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A class of linear-quadratic discrete regulators

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ABSTRACT

The optimality of linear-quadratic (LQ) discrete regulators with output feedback is investigated. It is established that when a certain matrix function is positive-semidefinite, an output feedback controller can be designed to yield optimal or (suboptimal) performance measure with respect to the standard LQ state regulator. Methods for selecting the output gain matrix are provided. A numerical example is worked out to illustrate the developed results.

INTRODUCTION

It is well known that the linear-quadratic (LQ) state feedback regulators have many important properties (Kosut 1970; Mahmoud & Singh 1984; Geering & Basar 1986; Zheng 1986). The usefulness of these properties, however, depends on two major factors: the availability of an accurate mathematical model of the system and the accessibility of all state variables for forming feedback signals. Therefore, the problems of optimal constrained state feedback have been addressed in the literature (Kosut 1970; Levine & Athans 1970; Wenk & Knapp 1980; Allwright 1982; Mahmoud & Singh 1984; Geromel & Melo 1984; Geering & Basar 1986; Shaked 1986; Zheng 1986).

The present paper studies the optimality and stability properties of discrete LQ problems with output feedback. Motivated by the results of Kosut (1970), Wenk & Knapp (1980), Allwright (1982), and Geering & Basar (1986) on structural properties of linear continuous-time regulators to attain a superior solution to the standard constrained LQ problems, the present work develops a sufficient condition for the optimality (or suboptimality) of linear LQ discrete regulators with output feedback. Stability properties of the feedback system are investigated. Methods for determining the output feedback gain matrix are given. A numerical example is presented to illustrate the results obtained.

It should be emphasized that the objective of our work albeit similar to the effort of Kosut (1970) and Levine & Athans (1970), our approach is markedly different in that we relate the LQ output regulator with the LQ state counterpart, and we focus on discrete-time systems. More interesting problems in the context of this paper are those that involve optimization of constrained feedback gain by parametric techniques. In this regard, Geromel & Melo (1984) have developed the discrete version of the result by Levine & Athans (1970) and have emphasized the clear connection between output feedback and decentralized feedback controls. Along this line, new results have been obtained in Siljak (1991). The thrust of our work is to design output stabilizing feedback controllers having a prescribed degree of suboptimality. This design has proved its utility and usefulness in the synthesis of digital controllers of many industrial processes (Mahmoud 1991).

BASIC PROPERTIES

We consider a class of linear-quadratic (LQ) discrete regulators described by

$$x_{k+1} = Ax_k + Bu_k, \qquad x_0 \text{ given} \tag{1}$$

$$y_k = Cx_k \tag{2}$$

$$J = \sum_{j=0}^{\infty} \beta^{2j}[y_j^t Q_0 y_j + u_j^t R u_j];$$

$$Q_0 = Q_0^t \geqslant 0, \qquad R = R^t > 0 \tag{3}$$

where $x_k \in \Re^n$ is the state, $u_k \in \Re^m$ is the control, $y_k \in \Re^r$ is the measured output and $\beta \geqslant 1$. The superscript t stands for vector (matrix) transposition. It is assumed that the pairs (A, B) and $(A, Q^{1/2})$ are reachable and observable, respectively. First, we recall the following well-known result.

Theorem 1. (Mahmoud & Singh 1984). The discrete LQ regulator (1)–(3) with state feedback has the optimal control and optimal performance measure given, respectively, by

$$u_k^* = -K^* x_k^* \tag{4a}$$

$$K^* = \beta^2 R^{-1} B^t P_0 [I + \beta^2 B R^{-1} B^t P_0]^{-1} \tag{4b}$$

$$J^* = x_0^t P_0 x_0 \tag{5}$$

where $P_0 = P_0^t > 0$ is the solution of the algebraic Riccati equation (ARE)

$$P_0 = C^t Q_0 C + \beta^2 A^t P_0 [I + \beta^2 B R^{-1} B^t P_0]^{-1} A \tag{6}$$

