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Novel approach to coprime factorization and minimal realization of transfer function matrices

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ABSTRACT

A new computationally simple algorithm is presented for calculating coprime matrix fraction description, and minimal state space representation of a multivariable linear system, specified by a transfer function matrix. The algorithm is based on a theorem derived in the paper which permits the calculation of the order of the minimal realization, as well as the observability indices (column degrees) directly from the coefficients of the given transfer function matrix. Existing minimal realization and coprime factorization algorithms are reviewed and compared with the proposed procedure. An illustrative example is given to show feasibility of the suggested method.

INTRODUCTION

The matrix fraction description (MFD) approach to linear multivariable system (MIMO) analysis and synthesis in frequency domain, has received a great deal of attention over the past few decades, and, as a result of that, several methods have been proposed for obtaining MFD's (Callier & Desoer 1982; Chen 1984; Datta & Gangopadhyay 1992; Forney 1975; Foster 1979; Kailath 1980; Kung *et al.* 1977; Patel 1981; Rosenbrock 1970; Sain 1975; Wang & Davison 1973; Wolovich 1974; Wolovich & Falb 1969). In particular, Rosenbrock (1970) used a method which requires elementary row operations involving multiplication and division by polynomials. The method is therefore difficult to implement on a computer and can be numerically unstable in Forney (1975); Foster (1979); Kung *et al.* (1977); Sain (1975); Wang & Davison (1973); and Wolovich & Falb (1969) the problem of solving a proper rational function matrix equation was cast in terms of finding a minimal polynomial basis for the right or left null space of a polynomial matrix (the minimal design problem MDP). These methods are computationally rather involved and prone to numerical difficulties. Patel (1981) developed a recursive algorithm which obtains coprime MFD from a state-space model in the "block Hessenberg" form, which can be derived using orthogonal transformations. Recently, Datta & Gangopadhyay (1992) proposed an algorithm to determine coprime MFD, for a given transfer function matrix by first, determining a minimal state-space description in an upper block Hessenberg form using either the Householder transformation, or

التحكم في الإهتزازات اللاخطية للجسور المعلقة بالكابلات بإستعمال الكتلة الموازنة الفعالة

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خلاصة

يعرض هذا البحث كيفية تصميم الكتلة الموازنة الفعالة للحد من الإهتزازات اللاخطية الناجمة من جراء تأثير الرياح على الجسور المعلقة بالكابلات. ويبين البحث أن طريقة التحكم بالسرعة المرتجعة ليست هي الطريقة المرغوبة لإمتصاص إهتزازات الجسر. أما طريقة فرض الأقطاب فقد أثبتت نجاحها بتزويد عامل إمتصاص جيد للجسر. ويستنتج بذلك أنه إذا ما صممت الكتلة الموازنة الفعالة بطريقة مجدية فإن ذلك سيؤدي الى تحكم أفضل بالإهتزازات من الكتلة الموازنة الغير فعالة.

the plane rotation. Then, using an orthogonal version of Wolovich-Falb's structure theorem (Wolovich & Falb 1969; Wolovich 1974), the coprime MFD is obtained. As in Patel (1981), the disadvantage of this method is that a recursive computation is needed. Also, both methods start from a non-minimal state-space representation of a relatively high order.

On the other hand, a fundamental problem in linear systems theory is to obtain minimal-order realizations of linear multivariable systems in the form of state equations for specified rational transfer function matrices. Such a realization may be useful in pole placement using state feedback, design of observers and system identification. Several methods for minimal realizations are available and a comparative study for eight of these methods can be found in Hashim *et al.* (1987). However, among these eight methods, only one which does not need the calculation of the rather involved Markov parameters. Unfortunately, this method has been characterized in a previous study to be numerically not reliable.

The purpose of this paper is to propose a computationally simple and easy to apprehend procedure, for coprime factorization and minimal realization of transfer function matrices of linear multivariable dynamic systems. The procedure is based on a theorem referred to as "Order" theorem, which permits direct determination of the order as well as the observability indices of a corresponding minimal realization. In the case of coprime factorization, the column degrees of a corresponding left coprime column reduced MFD are determined. The dual version of the theorem allows determination of the controllability indices of a minimal realization and row degrees of a right coprime row reduced MFD.

Unlike in the existing algorithms, the proposed procedure does not build a non minimal state space realization, which is subsequently reduced to the minimal order by using Hessenberg or Householder transformation (plane rotation ...). Also, it does not use concepts as unitary polynomial matrices, greatest common divisors, Smith-McMillan form, leading column/row coefficient matrix, ... etc. Only what is required in the proposed algorithm, is checking ranks of relatively low order matrices built directly from the coefficients of the given transfer function matrix, extraction of linearly independent and dependent columns and solving a system of linear algebraic equations.

The organization of the remainder of the paper is as follows: Section 2, gives necessary preliminary definitions and background. Section 3, describes a special state-space representation and its relation to MFD descriptions. Section 4, presents the order theorem for determining the order of the minimal realization and outlines a procedure for calculating minimal realization and left coprime MFD for the given transfer function matrix. This procedure is summarized as a simple algorithm in Section 5. Finally, Section 6 contains a computational example to illustrate the feasibility of the proposed method.

