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Best approximation in some function spaces

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ABSTRACT

Let X be a Banach space. A closed subspace Y of X is called proximinal in X if for every $x \in X$ there exists at least one $y \in Y$ such that $\|x - y\| \leqslant \|x - z\|$ for all $z \in Y$. It is shown that if $L^p(I, Y)$ has a continuous proximity map in $L^p(I, X)$, then Y has a continuous proximity map in X. Some other related results are also presented.

INTRODUCTION

Let X be a Banach space and Y be a (closed) subspace of X. For $x \in X$ let

$$d(x, Y) = \inf\{\|x - y\| : y \in Y\}.$$

The space Y is called proximinal in X if for every $x \in X$ there exists $y \in Y$ such that $d(x, Y) = \|x - y\|$.

For $x \in X$, set

$$P(x, Y) = \{y \in Y : \|x - y\| = d(x, Y)\}.$$

If for every $x \in X$, $P(x, X)$ contains exactly one element, then Y is called Chebyshev.

Let Y be a proximinal subspace of X and consider the set-valued map $\hat{P}: X \to 2^Y$ where

$$\hat{P}(x) = P(x, Y) = \{y \in Y : \|x - y\| = d(x, Y)\}.$$

A map $T: X \to Y$ such that $T(x) \in \hat{P}(x)$ is called a proximity map of Y in X. In general T is not continuous. We refer the reader to Singer (1970) for the basic theory of proximinality and proximity maps.

The objective of this paper is to prove some results on proximity maps and use them to prove the proximinality of some subspaces of function spaces. We shall be interested in vector valued L^p-spaces, continuous functions on some compact sets, and tensor products of Banach spaces.

PROXIMINALITY IN FUNCTION SPACES

Let (I, μ) be a probability measure space and X be a Banach space. For $1 \leqslant p < \infty$ let $L^p(I, X)$ denote the space of Bochner p-integrable functions with values in X. For

$f \in L^p(I, X)$ set

$$\|f\|_p = \left\{\int_I \|f(t)\|^p \, d\mu(t)\right\}^{1/p}$$

It is known that $(L^p(I, X), \|\cdot\|_p)$ is a Banach space, see Diestel & Uhl (1977). If Y is a closed subspace of X, then $L^p(I, Y)$ is a closed subspace of $L^p(I, X)$. Proximinality of $L^p(I, Y)$ in $L^p(I, X)$ was discussed by Khalil (1983), Light & Cheney (1985), Deeb & Khalil (1989) and Khalil & Deeb (1989).

Theorem 1.1. Let Y be a closed subspace of X and $P : L^p(I, X) \to L^p(I, Y)$ be a continuous proximity map. Then there exists a continuous proximity map $Q : X \to Y$.

Proof. For $x \in X$ define $f_x : I \to X$, where $f_x \in L^p(I, X)$, by $f_x(t) = x$ (a constant map) for all $t \in I$. Since $\mu(I) = 1$, it is clear that $f_x \in L^p(I, X)$. Since $L^p(I, Y)$ has a continuous proximity map P, it follows that $L^p(I, Y)$ is proximinal in $L^p(I, X)$.

Now define $Q : X \to Y$ by

$$Qx = \int_I (Pf_x)(t) d\mu(t), \qquad x \in X.$$

We claim that Q is a proximity map. To see this, let $x \in X$, then

$$\begin{aligned}
\|x - Qx\| &= \left\| x - \int_I (Pf_x)(t) d\mu(t) \right\| = \left\| \int_I [x - (Pf_x)(t)] d\mu(t) \right\| \\
&\leqslant \int_I \|x - (Pf_x)(t)\| \, d\mu(t) \\
&\leqslant \left\{\int_I \|x - (Pf_x)(t)\|^p \, d\mu(t)\right\}^{1/p} \\
&= \left\{\int_I \|f_x(t) - (Pf_x)(t)\|^p \, d\mu(t)\right\}^{1/p} = \|f_x - Pf_x\|_p \qquad (*) \\
&\leqslant \|f_x - f_z\|_p = \|x - z\|, \qquad \text{for all } z \in Y.
\end{aligned}$$

Hence, $Qx \in P(x, Y) = \hat{P}(x)$. Thus Q is a proximity map.