An important result is that the closed-loop system (1) and (4) is asymptotically stable in the sense that the eigenvalues of $(A - B^* K^*)$ are allocated within the unit disc in the complex plane. These eigenvalues are the zeros of the following return difference matrix (Shaked 1986):

$$F(z) = I + K^*[zI - A]^{-1} B \tag{7}$$

where I in the foregoing equations stands for the unit matrix. It is further shown (Allwright 1982; Geering & Basar 1986) that the optimal linear state feedback controller (4) possesses a dominant property in the sense that J^* in (5) is minimal for every initial state $x_0 \in \Re^n$ and more importantly, for any other linear state feedback controller $u_k = -Lx_k$ has a performance measure $J = x_0^t S x_0$ satisfying

$$J = x_0^t S x_0 \geqslant J^* = x_0^t P_0 x_0 \tag{8a}$$

or equivalently

$$S \geqslant P_0 \tag{8b}$$

Despite the foregoing good properties of the discrete LQ regulator with state feedback, the practical implementation suffers a major difficulty in view of the limited measured variables as expressed by (2). Our interest is, therefore, to consider the discrete LQ regulator with output feedback and to attempt to recover some of the attractive features of state feedback.

MAIN RESULTS

This section develops the main results concerning the LQ output regulators. Theorem 2 below gives a sufficient condition for a suboptimal gain matrix G_s. First, let us define

$$W = BR^{-1}B^tP_0 \tag{9a}$$

$$Q_s = C^tQ_0C + \beta^2A^tP_0[I + \beta^2W]^{-1}A - [A - BG_sC]^t[I + \delta W]^{-1}P_0[A - BG_sC] \tag{9b}$$

where $\delta \geqslant 1$ to be specified below.

Theorem 2. For the LQ discrete regulator (1)–(4), assume that the pairs (A, B) and $(A, Q^{1/2})$ are reachable and observable, respectively. Choose an output feedback gain G_s. If the matrix $Q_s \geqslant 0$ and the pair $(A, Q_s^{1/2})$ are observable then G_s yields a suboptimal performance measure J_s with a suboptimality index $\delta = J_s/J^*$.

Proof. We start with the LQ state regulator

$$x_{k+1} = (A + A_s)x_k + Bu_k; \qquad x_0 \text{ given} \tag{10}$$

$$J_s = \sum_{j=0}^{\infty} [x_j^t(\delta Q_s)x_j + u_j^tRu_j]; \qquad R = R^t > 0 \tag{11}$$

where for some G_s and using (9b) matrix, A_s is defined by

$$A_s = \delta WA - [I + \delta W]BG_sC \tag{12}$$

Applying Theorem 1 to system (10)–(12), yields

$$u_k^s = -K^sx_k \tag{13a}$$

$$K^s = R^{-1}B^tP[I + \beta^2BR^{-1}B^tP]^{-1}(A + A_s) \tag{13b}$$

where $P = P^t > 0$ is the solution of the ARE:

$$P = \delta Q_s + (A + A_s)^tP[I + BR^{-1}B^tP]^{-1}(A + A_s) \tag{14}$$

The substitution of (9) and (12) into (14) results in

$$P = \delta C^tQ_0C + \delta\beta^2A^tP_0[I + \beta^2W]^{-1}A - \delta[A - BG_sC]^t[I + \delta W]^tP_0[A - BG_sC] + D^tP[I + BR^{-1}B^tP]^{-1}D \tag{15a}$$

$$D = [I + \delta W][A - BG_sC] \tag{15b}$$

On rearranging the terms, we obtain

$$P = \delta C^tQ_0C + \delta\beta^2A^tP_0[I + \beta^2W]^{-1}A + D^t\{P[I + BR^{-1}B^tP]^{-1}[I + \delta W] - \delta P_0\}D \tag{16}$$

The comparison of (16) with (6) using (9a) shows that $P = \delta P_0$, from which it directly follows that

$$J_s = x_0^t P x_0 = \delta J^* \tag{17}$$

as desired. From (10) and (13a), we get the closed-loop system

$$x_{k+1} = (A + A_s - BK^s)x_k \tag{18}$$

A little algebra on (18) using (6b), (12) and (13b) results in

$$x_{k+1} = (A - BG_s C)x_k \tag{19}$$

which implies that G_s is the suboptimal output feedback gain matrix as desired. Thus if $Q_s \geqslant 0$ leading to $P = P^t > 0$ in (14), then G_s produces a suboptimal performance measure $J_s = \delta J^* \geqslant J^*$ and the proof is completed.