BACKGROUND

SYSTEM DESCRIPTIONS

Consider a linear time invariant multivariable system described by a $(p \times m)$ proper transfer function matrix given by:

$$G(s) = \{g_{ij}(s)\} = W(s)/d(s) \tag{1}$$

where $g_{ij}(s)$ are proper or strictly proper rational functions, while $W(s)$ and $d(s)$ are $(p \times m)$ polynomial matrix and scalar polynomial, respectively, which may be represented by:

$$W(s) = \{w_{ij}(s)\} = \sum_{h=0}^{r} W_h s^h \quad ; \quad d(s) = \sum_{h=0}^{r} d_h s^h \tag{2}$$

Note that integer r used in (2) is not necessarily equal to the order n of a corresponding minimal state space realization. Depending on the way how $G(s)$ is specified, r could be related to n by:

$$r < n \quad \text{or} \quad r > n \tag{3}$$

Alternate descriptions of system (1) are:

(a) Minimal order (irreducible) state space representation:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{4}$$

with:

$$G(s) = C(Is - A)^{-1}B + D$$

(b) Left coprime Matrix Fraction Description (MFD):

$$G(s) = D(s)^{-1}N(s) \tag{5}$$

with

$$D(s) = \sum_{h=0}^{k} D_h s^h \quad ; \quad N(s) = \sum_{h=0}^{k} N_h s^h \quad ; \quad k \leqslant n - p + 1$$

(c) Markov parameters:

$$G(s) = H(s^{-1}) = \sum_{h=0}^{\infty} H_h s^{-h} \tag{6}$$

with

$$H_0 = D \quad \text{and} \quad H_h = CA^{h-1}B, \quad \text{for} \quad h = [1, \infty]$$

The reason for including the matrix $H(s^{-1})$, in Eqn. 6, as one of the alternate system descriptions, will be clear later.

In Eqn. 4 $x(t)$, $u(t)$ and $y(t)$ are n-, m- and p-dimensional state, input and output vectors, respectively, while A, B, C and D are matrices of compatible dimensions. As it was mentioned earlier, system of order n is assumed to be minimal, i.e. the pairs $\{A, B\}$ and $\{A, C\}$ are assumed to be controllable and observable, respectively.

In Eqn. 5 the $(p \times p)$ and $(p \times m)$ polynomial matrices $D(s)$ and $N(s)$ are assumed to be left coprime with $D(s)$ being column reduced and monic.

COPRIME AND COLUMN REDUCED, MONIC POLYNOMIAL MATRICES

As it is known in Kailath (1980), left coprimness is defined as:

$$\mathit{rank}\,[D(s) : N(s)] = p, \quad \text{for all } s$$

which is a MIMO generalization of the condition that $D(s)$ and $N(s)$ do not have common terms, i.e. that there are no pole-zero cancellations.

A column reduced $(p \times p)$ polynomial matrix:

$$D(s) = [d_1(s) : \ldots : d_p(s)] \tag{7}$$

satisfies:

$$deg\,\{det\,[D(s)]\} = n = \sum_{i=1}^{p} n_i \tag{8}$$

where n_i, $i = [1,p]$, is the degree of the i-th column of $D(s)$. The column degree is defined as the highest power of s in all entries of the column.

The column reduced $D(s)$ is considered monic, if in each column $d_i(s)$ the polynomial with the degree n_i is a monic polynomial. In other words, from Eqn. 8 it may be concluded that the determinant $d(s)$ of a column reduced monic polynomial matrix $D(s)$ is the n-th order polynomial defined by:

$$d(s) = det\,[D(s)] \tag{9}$$

where the coefficient associated with the term s^n is ± 1.

AN ALTERNATE REPRESENTATION OF POLYNOMIAL MATRICES

In addition to defining a $(p \times m)$ polynomial matrix $X(s)$ by a usual symbol:

$$X(s) = \{x_{ij}(s)\} = \sum_{h=0}^{k} X_h s^h$$

already used in Eqns. 2 and 5, we will also need to express $X(s)$ by:

$$X(s) = I_p(s)^T X_c \quad \text{and} \quad X(s) = X_r I_m(s) \tag{10}$$

with:

$$X_r = [X_0 : X_1 : \ldots : X_k]; \qquad X_c = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_k \end{bmatrix}$$

$$I_p(s) = \begin{bmatrix} I_p \\ I_p s \\ \vdots \\ I_p s^k \end{bmatrix}; \qquad I_m(s) = \begin{bmatrix} I_m \\ I_m s \\ \vdots \\ I_m s^k \end{bmatrix}$$

where the dimensions of X_r, X_c, $I_p(s)$ and $I_m(s)$ are: $p \times (k+1)m$, $(k+1)p \times m$, $(k+1)p \times p$ and $(k+1)m \times m$, respectively. The indices "r" and "c" in X_r and X_c imply concatenation of real number submatrices X_h, $h = [0,k]$, in a row- and column-like matrices, respectively, while the indices "p" and "m" in $I_p(s)$ and $I_m(s)$ imply concatenation of $I_m s^h$ and $I_p s^h$, $h = [0,k]$, in a column-like matrix, respectively.

To be self-contained, before we give the main result we need to introduce a particular state space representation to be considered in the sequel, referred to as Modified Luenberger form, and to review some, more or less known, concepts which are related to this state space form such as:

- Observability indices,
- Crate diagram,

— Selector matrices,
— Relation between a state space and MFD descriptions.

This is done in the next section.

STATE SPACE REPRESENTATION

OBSERVABILITY INDICES

Consider the observability matrix Q_0 of an observable pair $\{A, C\}$, with a full row rank matrix C, i.e.:

$$Q_0 = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^k \end{bmatrix}; \qquad \text{rank}\,[C] = p \tag{11}$$

where $k = n - p + 1$.

