J. Univ. Kuwait (Sci.) **21**, 1994

Characterization of summability factors for Riesz methods

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ABSTRACT

In this paper we obtain necessary and sufficient conditions for $\Sigma\epsilon_n x_n$ to be summable $|\bar{N}, q_n|_s$ whenever Σx_n, $x_n \not\equiv 0$, is summable $|\bar{N}, p_n|_k$, where k and s satisfy $1 < k \leqslant s < \infty$, and so complete the result of Bor (1993) in the necessary and sufficient form.

1. INTRODUCTION

Let Σx_n be a given infinite series with $s = (s_n)$ as the sequence of its nth partial sums, and let $A = (a_{nv})$ be a normal matrix, i.e. lower-semi matrix of nonzero diagonal entries. We denote $A(s)$, the A-transform of the sequence s, as

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \qquad n = 0, 1, \ldots.$$

The series Σx_n is then said to be summable $|A|_k$, $k \geqslant 1$ if, according to Sarigöl (1991a, b)

$$\sum_{n=1}^{\infty} |a_{nn}|^{1-k} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

In the special case when A is a Riesz matrix (resp. $k = 1$), $|A|_k$ is reduced to $|\bar{N}, p_n|_k$ (resp. $|\bar{N}, p_n|$) summability (Bor 1985). By a Riesz matrix we mean one such that $a_{nv} = p_v/P_n$ for $0 \leqslant v \leqslant n$, and $a_{nv} = 0$ for $v > n$, where (p_n) is a sequence of positive real numbers for which $P_n = p_0 + p_1 + \cdots + p_n$, $P_n \to \infty$ as $n \to \infty$, $P_{-1} = p_{-1} = 0$.

Throughout the paper (p_n) and (q_n) will be positive sequences with $P_n \to \infty$ and $Q_n = q_0 + q_1 + \cdots + q_n \to \infty$ as $n \to \infty$, and $\Delta\epsilon_n = \epsilon_n - \epsilon_{n+1}$ for $n = 0, 1, \ldots$.

$|\bar{N}, p_n|_k$ and $|\bar{N}, q_n|_k$, $k \geqslant 1$ are, in general, known to be independent. More recently Bor (1993) has used a multiplier sequence (ϵ_n) to determine necessary conditions for $\Sigma\epsilon_n x_n$ to be $|\bar{N}, q_n|_k$ summable whenever Σx_n is $|\bar{N}, p_n|_k$ summable, and generalized some results due to the author (Sarigöl 1991a, b). He has proved

Theorem A. Suppose (p_n) and (q_n) are positive sequences such that $P_n \to \infty$ as $n \to \infty$ and $Q_n \to \infty$ as $n \to \infty$. If $\Sigma\epsilon_n x_n$ is summable $|\bar{N}, q_n|_k$, $k \geqslant 1$, whenever Σx_n,

$x_n \not\equiv 0$, is summable $|\bar{N}, p_n|_k$, then

(i) $$\epsilon_n = O\left\{\left(\frac{p_n Q_n}{q_n P_n}\right)^{1/k}\right\},$$

(ii) $$\Delta\epsilon_n = O\left\{\left(\frac{p_n}{P_n}\right)^{1/k}\right\}.$$

2. MAIN RESULT

It is the purpose of this paper to complete Theorem A and some known results in the necessary and sufficient form, using the functional analytic techniques. In what follows we prove the following theorem which gives us much more than we need.

Theorem 1. Let $1 < k \leqslant s < \infty$. Then, $\Sigma\epsilon_n x_n$ is summable $|\bar{N}, q_n|_s$ whenever Σx_n, $x_n \not\equiv 0$, is summable $|\bar{N}, p_n|_k$, if and only if

(i) $$\epsilon_n = O\left\{\left(\frac{Q_n}{q_n}\right)^{1/s}\left(\frac{p_n}{P_n}\right)^{1/k}\right\},$$

(ii) $$\sum_{v=1}^{m}\left|\frac{P_v}{p_v}\Delta(Q_{v-1}\epsilon_v) + Q_v\epsilon_{v+1}\right|^{k^*}\frac{p_v}{P_v} = O(Q_m^{k^*}),$$ (1)

where $1/k + 1/k^* = 1$.