Now for continuity of Q, let $y_n \to y$ in Y. Since P is continuous, we have $Pf_{y_n} \to Pf_y$ in $L^p(I, Y)$. Equation (*) implies that

$$\|y - Qy_n\| \leqslant \|f_y - Pf_{y_n}\|.$$

Hence, $\lim_n \|y - Qy_n\| = 0$, and thus establishing the continuity of Q.

Let K be a compact Hausdorff space, X be a Banach space, and $C(K, X)$ be the space of continuous functions on K with values in X. For $f \in C(K, X)$ we write

$$\|f\|_\infty = \sup_{t \in K} \|f(t)\|.$$

Theorem 1.2. Let K be a compact Hausdorff space and Y be a closed subspace of the Banach space X. Then

(i) If Y has a continuous proximity map, then $C(K, Y)$ is proximinal in $C(K, X)$.
(ii) If $C(K, Y)$ has a continuous proximity map, then Y has a continuous proximity map in X.

Proof. For the proof of (i) we refer to Theorem 2.1 of Light & Cheney (1985). To prove (ii), let P be a continuous proximity map of $C(K, Y)$ in $C(K, X)$. For $x \in X$, the function $f_x(t) = x$ for all $t \in K$ is continuous. Define $Q : X \to Y$ by

$$Qx = (Pf_x)(t_0) \qquad \text{for some fixed } t_0 \in K.$$

Then

$$\begin{aligned}\|x - Qx\| &= \|f_x(t_0) - (Pf_x)(t_0)\| \\ &\leqslant \|f_x - Pf_x\| \\ &\leqslant \|f_x - f_z\| = \|x - z\| \qquad \text{for all } z \in Y.\end{aligned}$$

Hence Q is a proximity map, and Y is proximinal in X.

For continuity, let $x_n \to x$ in X, then $\|x_n - x\| = \|f_{x_n} - f_x\|_\infty$. By the continuity of P, we have $\lim_n \|Pf_{x_n} - Pf_x\|_\infty = 0$. Hence,

$$\begin{aligned}\|Qx_n - Qx\| &= \|Pf_{x_n}(t_0) - Pf_x(t_0)\| \\ &\leqslant \|Pf_{x_n} - Pf_x\|_\infty .\end{aligned}$$

Thus, $\lim_n \|Qx_n - Qx\| = 0$ and therefore Q is continuous.

PROXIMINALITY IN TENSOR PRODUCT SPACES

Let X and Y be a Banach space, and let $X \hat{\otimes} Y$ denote the completion of the projective tensor product of X with Y, while $X \check{\otimes} Y$ stands for the completion of the injective tensor product. We refer to Diestel & Uhl (1977) for the basic theory of tensor products of Banach spaces.

Theorem 2.1. Let X and Y be Banach spaces, and let G and H be subspaces of X and Y respectively. If $X \check{\otimes} H + G \check{\otimes} Y$ is proximinal in $X \check{\otimes} Y$, then G and H are proximinal in X and Y respectively.

Proof. We will prove that G is proximinal in X. The case for H will be similar, and so will be omitted.

Let $x \in X$ and e be a fixed element in Y such that $d(e, H) = 1$. Then $x \otimes e \in X \check{\otimes} Y$. By assumption there exists $w \in X \check{\otimes} H + G \check{\otimes} Y$ such that

$$\|x \otimes e - w\| \leqslant \|x \otimes e - z\| \qquad \text{for all } z \in X \check{\otimes} H + G \check{\otimes} Y.$$

In particular,

$$\|x \otimes e - w\| \leqslant \|x \otimes e - g \otimes e\| \qquad \text{for all } g \in G.$$

Let y^* in Y^* (the dual of Y) be chosen so that

$$y^*(e) = \|y^*\| = 1 \qquad \text{and} \qquad y^*(y) = 0 \qquad \text{for all } y \in Y.$$

If I denotes the identity of X, then

$$\begin{aligned}\|x-(I\otimes y^*)(w)\| &= \|(I\otimes y^*)(x\otimes e-w)\|\\ &\leqslant \|I\otimes y^*\|\,\|x\otimes e-w\|\\ &\leqslant \|x\otimes e-g\otimes e\|\\ &\leqslant \|x-g\| \qquad\qquad \text{for all } g\in G,\end{aligned}$$

since $\|I\otimes y^*\| = \|I\|\cdot\|y^*\|$.