By virtue of observability, i.e. rank $[Q_0] = n$, there are n linearly independent rows in Q_0. Assume that checking for linearly independent rows started from the "top" of Q_0. Thus, due to Eqn. 11, the first p rows, i.e. the rows c_i of the matrix C, are linearly independent. The remaining $n - p$ linearly independent rows are located somewhere in the blocks $CA^j, j = [1, k]$. Let the locations of the linearly independent rows in Q_0 be denoted by unities in the $(k + 1)p$ dimensional "selector" vector v_{li}. The zeros in v_{li} indicate linearly dependent rows. Among all rows from Q_0 of the form

$$c_i A^j \qquad i = [1, p],\ j = [0, k]$$

there are v_i linearly independent rows "associated" with c_i. These rows are given by:

$$c_i A^j \qquad \text{for } j = [0, v_i - 1] \tag{12}$$

It is known that the integer v_i is called the observability index of the i-th output, and the set:

$$v = \{v_1, \ldots, v_i, \ldots, v_p\} \tag{13}$$

is referred to as the set of unique observability indices of the pair $\{A, C\}$. In the sequel we will need also:

$$v_x = \max\{v_i\} \qquad i \in [1, p] \tag{14}$$

CRATE DIAGRAM

Observability indices and linearly independent and dependent rows in Q_0 may be visualized by considering the Crate diagram (Chen 1984; Kailath 1980) consisting of a table (matrix $E = \{e_{ji}\}$) with $(k + 1)$ rows and p columns, shown in Fig. 1. Each element $e_{ji}, j = [0, k], i = [1, p]$, represents a particular row $c_i A^j$ of Q_0. If a particular row $c_i A^j$ is linearly independent with respect to the preceding $jp + (i - 1)$ rows:

$$c_1, \ldots, c_p, c_1 A, \ldots, c_p A^{j-1}, c_1 A^j, \ldots, c_{i-1} A^j \tag{15}$$

then the unity is placed at the location e_{ji}.

j \ i	1	2	3
0	1	1	1
1	1	X	1
2	1		X
3	1		
4	X		
5			

Fig. 1. Crate diagram of a system having observability indices equal to $v = \{4, 1, 2\}$.

Example of a 7-th order system with $p = 3$ having observability indices equal to:

$$v = \{v_1, v_2, v_3\} = \{4, 1, 2\} \tag{16}$$

is shown in Fig. 1.

From Fig. 1 it may be seen that linearly dependent rows are represented in Crate diagram as follows:

— In each column, corresponding to a particular row c_i of C, the element e_{ji} immediately after the last unity, i.e. for $j = v_i$, is denoted by "X".
— The remaining locations e_{ji} are left "blank".

SELECTOR VECTORS AND SELECTOR MATRICES

From the crate diagram we need to define the following four selector vectors:

- $(k+1)p$—dimensional row v_{li} indicating the locations of linearly independent rows in Q_0. The vector v_{li} is obtained by:
  — setting zeros to all blank and "X" elements, and
  — concatenating all $(k+1)$ rows into the row v_{li}.
- $(k+1)p$—dimensional row v_{ld} indicating locations of the "first" linearly dependent rows in Q_0. The vector v_{ld} is obtained by:
  — setting zeros to all "X" elements,
  — setting unities to all blank elements,
  — complementing all elements, i.e. converting zeros to unities and vice versa, and
  — concatenating all $(k+1)$ rows into the row v_{ld}.

The next two n-dimensional vectors denoted by v_i and v_a are obtained from the Crate diagram in the following way:

- Vector v_i:
  — disregarding the first row containing p unities,
  — disregarding all blank elements,
  — setting zeros to all "X" elements, and
  — concatenating reduced rows into the row v_i.
- Vector v_a:
  — complementing all elements of v_i, i.e. converting zeros to unities and vice versa.

In the example, represented by Fig. 1, the above mentioned four selector vectors become:

$$\begin{aligned} v_{li} &= [1\,1\,1 : 1\,0\,1 : 1\,0\,0 : 1\,0\,0 : 0\,0\,0 : 0\,0\,0] \\ v_{ld} &= [0\,0\,0 : 0\,1\,0 : 0\,0\,1 : 0\,0\,0 : 1\,0\,0 : 0\,0\,0] \\ v_i &= [1\,0\,1 : 1\,0 : 1 : 0] \\ v_a &= [0\,1\,0 : 0\,1 : 0 : 1] \end{aligned} \tag{17}$$

Thus, vectors v_{li}, v_{ld}, v_i and v_a have n, p, $n-p$ and p unities, respectively.

There is a unique one-to-one correspondence between the above selector vectors and the corresponding set of observability indices v, Eqn. 16, or equivalently, the location of unity elements in the crate diagram, (Fig. 1). As it will be shown later, the selector vectors greatly facilitate calculation of the observable form as well as the left coprime column reduced MFD based on the set v. In particular, the selector matrices given by Eqn. 18 below, which are derived from the associated selector vectors by a corresponding selection of columns from an appropriately dimensioned identity matrix, are actually used in obtaining the observable form and left coprime MFD.

$$S_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T;$$

$$S_{li} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{18a}$$

$$S_a = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T;$$

$$S_{ld} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{18b}$$

Note that the columns in $\{S_i, S_a\}$ as well as in $\{S_{li}, S_{ld}\}$ are orthonormal, i.e.: $S_i^{\mathrm{T}} S_a = 0$ and $S_{li}^{\mathrm{T}} S_{ld} = 0$.

The selection of rows (columns) of a matrix may be accomplished by pre- (or post-) multiplication by a corresponding selector matrix. Thus, for instance, since S_i in Eqn. 18 is a (7×4) selector matrix, the product $S_i^T M$, where M is a (7×7) matrix, creates a (4×7) matrix containing the rows from M with indices 1, 3, 4 and 6.

Given a set of observability indices, a simple algorithm may be developed to calculate all four selector matrices (Bingulac & VanLandingham 1993).