It must be emphasized that Theorem 2 provides only a sufficient condition for an LQ output regulator to be identical with an optimal LQ state regulator.

Corollary. Under conditions of Theorem 2, the closed-loop system (19) is asymptotically stable in the large.

The validity of this result stems from the results of Theorem 1 and the equivalence of the suboptimal output feedback controller and an optimal state feedback regulator. It may appear that the condition of Theorem 2 is conservative for the stability of (19). A less conservative condition is established below.

Theorem 3. Consider that the solution of the LQ state regulator (1)–(3) is available and an output feedback gain matrix G is selected. Define the weighting matrix

$$\begin{aligned} Q_t = Q - C^tG^tB^tP_0 BGC - A^tP_0 A + A^tP_0 BGC \\ + \beta^2 A^tP_0[I + \beta^2 W]^{-1}A + C^tG^tB^tP_0 A \end{aligned} \tag{20}$$

If $Q_t \geqslant 0$ and the pair $(A, Q_t^{1/2})$ are observable, then G is a stabilizing output feedback gain matrix.

Proof. Given a gain matrix G, then system (1) and (2) under the action of the output feedback $y_k = -Gu_k$ becomes

$$x_{k+1} = (A - BGC)x_k; \qquad x_0 \text{ given} \tag{21}$$

For system (21), let $V_k = x_k^t P_t x_k$ be a Lyapunov function candidate. Then, it follows (Mahmoud & Singh 1984) that the corresponding Lyapunov equation takes the form

$$P_t = (A - BGC)^tP_t(A - BGC) + Q_t \tag{22}$$

The substitution of (20) into (22) with some manipulations yields

$$\begin{aligned} \{\beta^2 A^tP_0[I + \beta^2 W]^{-1}A + C^tQ_0 C - P_t\} + A^t(P_t - P_0)A \\ + C^tG^tB^t(P_t - P_0)BGC - C^tG^tB^t(P_t - P_0)A - A^t(P_t - P_0)BGC = 0 \end{aligned} \tag{23}$$

A comparison of (23) with (6) indicates that the solution of (22) is $P_t = P_0$. Since $P_0 = P_0^t > 0$, then $P_t = P_t^t > 0$ exists and G is a stabilizing output feedback gain matrix.

Now we proceed to examine methods for selecting the output matrix G_s (or G). Recall from (19) that the closed-loop eigenvalues are given by

$$\begin{aligned} \det [zI - A + BG_s C] &= \det [(zI - A)(I + [zI - A]^{-1} BG_s C)] \\ &= \det [zI - A] \det [I + [zI - A]^{-1} BG_s C] \\ &= \det [zI - A] \det [I + G_s C[zI - A]^{-1} B] \\ &= 0 \end{aligned} \tag{24}$$

It is well known (Brogan 1991) that when the system (A, B, C) is output-reachable, then G_s can be obtained by placing r closed-loop eigenvalues close to desired locations.