But $w=u+v$, for some $u\in X\check{\otimes} H$ and $v\in G\check{\otimes} Y$. Hence $u=\lim_n u_n$, where u_n is a finite rank element of $G\check{\otimes} Y$, and the convergence is in the injective norm. Hence $(I\otimes y^*)(u_n)=0$ for all n and consequently $(I\otimes y^*)(u)=0$. Thus

$$(I\otimes y^*)(w)=(I\otimes y^*)(c)\in G,$$

and therefore, G is proximinal.

Corollary 2.2. If $X\check{\otimes} H+G\check{\otimes} Y$ has a continuous proximity map, then G and H have continuous proximity maps.

Proof. Let e and y^* be as in the proof of Theorem 2.1, and P be the continuous proximity map of $X\check{\otimes} H+G\check{\otimes} Y$. Define $Q: X\to G$ by $Q(x)=(I\otimes y^*)\otimes P(x\otimes e)$. Then as in the proof of Theorem 2.1, we can show that

$$\|x-Q(x)\|\leqslant\|x-g\| \qquad \text{for all } g\in G.$$

The case for H is similar.

Corollary 2.3. Let K_1, K_2 be compact spaces and G, H be subspaces of $C(K_1)$ and $C(K_2)$ respectively. If $C(K_1)\check{\otimes} H+G\check{\otimes} C(K_2)$ has a continuous proximity map in $C(K_1\times K_2)$, then $C(K_1)\check{\otimes} H$ and $G\check{\otimes} C(K_2)$ have continuous proximity maps.

Proof. The proof follows from Corollary 2.2 and Theorem 2.1 of Light & Cheney (1985).

Remark 2.4. Theorem 2.1 is true for the projective tensor product and the proof is the same as for the injective case.

Corollary 2.5. Let (I_1, μ_1) and (I_2, μ_2) be finite measure spaces, and G, H be subspaces of $L^1(I_1)$ and $L^1(I_2)$ respectively. If $L^1(I_1)\otimes H+G\hat{\otimes} L^1(I_2)$ has a continuous proximity map in $L^1(I_1\times I_2)$, then $L^1(I_1)\hat{\otimes} H$ and $G\hat{\otimes} L^1(I_2)$ are proximinal in $L^1(I_1\times I_2)$.

Proof. Theorem 2. 1 and Remark 2.4 taken together imply that G (respectively H) has a continuous proximity map. The result now follows from Corollary 2.11 of Light & Cheney (1985).

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REFERENCES

Deeb, W. & Khalil, R. 1989. Best approximation in $L^p(I, X)$, $0 < p < 1$. Journal of Approximation Theory **58**: 68–77.

Diestel, J. & Uhl, J.R. 1977. Vector measures. Mathematical Surveys 15. American Mathematical Society, Providence, Rhode Island.

Khalil, R. 1983. Best approximation in $L^p(I, X)$. Mathematical Proceedings of the Cambridge Philosophical Society **94**: 277–79.

Khalil, R. & Deeb, W. 1989. Best approximation in $L^p(I, X)$ *II*. Journal of Approximation Theory **59**: 296–99.

Light, W. & Cheney, E. 1985. Approximation theory in tensor product spaces. Lecture Notes in Mathematics, vol. 1169. Springer-Verlag.

Singer, I. 1970. Best approximation in normed linear spaces. Springer-Verlag.

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أفضل تقريب في بعض فضاءات الدوال

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خلاصة

ليكن X فضاء باناخ و Y فضاء مغلقا جزئيا من X. نسمي Y فضاء قـرابيا في X إذا تحقق الشرط التالي:

لكل $x \in X$ يوجد على الأقل $y \in Y$، بحيث تتحقق المتباينة $\|x-y\| \leqslant \|x-z\|$ من أجل كـل $z \in Y$.

سندرس في هذا البحث القرابية في بعض فضاءات الدوال. وبشكل خاص، سنبرهن على أنه إذا كان لـ $L^p(I,Y)$ تطبيق قرابي متصل في $L^p(I,X)$، عندئذ يكون للفضاء Y تطبيق قرابي متصل في X.

وسنعرض في هذا البحث نتائج أخرى.