MODIFIED LUENBERGER OBSERVABLE FORM

Let us now select all n linearly independent rows from Q_0 into a similarity transformation matrix T. The ordering of these rows in T is done according to the rows of the Crate diagram, i.e. according to the order as they are found to be linearly independent. In other words, no additional ordering of the rows $c_i A^j$ is required, as it is the case in the Luenberger canonical forms (Luenberger 1967).

In the above example the matrix T is obtained by concatenating the following rows:

$$\{c_1, c_2, c_3, c_1 A, c_3 A, c_1 A^2, c_1 A^3\}$$

in that order.

Using the obtained T, the following observable form is obtained

$$\begin{aligned} A_0 &= TAT^{-1} \\ B_0 &= TB \\ C_0 &= CT^{-1} \\ D_0 &= D \end{aligned} \tag{19}$$

It may be easily verified that the matrices C_0 and A_0 in Eqn. 19 have the following structure:

$$C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}; \qquad A_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ x & x & x & x & y & y & y \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ x & x & x & x & x & x & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ x & x & x & x & x & x & x \end{bmatrix} \tag{20}$$

where "x" and "y" represent possible non-zero/non-unity quantities. The structure of the pair $\{A_0, C_0\}$ is characterized by the following points:

- The first p rows of the $(n \times n)$ identity matrix I_n are in the matrix C_0,
- The last $n - p = 4$ rows from I_n are located in A_0 at locations specified by the unities in the selector vector v_i.
- At locations specified by the $p = 3$ unities in the selector vector v_a, the matrix A_0 contains rows of elements which are not necessarily of zero or unit value. For the future use, these rows may be concatenated into a $(p \times n)$ matrix A_r given by:

$$A_r = \begin{bmatrix} x & x & x & x & y & y & y \\ x & x & x & x & x & x & y \\ x & x & x & x & x & x & x \end{bmatrix}$$

As it was shown in (Bingulac & VanLandingham 1993), if the system is represented by a possible set of admissible pseudo observability indices, then the quantities "y" above are different from zero. However, if a unique set of observability indices is used, then the quantities "y" are by definition equal to zero (Bingulac & Krtolica 1987). The issue of the pseudo observability indices is discussed in (Al-Muthairi & Bingulac 1994).

According to the definition of the selector matrices S_a and S_i, (Eqn. 18), it may be verified that:

$$A_r = S_a^T A_0 \quad \text{and} \quad A_0 = S_a A_r + S_i A_2 \tag{21}$$

where

$$A_2 = [O : I_{n-p}]$$

- The observability matrix Q_{00} of the pair $\{A_0, C_0\}$, i.e.

$$Q_{00} = \begin{bmatrix} C_0 \\ C_0 A_0 \\ \vdots \\ C_0 A_0^k \end{bmatrix}$$

  contains all n rows of I_n at locations specified by $n = 7$ unities in the selector vector v_{li}.
- The $p = 3$ rows of A_0 containing not necessarily zero nor unit elements, i.e. rows of the $(p \times n)$ matrix A_r, (Eqn. 21), appear in Q_{00} at locations specified by unities in the selector vector v_{ld}. In other words:

$$S_{li}^T Q_{00} = I_n \quad \text{and} \quad S_{ld}^T Q_{00} = A_r \tag{22}$$

  Equations 21 and 22 will be extensively used in later developments.

RELATION BETWEEN STATE-SPACE REPRESENTATION AND MFD

This section gives a one-to-one correspondence between the observable form and the MFD. Therefore, given one form, the other may be derived accordingly. The presentation considers only the left coprime MFD and the observable form. The right coprime MFD and the controllable form can be derived in a dual sense.

Derivation of MFD:

Consider a minimal n-th order state space representation given by Eqn. 4 where the matrices $\{A, B, C, D\}$ are in the observable form, given by Eqn. 19. It has been shown by Al-Muthairi & Bingulac (1994) that a left coprime MFD described by Eqn. 5 can be obtained by:

$$D(s) = D_r I_p(s) \quad ; \quad N(s) = N_r I_m(s) \tag{23}$$

where

$$D_r = S_{ld}^T - A_r S_{li}^T \quad ; \quad N_r = H_2 - A_r H_1 \tag{24}$$

and

$$H_1 = S_{li}^T H \quad ; \quad H_2 = S_{ld}^T H \tag{25}$$

The matrix H is a $[(k+1)p \times (k+1)m]$ lower block triangular matrix containing the

Markov parameters and is given by:

$$H = \begin{bmatrix} D & & & & & \\ CB & D & & & & \\ CAB & CB & D & & & \\ \vdots & \vdots & \vdots & \ddots & D & \\ CA^{k-1}B & CA^{k-2}B & CA^{k-3}B & \cdots & CB & D \end{bmatrix} \tag{26}$$

with $k = v_x$ (Eqn. 14).

Properties of $D(s)$:
Comparing the expression for D_r given by Eqn. 24 with that derived from Eqn. 10, i.e.

$$D_r = S_{ld}^T - A_r S_{li}^T = [D_0 : D_1 : \ldots : D_k]$$

it may be concluded that D_r contains:

- n columns with non-zero/non-unity elements, i.e. n columns from the $(p \times n)$ matrix $-A_r$
- p columns of the $(p \times p)$ identity matrix I_p, and
- $kp - n$ columns of zeros.