Method 1. Let the desired eigenvalues be $\{z_1, \ldots, z_r\}$ and define $\Phi(z) = C[zI - A]^{-1}B$. Then form the $(rxrm)$ matrix

$$[\Phi(z_1)\ \Phi(z_2) \ldots \Phi(z_r)]$$

and select r independent columns $\phi_{jk}(z_k)$, $k = 1, \ldots, r$ to constitute an (rxr) matrix M given by

$$M = [\phi_{j1}(z_1) \ldots \phi_{jr}(z_r)] \tag{25}$$

Denote by E the matrix whose columns are unit vectors selected from the identity matrix I corresponding to the vectors $\phi_{jk}(z_k)$. The gain matrix G_s is then given by Brogan (1991)

$$G_s = -EM^{-1} \tag{26}$$

Method 2. Select G_s to satisfy

$$G_s C = \delta R^{-1} B^t P_0 [I + \delta BR^{-1} B^t P_0] A \tag{27}$$

where P_0 is the solution of (6a). Note that to determine G_s, the output matrix C must be fully known and of full rank. A direct consequence of using (27) in (12) and manipulating with the aid of the matrix inversion lemma (Mahmoud & Singh 1984)

$$R^{-1} B^t P_0 [I + BR^{-1} B^t P_0]^{-1} = [R + B^t P_0 B]^{-1} B^t P_0 \tag{28}$$

it follows that $A_s = 0$, which leads to $K^s = K^*$. Thus, the choice (27) improves the suboptimality of the LQ output regulator.

Method 3. Since $G_s C$ is an (mxn) matrix and only r columns of it can be made identical to the state feedback K^*, it is readily evident that a total of $[n!/r!\,(n - r)!]$ schemes is generally available. One can then select any of these schemes that meets the sufficient condition (9). In the particular case that the output matrix C is freely selected for plants with numbers of inputs and outputs ($r = m$), we introduce some coordinate transformation in order to yield

$$C = [C_1\ C_2]; \qquad C_1 \in \Re^{mxm} \tag{29a}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}; \qquad B_1 \in \Re^{mxm} \tag{29b}$$

$$\det [C_1] \neq 0, \qquad \det [B_1] \neq 0 \tag{29c}$$

with C_2 containing free parameters. Now we have the following result:

Theorem 4. Given the solution of the standard LQ state regulator (1)–(6) along with the identification (29), then the optimal output feedback gain matrix G_0 and the corresponding output matrix block C_2 are given by

$$G_0 = R^{-1}B_1^t P_0[I + BR^{-1}B^tP_0]^{-1}AC_1^{-1} \tag{30}$$

$$C_2 = G_0^{-1}R^{-1}B_2^t P_0[I + BR^{-1}B^tP_0]^{-1}A \tag{31}$$

Proof. For the optimal output feedback, we set

$$\delta = 1, \qquad \beta = 1$$

and enforce

$$\begin{aligned} G_0 C &= G_0[C_1 C_2] \\ &= K^* \\ &= [K_1^* K_2^*] \end{aligned} \tag{32}$$

Using (4b), (29b) in (32) we obtain (30) and (31) directly. Note in this case that $G_s = G_0$, and more importantly, utilizing (6b) and (32) in (12) results in $A_s = 0$. Consequently Q_s reduces to

$$Q_s = Q \geqslant 0$$

Therefore we conclude that the gains (30) and (31) guarantee the optimality of the LQ regulator.

EXAMPLE

A fourth-order system of the type (1)–(3) has

$$A = \begin{bmatrix} 0{\cdot}7521 & 0{\cdot}0074 & 0{\cdot}0589 & 0{\cdot}0887 \\ 0{\cdot}2385 & 0{\cdot}7526 & 0{\cdot}0634 & 0{\cdot}1790 \\ 0{\cdot}1498 & 0{\cdot}0748 & 0{\cdot}5441 & 0{\cdot}2173 \\ 0{\cdot}0788 & 0{\cdot}0728 & -0{\cdot}0942 & 0{\cdot}8148 \end{bmatrix}; \qquad B = \begin{bmatrix} 0{\cdot}0950 & 0{\cdot}1774 \\ 0{\cdot}0259 & 0{\cdot}1163 \\ 0{\cdot}0954 & 0{\cdot}0956 \\ 0{\cdot}0892 & 0{\cdot}0070 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}; \qquad R = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$$

$$Q = \begin{bmatrix} 5 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 1 & 5 \end{bmatrix}; \qquad \beta = 1$$