3. SOME LEMMAS

We make use of the following lemmas in the proof of the theorem.

Lemma 1 (Sarigöl 1991a, b). Suppose that $k > 0$ and $p_n > 0$, $P_n \to \infty$ as $n \to \infty$. Then there exists two (strictly) positive constants M and N, depending only on k, such that for all $v \geqslant 1$

$$\frac{M}{P_{v-1}^k} \leqslant \sum_{n=v}^{\infty}\frac{p_n}{P_n P_{n-1}^k} \leqslant \frac{N}{P_{v-1}^k},$$

where M and N are independent of (p_n).

Lemma 2. Let $1 < k \leqslant s < \infty$. If the matrices B and C are defined by

$$b_{nv} = \begin{cases} \left(\frac{q_n}{Q_n}\right)^{1/s}\frac{1}{Q_n}\left\{\frac{P_v}{p_v}\Delta(Q_{v-1}\epsilon_v) + Q_v\epsilon_{v+1}\right\}\left(\frac{p_v}{P_v}\right)^{1/k^*}, & \text{if } 1 \leqslant v \leqslant n-1 \\ 0, & \text{if } v \geqslant n, \end{cases}$$

and

$$c_{nv} = \begin{cases} \left(\frac{q_n}{Q_n}\right)^{1/s}\left(\frac{P_n}{p_n}\right)^{1/k}\epsilon_n, & \text{if } v = n \\ 0, & \text{if } v \neq n, \end{cases}$$

then

(i) B maps l_k into l_s iff (1ii) holds,

(ii) C maps l_k into l_s iff (1i) holds. (2)

This lemma is easily proved by using Theorem 2(iii) of Bennett (1987) and Lemma 1.

Lemma 3. Let $1 \leqslant k \leqslant s < \infty$. If $\Sigma \epsilon_n x_n$ is summable $|\bar{N}, q_n|_s$ whenever Σx_n, $x_n \not\equiv 0$, is summable $|\bar{N}, p_n|_k$, then (1i) holds.

Proof. Let $(u_n(x))$ and $(t_n(x))$ be the sequences of Riesz means $(\bar{N}, p_n)$ and $(\bar{N}, q_n)$ of the series Σx_n and $\Sigma \epsilon_n x_n$, respectively. Then we have

$$\begin{aligned} u_n(x) &= \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{i=0}^{v} x_i, \\ U_n(x) &= u_n(x) - u_{n-1}(x) = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} x_v, \qquad P_{-1} = 0, \\ U_0(x) &= x_0, \end{aligned} \tag{3}$$

and similarly

$$\begin{aligned} t_n(x) &= \frac{1}{Q_n} \sum_{v=0}^{n} q_v \sum_{i=0}^{v} \epsilon_i x_i, \\ T_n(x) &= t_n(x) - t_{n-1}(x) = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} \epsilon_v x_v, \qquad T_0(x) = \epsilon_0 x_0. \end{aligned} \tag{4}$$

For $s, k \geqslant 1$, define $F = \{x = (x_i)\colon \Sigma x_i$ is summable $|\bar{N}, p_n|_k$ and $G = \{(\epsilon_i x_i)\colon \Sigma \epsilon_i x_i$ is summable $|\bar{N}, q_n|_s\}$. Then it is routine to verify that these are Banach spaces and also K-spaces (i.e., the coordinate functionals are continuous) if normed

$$\|x\|_F = \left\{ \sum_{n=0}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |U_n(x)|^k \right\}^{1/k}, \tag{5}$$

and

$$\|x\|_G = \left\{ \sum_{n=0}^{\infty} \left(\frac{Q_n}{q_n} \right)^{s-1} |T_n(x)|^s \right\}^{1/s}, \tag{6}$$

respectively. Since $F \subset G$ by the hypothesis, $\|x\|_F < \infty$ implies $\|x\|_G < \infty$. Consider for each n a functional $T_n\colon F \to \mathbb{C}$ defined by (4), which is a bounded linear. Now put

$$f_n(x) = \left\{ \sum_{i=0}^{n} \left(\frac{P_i}{p_i} \right)^{k-1} |T_i(x)|^k \right\}^{1/k} \quad \text{for each } n.$$