From Eqn. 24 it may be verified that the locations of non-zero/non-unity columns and the columns of I_p mentioned above are defined by the locations of the unities in the selector vectors v_{li} and v_{ld}, respectively, generated by the set of observability indices v. Thus, using the same line of proof as in Chen (1984), it may be concluded that $D(s)$ is column reduced and monic and that:

$$deg\,\{det\,[D(s)]\} = n$$

where n is the order of the minimal representation, Eqn. 19, used in Eqn. 26. Moreover, the column degrees n_i, $i = [1, p]$, of $D(s)$ are equal to the observability indices v_i, i.e.:

$$n_i = v_i, \quad \text{for} \quad i = [1, p] \tag{27}$$

Derivation of state-space representation:
Given the column reduced monic $D(s)$ and combining Eqns. 21 and 24, an expression to calculate A, i.e. A_0 in Eqn. 20, can be found to be:

$$A = S_i A_2 - S_a D_r S_{li} \quad \text{or:} \quad D_r = S_{ld}^T - S_a^T A S_{li}^T \tag{28}$$

where

$$A_2 = [O : I_{n-p}]$$

In addition to that, from Eqn. 20, C becomes: $C = [I_p : O]$.

The matrix D may be obtained by taking the last p rows from W_c, or last m columns from W_r, i.e. by partitioning:

$$W_c = \begin{bmatrix} X \\ -- \\ D \end{bmatrix} \begin{matrix} \}rp \\ \\ \}p \end{matrix} \quad \text{or} \quad W_r = [Y \,\vdots\, \underbrace{D}_{m}\,] \tag{29}$$

where: $W(s) = I_p(s)^T W_c = W_r I_m(s)$.

As far as the calculation of the matrix B in the representation given by Eqn. 19 goes, it may be verified that B may be calculated by:

$$B = Q_{ca} N_c \tag{30}$$

where Q_{ca} is the controllability matrix of the auxiliary pair $\{A, S_a\}$ having $(k+1)p$ columns, while N_c is the "column-like" $[(k+1)p \times m]$ matrix obtained by rearrangement of $(p \times m)$ submatrices $N_j, j = [0, k]$, from N_r into N_c, see Eqn. 10.

ORDER THEOREM

Given a transfer function matrix $G(s)$ (Eqns. 1 and 2), the order n of the minimal representation $\{A_0, B_0, C_0, D_0\}$ (Eqns. 4 and 19), as well as the degree of the polynomial $det\{D(s)\}$, of the $(p \times p)$ column reduced monic polynomial matrix $D(s)$, defining left coprime MFD, Eqn. 5, may be calculated from:

$$n = n_k = \text{rank}\, H_2[k] \tag{31}$$

where:

$$H_2[k] = W[k] N_D[k] \tag{32}$$

$$D[k] = \begin{bmatrix} \tilde{D}_0 & \tilde{D}_1 & \cdots & \tilde{D}_r & & \\ & \ddots & \ddots & & \ddots & \\ & & \tilde{D}_0 & \tilde{D}_1 & \cdots & \tilde{D}_r \end{bmatrix}$$

$$W[k] = \begin{bmatrix} W_0 & W_1 & \cdots & W_r & & \\ & \ddots & \ddots & & \ddots & \\ & & W_0 & W_1 & \cdots & W_r \end{bmatrix}$$

and k is the smallest integer satisfying:

$$n_k = n_{k+1} \tag{33}$$

Matrix $N_D[k]$, appearing in Eqn. 32 is a $[(r+k+1)m \times rm]$ "null" space matrix satisfying:

$$D[k] N_D[k] = 0 \tag{34}$$

while $D[k]$ and $W[k]$ are Sylvester type matrices of the dimensions:

$$[(k+1)m \times (r+k+1)m] \quad \text{and} \quad [(k+1)p \times (r+k+1)m]$$

respectively, with $\tilde{D}_i = d_i I_m$, $i = [0, r]$.

Integer k satisfying Eqn. 33, is equal to k in Eqn. 5, as well to v_x in Eqn. 14. Once the integer k satisfying (Eqn. 33) is determined, the matrices $D[k]$ and $W[k]$, having $(r+k+1)m$ columns, may be truncated so to use blocks $\tilde{D}_h$ and W_h, for $h = [0, k+1]$ only, instead of $[0, r]$, $r > k$. In other words, without loss of generality it may be assumed that in Eqn. 32, $r = k+1$, leading to the total number of columns in $D[k]$ and $W[k]$ equal to $2(k+1)m$. The reason for this assumption will be clear later.

Observability indices v, (Eqn. 13), defining the structure of the observable form, (Eqn. 19), as well as the column degrees n_i, $i = [1, p]$ (Eqns. 8 and 27), defining the MFD. Equation 5, may be determined by checking linear independence of $(k+1)p$ rows of $H_2[k]$, Eqn. 31, starting from the "top" in exactly the same manner as it was done in the case of the observability matrix Q_0.

Recall that there is a unique one-to-one correspondence between the observability indices v and the selector vector v_{li}, (Eqn. 17). Thus, having the selector vector v_{li}, the observability indices v may be obtained by building "back" the $[(k+1) \times p]$ matrix E, representing the Crate diagram (Fig. 1) and by multiplying E with the $(k+1)$ dimensional row containing $k+1$ unities, i.e.:

$$v = qE; \quad \text{where} \quad q = [1\,1\ldots 1] \tag{35}$$

Proof. To prove the Order Theorem it is sufficient to show that the matrix $H_2[k]$ (Eqns. 31 and 32), may be represented by:

$$H_2[k] = H[k]T \tag{36}$$

where $H[k]$ is the "standard" Hankel matrix of the structure:

$$H[k] = Q_0 Q_c = \begin{bmatrix} H_1 & \cdots & H_{k+1} \\ \vdots & & \vdots \\ H_{k+1} & \cdots & H_{2k+1} \end{bmatrix} \tag{37}$$

which is used extensively in minimal realization procedures given Markov parameters $H_h, h = [0, 2k+1]$ (Chen 1984; Ho & Kalman 1966), while T is a non singular square matrix of the order $(k+1)m$.

