g_i = \pi_{2i}h_i\pi_{2i+1}$ for $i = 1, \ldots, n-1$, and $g_n = \pi_{2n}h_n$.

Proof. $w_1 \cdot w_2^{-1} = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_{n-1} \cdot g_{n-1} \cdot y_n \cdot g_n h_m^{-1} \cdot \bar{X}_M \cdot h_{m-1}^{-1} \cdot \ldots \cdot \bar{X}_1 \cdot h_0^{-1}$.

Since $[w_1] = [w_2]$, therefore $w_1 \cdot w_2^{-1}$ has value 1, the identity element of G. By $w_1 \cdot w_2^{-1}$ the above theorem is not reduced.

Let

$$\tilde{X}_I = \begin{cases} X_I & \text{if } G(\bar{X}_I, \bar{X}_I) = \phi \\ \bar{X}_I & \text{if } G(\bar{X}_I, X_I) \neq \phi \end{cases} \quad \text{for } i = 1, \ldots, m.$$

Since w_1 and w_2 are reduced, therefore $y_n = \tilde{X}_M$ and $g_n h_m^{-1} \in G_{\dot{y}_n}$. Therefore $g_n = \pi_{2n}h_m$, where $\pi_{2n} \in G_{\dot{y}_n}$, and π_{2n} is unique. From this we get the word $w^* = g_0 \cdot y_1 \cdot g_1 \cdot \ldots \cdot y_{n-1} \cdot g_{n-1}\bar{\phi}_{y_{n-1}} \cdot (\pi_{2n})h_m^{-1} \cdot \tilde{X}_{M-1} \cdot h_{m-1}^{-1} \cdot \ldots \cdot \tilde{X}_1 \cdot h_0^{-1}$ such that w^* has value 1.

Similar to above we have $y_{n-1} = \tilde{X}_{M-1}$ and $g_{n-1}\bar{\phi}_{y_{n-1}}(\pi_{2n})h_m^{-1} \in G_{\dot{y}_{n-1}}$. Therefore $g_{n-1} = \pi_{2n-1}h_m\bar{\phi}_{y_{n-1}}(\pi_{2n}^{-1}) = \pi_{2n-1}h_{m-1}\bar{\phi}_{y_{n-1}}(\pi_{2n}^{-1}) = \pi_{2n-2}h_{m-1}\pi_{2n-1}$ where $\pi_{2n-2} \in G_{\dot{y}_{n-1}}$ and $\pi_{2n-1} = \bar{\phi}_{y_{n-1}}(\pi_{2n}^{-1})$, which is in G_v, and is unique.

In the same way we get $y_{n-2} = \tilde{X}_{M-2}$, $g_{n-2} = \pi_{2n-4}h_{m-2}\pi_{2n-3}$, where $\pi_{2n-3} = \bar{\phi}_{y_{n-1}}(\pi_{2n-2}^{-1})$, and $\pi_{2n-4} \in G_{\dot{y}_{n-2}}$. By induction we take $y_i = \tilde{X}_i$, $\pi_{2i-1} = \bar{\phi}_{y_i}(\pi_{2i}^{-1})$, and $\pi_{2i} = g_i\pi_{2i+1}^{-1}h_i^{-1}$, for $i = 0, \ldots, n-1$, with convention $\pi_0 = 1$. So $y_1 = \tilde{X}_I$, $g_1 = \pi_2 h_1\pi_3$ and $g_0 = h_0\pi_1$.

From above we have $n = m$, and hence $|w_1| = |w_2|$. The sequence $\pi_1, \pi_2, \ldots, \pi_{2n}$ is the required sequence. This completes the proof of Corollary 2.

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مبرهنة الصيغة الطبيعية لزمر تؤثر في شجرات ذوات انعكاسات

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خلاصة

لقد تم في هذا البحث تعميم مبرهنة الصيغة الطبيعية لزمر تؤثر في شجرات بدون انعكاسات إلى حالة وجود الانعكاسات . وهذه النتيجة توسع الحالات المعروفة في الزمر الحرة والحواصل الشجرية والزمر *HNN* .