Then f_n is a seminorm on F. Moreover it is bounded on F because of the fact that each T_i is a bounded linear functional on F. Thus if we define

$$f(x) = \lim f_n(x) = \|x\|_G < \infty, \tag{7}$$

then f is bounded on F, by the resonance (Wilansky 1964), i.e. there exists M

such that

$$\|x\|_G \leqslant M\|x\|_F,$$

for all $x \in F$. By applying (3) and (4) to $x = e_v - e_{v+1}$ (e_v is the vth coordinate vector), respectively, we get

$$U_n(x) = \begin{cases} 0, & \text{if } n < v \\ p_v/P_v, & \text{if } n = v \\ -p_v p_n/P_n P_{n-1}, & \text{if } n > v, \end{cases}$$

and

$$T_n(x) = \begin{cases} 0, & \text{if } n < v \\ \epsilon_v q_v/Q_v, & \text{if } n = v \\ \Delta(Q_{v-1}\epsilon_v) q_n/Q_n Q_{n-1}, & \text{if } n > v, \end{cases}$$

and so (5) and (6) gives

$$\|x\|_F = \left\{ p_v/P_v + p_v^k \sum_{n=v+1}^{\infty} p_n/P_n P_{n-1}^k \right\}^{1/k},$$

$$\|x\|_G = \left\{ |\epsilon_v|^s q_v/Q_v + |\Delta(Q_{v-1}\epsilon_v)|^s \sum_{n=v+1}^{\infty} q_n/Q_n Q_{n-1}^s \right\}^{1/s}.$$

Thus it follows from (7) that (1i) holds, by Lemma 1.

We are now in a position to prove the theorem.

4. PROOF OF THE THEOREM

We use the symbols that are defined in Lemma 3. It follows from the inversion of (3) that

$$\begin{aligned} T_n(x) &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1}\epsilon_v \left(\frac{P_v}{p_v} U_v(x) - \frac{P_{v-2}}{p_{v-1}} U_{v-1}(x) \right) \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left\{ \frac{P_v}{p_v} \Delta(Q_{v-1}\epsilon_v) + Q_v \epsilon_{v+1} \right\} U_v(x) + \frac{q_n Q_n \epsilon_n}{Q_n p_n} U_n(x). \end{aligned}$$

Put $U_n^*(x) = (P_n/p_n)^{1-1/k} U_n(x)$ and $T_n^*(x) = (Q_n/q_n)^{1-1/s} T_n(x)$ for all $n \geqslant 1$. Then

$$\begin{aligned} T_n^*(x) &= (q_n/Q_n)^{1/s}(1/Q_n) \sum_{v=1}^{n-1} \left\{ \frac{P_v}{p_v} \Delta(Q_{v-1}\epsilon_v) + Q_v \epsilon_{v+1} \right\} (p_v/P_v)^{1/k^*} U_v^*(x) \\ &\quad + (q_n/Q_n)^{1/s}(P_n/p_n)^{1/k} \epsilon_n U_n^*(x) = \sum_{v=1}^{\infty} a_{nv} U_v^*(x), \end{aligned}$$

where

$$a_{nv} = \begin{cases} (q_n/Q_n)^{1/s}(1/Q_{n-1}) \left\{ \frac{P_v}{p_v} \Delta(Q_{v-1}\epsilon_v) + Q_v \epsilon_{v+1} \right\} (p_v/P_v)^{1/k^*}, & 1 \leqslant v \leqslant n-1 \\ (q_n/Q_n)^{1/s}(P_n/p_n)^{1/k} \epsilon_n, & v = n \\ 0, & v > n. \end{cases}$$

Now $\Sigma\epsilon_n x_n$ is summable $|\bar{N}, q_n|_s$ whenever Σx_n is summable $|\bar{N}, p_n|_k$ if and only if

$$\sum |T_n^*(x)|^s < \infty \text{ whenever } \sum |U_n^*(x)|^k < \infty,$$

or, equivalently $A = (a_{nv}) \in (l_k, l_s)$. Moreover

$$T_n^*(x) = \sum_{v=1}^{n-1} b_{nv} U_v^*(x) + c_{nn} U_n^*(x), \tag{8}$$

i.e. $A = B + C$. Now if $B, C \in (l_k, l_s)$, it immediately follows from (8) that $A \in (l_k, l_s)$. However, if $A \in (l_k, l_s)$ then (1i) holds by Lemma 3, and so $C \in (l_k, l_s)$ by Lemma 2. Therefore we get that

$$A \in (l_k, l_s) \text{ iff } B, C \in (l_k, l_s),$$

which completes the proof of the theorem together with Lemma 2.