To this end:

(a) Combining Eqns. 1, 2 and 5 the following expression may be obtained:

$$D(s)W(s) = N(s)\tilde{D}(s), \qquad \tilde{D}(s) = I_m d(s)$$

or, equivalently,

$$[N_r \mid D_r]\begin{bmatrix} D[k] \\ \hline -W[k] \end{bmatrix} = 0 \tag{38}$$

(b) On the other hand, combining Eqns. 5 and 6 and with the help of Eqn. 10, i.e.

$$D(s)H(s^{-1}) = N(s)$$

and

$$\sum_{i=0}^{k} D_i s^i \sum_{j=0}^{\infty} H_j s^{-j} = \sum_{i=0}^{k} N_i s^i$$

we may write:

$$[N_r \mid D_r]\left[\begin{array}{c|c} 0 & I_{(k+1)m} \\ \hline -\tilde{H}_2[k] & -\tilde{H}_1[k] \end{array}\right] = 0 \tag{39}$$

where the matrices $\tilde{H}_1[k]$ and $\tilde{H}_2[k]$ are given by:

$$\tilde{H}_1[k] = \begin{bmatrix} H_0 & & \\ \vdots & \ddots & \\ H_k & \cdots & H_0 \end{bmatrix}$$

$$\tilde{H}_2[k] = \begin{bmatrix} H_{k+1} & \cdots & H_1 \\ \vdots & & \vdots \\ H_{2k+1} & \cdots & H_{k+1} \end{bmatrix}$$

It is worth mentioning that $\tilde{H}_1[k]$ above is equal to the matrix H in Eqn. 26. Note that $\tilde{H}_2[k]$ may be related to $H[k]$ by:

$$\tilde{H}_2[k] = H[k]P; \quad \text{with} \quad P = \begin{bmatrix} & & I_m \\ & \cdot^{\cdot^{\cdot}} & \\ I_m & & \end{bmatrix} \tag{40}$$

where P represents simple column permutation of $H[k]$.

Post multiplying Eqn. 38 with a non singular matrix given by:

$$[N_D[k] : D[k]^{\dagger}] \tag{41}$$

where $N_D[k]$ is defined by Eqn. 34, while $D[k]^{\dagger}$ is a pseudo inverse of the full row rank matrix $D[k]$, Eqns. 32 and 38 lead to:

$$[N_r \mid D_r] \left[\begin{array}{c|c} 0 & I_{(k+1)m} \\ \hline -H_2[k] & -H_1[k] \end{array}\right] = 0 \tag{42}$$

Obviously, $H_2[k] = W[k]N_D[k]$ and $H_1[k] = W[k]D[k]^{\dagger}$. Recall that $D[k]^{\dagger}$ satisfies:

$$D[k]D[k]^{\dagger} = I_{(k+1)m}$$

From Eqns. 39 and 42 it may be concluded that there is a nonsingular $2(k+1)m$ order matrix V of the structure:

$$V = \left[\begin{array}{c|c} V_2 & V_1 \\ \hline 0 & I_r \end{array}\right] \tag{43}$$

which by post multiplying Eqn. 39 gives Eqn. 42. Thus, $H_2[k]$ may be expressed by:

$$H_2[k] = \tilde{H}_2[k]V_2 \tag{44}$$

using Eqn. 40, finally verifies Eqn. 36, since:

$$H_2[k] = H[k]PV_2$$

Having established that $H_2[k]$ satisfies Eqn. 36, and considering a sequence of column permutations, represented by Eqns. 40 and 44, it may be concluded that the locations of the first n linearly independent rows of the matrix $H_2[k]$ define the selector vector v_{li}, in exactly the same way as the linearly independent rows of Q_0 define vector v_{li} and observability indices.

Calculation of the MFD:

Substituting Eqn. 24 into Eqn. 42, the following expressions may be obtained:

$$(S_{ld}^T - A_r S_{li}^T)H_2[k] = 0; \qquad D_r H_1[k] = N_r \tag{45}$$

leading to:

$$A_r T_1 = T_2 \tag{46}$$

where:

$$T_1 = S_{li}^T H_2[k] \quad \text{and} \quad T_2 = S_{ld}^T H_2[k]$$

Equations 45 and 46 may be used to calculate matrices N_r and A_r, which according to Eqns. 24 and 10 permits calculation of the left coprime MFD $\{D(s), N(S)\}$, where $D(s)$ is column reduced and monic.

The computational algorithm, given in the sequel, clarifies all issues of calculating both the MFD and the state space representation in the form given by Eqns. 19 and 20, given a $(p \times m)$ transfer function matrix $G(s)$ (Eqn. 1).

ALGORITHM

COPRIME FACTORIZATION

Step 1.
Define $(p \times m)$ transfer function matrix $G(s) = W(s)/d(s)$.
Set $0 \Rightarrow k$, $0 \Rightarrow n$.

Step 2.
Set $k + 1 \Rightarrow k$, build matrices $D[k]$, $W[k]$ and
Set $W[k]N_D[k] \Rightarrow H_2[k]$, Eqn. 32.

Step 3.
Set rank of $H_2[k] \Rightarrow n_k$

Step 4.
If $n = n_k$, go to Step 5; Else, set $n_k \Rightarrow n$ and go to Step 2.

Step 5.
Build selector vector v_{li}, (Eqn. 17), indicating linearly independent rows in $H_2[k]$.

Step 6.
From v_{li} determine observability indices v (Eqn. 35).

Step 7.
Using v build the selector matrices S_a, S_i, S_{li} and S_{ld} (Eqn. 18).

Step 8.
Set $S_{li}^T H_2[k] \Rightarrow T_1$, $S_{ld}^T H_2[k] \Rightarrow T_2$, (Eqn. 46).

Step 9.
Solve: $A_r T_1 = T_2$, for unknown A_r.