The solution of (6) using PRO-MATLAB takes the form

$$P = \begin{bmatrix} 15{\cdot}3951 & 5{\cdot}6514 & 1{\cdot}1676 & 8{\cdot}8375 \\ 5{\cdot}6514 & 7{\cdot}5251 & 2{\cdot}1526 & 4{\cdot}7323 \\ 1{\cdot}1676 & 2{\cdot}1526 & 5{\cdot}8347 & 1{\cdot}5587 \\ 8{\cdot}8375 & 4{\cdot}7323 & 1{\cdot}5587 & 18{\cdot}0503 \end{bmatrix}$$

then from (4b), we obtain

$$K^* = \begin{bmatrix} 0{\cdot}7667 & 0{\cdot}3739 & 0{\cdot}1191 & 1{\cdot}1341 \\ 0{\cdot}2077 & 0{\cdot}1229 & 0{\cdot}0507 & -0{\cdot}0284 \end{bmatrix}$$

According to Method 1, by arbitrarily selecting

$$z_1 = 0{\cdot}6, \quad z_2 = 0{\cdot}7 \quad \text{and} \quad z_3 = 0{\cdot}8$$

then the solution of (26) yields

$$G_s = \begin{bmatrix} 3{\cdot}5761 & -0{\cdot}3354 & -1{\cdot}0794 \\ -1{\cdot}8621 & 0{\cdot}3878 & 0{\cdot}8057 \end{bmatrix} \tag{33}$$

With $\delta = 1{\cdot}1$, we proceed to solve (27) and get

$$G_s = \begin{bmatrix} 0{\cdot}7899 & 0{\cdot}3686 & 0{\cdot}0956 \\ 0{\cdot}1136 & 0{\cdot}0216 & -0{\cdot}0567 \end{bmatrix} \tag{34}$$

which satisfies condition (9) since simple calculations show that

$$\lambda(Q_s) = \{6{\cdot}7835,\ 1{\cdot}9689,\ 4{\cdot}5133,\ 3{\cdot}3348\}$$

According to Method 3, we select the first three columns of K^* to yield

$$G_s = \begin{bmatrix} 3{\cdot}5761 & -0{\cdot}3354 & -1{\cdot}0794 \\ -1{\cdot}8621 & 0{\cdot}3878 & 0{\cdot}8057 \end{bmatrix} \tag{35}$$

which also satisfies condition (9) since in this case we have

$$\lambda(Q_s) = \{9{\cdot}6331,\ 1{\cdot}9881,\ 4{\cdot}3425,\ 3{\cdot}3477\}$$

It remains to emphasize that the resulting G_s in (33), (34) and (35) are different but all of them satisfy the developed sufficient condition. The main reason for this variation is that the condition is in the form of inequality to ensure the positive definiteness of a matrix.

CONCLUSIONS

A sufficient condition has been established in order to guarantee the optimality (or suboptimality) of linear-quadratic (LQ) discrete output regulators. Methods for selecting the feedback gain matrix are presented.

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مجموعة من المنظمات الخطية المتقطعة ذات المعيار التربيعي

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خلاصة

تبحث هذه المقالة في مدى أمثلية المنظمات الخطية المتقطعة ذات المعيار التربيعي عند إستخدام التغذية العكسية المخرجية. ولقد تم إثبات أن تصميم محكومة التغذية العكسية المخرجية يعتمد على قيمة دالة مصفوفة. فإذا جاءت قيمة الدالة شبه موجبة فإن التصميم يعطي معيارا ذا أداء أمثل (أو شبه أمثل)، وذلك بالنسبة لمنظمة الحالة الخطية ذات المعيار التربيعي. ولقد تم عرض طرق إختيار مصفوفة الزيادة المصاحبة للمحكومة المصممة، ثم طبقت هذه الطرق على مثال عددي، وذلك لإيضاح النتائج التي تم التوصل إليها.