5. COROLLARIES

Theorem 1 also includes some known results as a special case, which establish some relations between summability fields of absolute weighted means. We now list them.

Corollary 1. Let $1 < k < \infty$. Then every $|C, 1|_k$ summable series is also summable $|\bar{N}, p_n|_k$ if and only if

$$\begin{aligned} &\text{(i)} && np_n = O(P_n), \\ &\text{(ii)} && \sum_{v=1}^{m} |P_v - (v+1)p_v|^{k^*}(1/k) = O(P_m^{k^*}), \end{aligned} \tag{9}$$

where k^* is conjugate of k.

Proof. Use Theorem 1 with $p_n = \epsilon_n = 1$, $k = s$, and $p_n = q_n$.

We note that Bor (1985) established the sufficiency of the conclusion of Corollary 2 using (9i) and condition $P_n = O(np_n)$, which are stronger than condition (9), and the author also showed that (9i) is necessary for the conclusion but not condition $P_n = O(np_n)$.

Applying Theorem 1 with $p_n = q_n$ and $\epsilon_n = 1$ yields

Corollary 2. Let $1 < k < s < \infty$. Then every $|\bar{N}, p_n|_k$ summable series is also summable $|\bar{N}, p_n|_s$ if and only if

$$P_n = O(p_n). \tag{10}$$

We now provide example to illustrate a situation in which the sequence (p_n) satisfies condition (10). Consider the sequence (p_n) defined by $p_v = x^v$ for $v = 0, 1, \ldots$, where $x > 1$. A few calculations reveal that $P_v/p_v \sim x(x-1)^{-1}$ (i.e., P_v/p_v is asymptotic to $x(x-1)^{-1}$), which gives (10).

Corollary 3. Let $1 < k < \infty$. If

$$\begin{aligned} &\text{(i)} && p_n Q_n = O(q_n P_n) \\ &\text{(ii)} && q_n P_n = O(p_n Q_n) \end{aligned} \tag{11}$$

satisfy, then summability $|\bar{N}, p_n|_k$ is equivalent to summability $|\bar{N}, q_n|_k$.

Proof. Apply Theorem 1 with $\epsilon_n = 1$ and $k = s$. Then (1i) and (1ii) reduce to (11ii), and

$$\sum_{v=1}^{m} \left| Q_v - \frac{P_v q_v}{p_v} \right|^{k^*} (p_v/P_v) = O(Q_m^{k^*}), \tag{12}$$

respectively. Since Hypotheses (11i) and (11ii) guarantee that (12) holds, summability $|\bar{N}, p_n|_k$ implies summability $|\bar{N}, q_n|_k$. It follows from the sufficiency of Theorem 1 by interchanging the roles of p_n and q_n that $|\bar{N}, q_n|_k$ implies $|\bar{N}, p_n|_k$, completing the proof.

Using the necessity and sufficiency of Theorem 1 with $\epsilon_n = 1$ and $k = s$, a similar argument leads to

Corollary 4. Let $1 < k < \infty$. Summability $|\bar{N}, p_n|_k$ is equivalent to summability $|\bar{N}, q_n|_k$ if and only if (11i) and (11ii) hold.

Corollaries 3 and 4 have been given by Bor and Thorpe (1987) and Sarigöl (1992).

ACKNOWLEDGEMENT

This paper was supported by TBAG-C2 (TÜBİTAK).

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(*Received 28 November 1992, revised 13 October 1993*)

تمييز عوامل قابلية الجمع
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نحصل في هذا البحث على شروط لازمة وكافية من أجل أن تكون $\Sigma\varepsilon_n x_n$ قابلة للجمع وفق $|\bar{N},q_n|_s$ عندما تكون Σx_n, $x_n/0$ قابلة للجمع وفق $|\bar{N},P_n|k$ حيث تحقق s و k المتباينات $1<k\leq s<\infty$.

وبذا نكون قد أتممنا نتيجة بور *(1993)* بالشكل اللازم والكافي.