Step 10.
Set $S_{ld}^T - A_r S_{li}^T \Rightarrow D_r$ and $D_r W[k] D[k]^\dagger \Rightarrow N_r$ (Eqn. 45).
D_r and N_r define left coprime MFD $\{D(s), N(s)\}$, with $D(s)$ column reduced and monic (Eqn. 10).

Minimal realization
Step 11.
Partition: $I_n \Rightarrow \begin{bmatrix} C \\ \hline A_2 \end{bmatrix}$; C has p rows (Eqn. 20).

Step 12.
Set $S_i A_2 + S_a A_r \Rightarrow A$ (Eqn. 21).

Step 13.

Partition: $W_c \Rightarrow \left[\frac{X}{D}\right]$; D has p rows (Eqn. 29).

Step 14.

Build Q_{ca} and N_c (Eqn. 30) and set $Q_{ca}N_c \Rightarrow B$.

ILLUSTRATIVE EXAMPLE

Consider the (2 × 2) transfer function matrix $G(s) = W(s)/d(s)$ where

$$d(s) = 4 + 12s + 13s^2 + 6s^3 + s^4;$$

$$W(s) = \begin{bmatrix} s & 4 + 13s + 15s^2 + 7s^3 + s^4 \\ -s - 2s^2 - s^3 & -s - 2s^2 - s^3 \end{bmatrix}$$

This example was derived from Kailath 1980, by making the strictly proper $G(s)$ proper.

Matrices $\tilde{D}_i$'s and W_i's used in building $D[k]$ and $W[k]$ (Eqn. 32), are:

$$[\tilde{D}_0\tilde{D}_1\tilde{D}_2\tilde{D}_3\tilde{D}_4] = \left[\begin{array}{cc|cc|cc|cc|cc} 4 & 0 & 12 & 0 & 13 & 0 & 6 & 0 & 1 & 0 \\ 0 & 4 & 0 & 12 & 0 & 13 & 0 & 6 & 0 & 1 \end{array}\right]$$

$$[W_0W_1W_2W_3W_4] = \left[\begin{array}{cc|cc|cc|cc|cc} 0 & 4 & 1 & 13 & 0 & 15 & 0 & 7 & 0 & 1 \\ 0 & 0 & -1 & -1 & -2 & -2 & -1 & -1 & 0 & 0 \end{array}\right]$$

The Algorithm gave for integer k and order n of the minimal realization the following values

$$k = 3; \qquad n = 5$$

Matrix $H_2[k]$; given by: $W[k]N_D[k]$ (Eqn. 32), is:

$$H_2[k] = \begin{bmatrix} 0.441 & 0.249 & 0.054 & 0.067 & 0.191 & -0.177 & 0.056 & -0.059 \\ -0.115 & 0.062 & -0.026 & 0.120 & 0.111 & 0.202 & 0.037 & 0.055 \\ -0.107 & -0.432 & -0.110 & -0.126 & -0.273 & 0.232 & -0.075 & 0.065 \\ 0.161 & -0.080 & 0.023 & -0.174 & -0.142 & -0.266 & -0.033 & -0.057 \\ -0.113 & 0.537 & 0.170 & 0.124 & 0.209 & -0.228 & 0.047 & -0.020 \\ -0.181 & 0.072 & 0.011 & 0.217 & 0.122 & 0.255 & -0.018 & 0.008 \\ 0.171 & -0.527 & -0.211 & -0.065 & 0.065 & -0.030 & 0.030 & -0.178 \\ 0.081 & 0.031 & -0.137 & -0.172 & 0.080 & 0.044 & 0.204 & 0.196 \end{bmatrix}$$

Observability indices were obtained as: $v = \{3 \quad 2\}$. The corresponding selector matrices are:

$$S_a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad S_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$S_{li} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \qquad S_{ld} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Matrices T_1, T_2 and A_r (Eqns. 45 and 46) are as follows:

$$T_1 = \begin{bmatrix} 0.441 & 0.249 & 0.054 & 0.067 & 0.191 & -0.177 & 0.056 & -0.059 \\ -0.115 & 0.062 & -0.026 & 0.120 & 0.111 & 0.202 & 0.037 & 0.055 \\ -0.107 & -0.432 & -0.110 & -0.126 & -0.273 & 0.232 & -0.075 & 0.065 \\ 0.161 & -0.080 & 0.023 & -0.174 & -0.142 & -0.266 & -0.033 & -0.057 \\ -0.113 & 0.537 & 0.170 & 0.124 & 0.209 & -0.228 & 0.047 & -0.020 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} -0.181 & 0.072 & 0.011 & 0.217 & 0.122 & 0.255 & -0.018 & 0.008 \\ 0.171 & -0.527 & -0.211 & -0.065 & 0.065 & -0.030 & 0.030 & -0.178 \end{bmatrix}$$

$$A_r = \begin{bmatrix} 0 & -4 & 0 & -4 & 0 \\ -2 & -2 & -5 & -1 & -4 \end{bmatrix}$$

Note that T_1 contains the first 5 rows from $H_2[k]$, while the 6th and 7th columns are in T_2.

Using Eqns. 24, 45 and 46 matrices D_r and N_r obtained as:

$$D_r = \begin{bmatrix} 0 & 4 & 0 & 4 & 0 & 1 & 0 & 0 \\ 2 & 2 & 5 & 1 & 4 & 0 & 1 & 0 \end{bmatrix}; \qquad N_r = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 5 & 0 & 5 & 0 & 1 \end{bmatrix}$$

which finally gave the following left coprime $\{D(s), N(s)\}$ with column reduced, monic $D(s)$.

$$D(s) = \begin{bmatrix} 0 & 4 + 4s + s^2 \\ 2 + 5s + 4s^2 + s^3 & 2 + s \end{bmatrix}; \qquad N(s) = \begin{bmatrix} -s & -s \\ 0 & 2 + 5s + 5s^2 + s^3 \end{bmatrix}$$

The minimal state space representation in the form given by Eqn. 20 was obtained as:

$$\left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right] = \left[\begin{array}{ccccc|cc} 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & -4 & 0 & -4 & 0 & 4 & 4 \\ -2 & -2 & -5 & -1 & -4 & 1 & 12 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

with observability indices $v = \{3, 2\}$, where matrices A_0, B_0, C_0 and D_0 were calculated using Eqns. 21, 30, 20 and 29, respectively. The controllability matrix Q_{ca} of the pair $\{A_0, S_a\}$ used in calculating B_0 is:

$$Q_{ca} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 0 & -4 & 0 & 12 & 0 \\ 0 & 0 & 0 & 1 & -1 & -4 & 6 & 11 \\ 1 & 0 & -4 & 0 & 12 & 0 & -32 & 0 \\ 0 & 1 & -1 & -4 & 6 & 11 & -23 & -26 \end{bmatrix}$$

Note that the matrix A_r appears at the bottom of the matrix A_0, and that all $n = 5$ columns of $-A_r$ appear as the first n columns in D_r. As it was explained earlier, this is a direct consequence of the utilized observable form given by Eqns. 19 and 20. To calculate the right coprime MFD $\{N(s), D(s)\}$ with $D(s)$ row reduced and monic as well as a minimal realization in a controllable form, a dual version of the Algorithm was applied. Using the same $G(s)$ given above, the following MFD and $\{A, B, C, D\}$ was obtained:

$$N_c = \begin{bmatrix} 4 & -2 \\ 0 & 0 \\ 5 & -1 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \qquad D_c = \begin{bmatrix} 0 & 2 \\ 4 & -2 \\ 0 & 5 \\ 4 & -1 \\ 0 & 4 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix};$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccccc|cc} 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -4 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & -5 & 0 & 0 \\ 0 & 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 & 0 \\ \hline 0 & 1 & 0 & -4 & 1 & 0 & 1 \\ -1 & -1 & 4 & 4 & -12 & 0 & 0 \end{array}\right]$$

The above given N_c and D_c define the obtained right coprime MFD $\{N(s), D(s)\}$ by:

$$N(s) = I_p(s)^T N_c \quad \text{and} \quad D(s) = I_m(s)^T D_c.$$

Controllability indices as well as row degrees of the MFD are: $v = \{3 \quad 2\}$. Note that the first $n = 5$ rows in D_c are equal to the last $m = 2$ columns in A, multiplied by -1.

CONCLUSIONS

In this paper an extremely simple computational procedure for both coprime factorization and minimal realization of transfer function matrices is proposed. It is based on the Order Theorem derived in the paper, which permits direct determination of the order of a minimal state space representation as well as the observability indices of the utilized observable form.

In calculating the order and observability indices the suggested algorithm operates on a matrix having only $(k+1)p$ rows and rm columns, m and p being dimensions of input and outputs vectors, respectively, while k is the maximum value of the set of observability indices (column degrees) and r is the order of the characteristics polynomial of the given transfer function matrix. In fact, our approach starts from finding a minimal basis for the left null space of $[I_p d(s) : -W(s)^T]^T$ which was characterized in Patel 1981, as an approach "involving operations on matrices whose orders may often be much higher than the dimension of the state space". In our approach, however, we avoided this disadvantage by judicious choice and exploitation of properties of various systems descriptions, as state space, transfer function matrix, MFD's and Markov parameters. By exploiting these properties we arrived at an efficient procedure which proved to be sufficiently accurate and easy to implement.

In this way, from a computational point of view, our procedure approached simplicity of the celebrated Ho-Kalman minimal realization algorithm, which was characterized by Ho & Kalman (1966) as "the simplest method for computing a realization that will ever be found". Since the approach given here calculates not only the minimal state space realization, but also a coprime MFD, it seems that our approach is quite comparable, if not more convenient, than the majority of existing minimal realization and coprime factorization procedures. It is believed that this algorithm will prompt additional research toward development of even simpler algorithms.

Finally, by employing a state representation of the structure given by Eqns. 19 and 20, which is, in a way, very close to a corresponding MFD, the problem of calculating the minimal state space representation given a coprime, column reduced MFD becomes almost trivial. The same applies for the calculation of the coprime MFD given a minimal state space representation. More details on the relationship between state space and MFD descriptions are given in (Bingulac & VanLandingham 1993).

All calculations are performed on an IBM PS/2 using the Control System CAD Package/Language L-A-S (Bingulac 1988; Jamshidi *et al.* 1992).

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طريقة جديدة للتحليل الأولي المساعد وأقل تحقيق لمصفوفة دالة النقل

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خلاصة

يقدم هذا البحث خوارزم حسابي بسيط لحساب وصف كسر المصفوفة الأولي، وأقل عرض لحالة الفضاء الخاصة بالنظام الخطي متعدد المتغيرات، والذي يتم تعريفه بمصفوفة دالة النقل. وقد تم بناء الخوارزم على أساس نظرية تم إستنتاجها في هذه المقالة، وهي تجيز حساب درجة أقل تحقيق وكذلك فهارس المراقبة (تدريج الأعمدة) مباشرة من معاملات مصفوفة دالة النقل المعطاة. وقد تم مراجعة ومقارنة الخوارزمات المعروفة لأقل تحقيق والتحليل الأولي بالطريقة المقترحة. وتم وضع مثال مفصل لعرض كفاءة الطريقة المقترحة.