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Distribution of wave steepness

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ABSTRACT

Available theoretical expressions describing the statistical distribution of wave steepness do not compare favorably with empirical data. The observed histograms of wave steepness tend to fit the log-normal probability law. The practical utility of this result in engineering design requires that the mean and root-mean-square values of wave steepness be predicted accurately. Herein, theoretical approximations describing these two key statistics are derived, and the resulting log-normal distribution is compared with observational data, obtaining good agreement.

1. INTRODUCTION

Most kinematic and dynamic properties of sea waves can be expressed in terms of certain principal wave characteristics such as wave height H, wave length L, and wave period T. For progressive wave trains in deep water, temporal and spatial characteristics are further interrelated by way of a dispersion relation, requiring that $L = gT^2/2\pi$ in which g stands for the gravitational acceleration. Wave steepness is defined as the ratio

$$s = \frac{H}{L} = \frac{2\pi H}{gT^2} \tag{1}$$

This parameter is of both practical and theoretical relevance in maritime engineering and design. In particular, severe sea states are characterized with the occurrence of high waves which tend to be quite steep. The extremal inertia forces induced by such waves on fixed or floating offshore structures, and also wave run-up or overtopping of coastal dikes and breakwaters depend on the steepness of waves. Unfortunately, however, sea waves are random, with individual waves maintaining their identities only for short intervals of time. Accordingly, the basic wave properties vary randomly from one wave to the next. This in turn implies that wave steepness, regarded as a function of wave height and period, is also random, and thus requires a statistical description.

In theory, the statistical distribution of s can be derived by way of a standard transformation of variables from the joint distribution of wave heights and periods. In practice, however, the joint distribution required is not known exactly, but there are several theoretical approximations to it, most notably by Longuet-Higgins (1975,

1983) and Cevanie *et al.* (1976). A systemic analysis of wave steepness based on these approximations has been carried out by Hamadeh (1989). His analysis shows that certain approximations inherent in Longuet-Higgins' (1975, 1983) theories render them quite unsuitable for use in the present context, implying that s must have an unbounded mean-square value. In contrast, the use of the theory of Cevanie *et al.* (1976) leads to a closed form expression describing the probability density of s in the form

$$p(s) = C_1 s \exp(-C_2 s^2)[1 + \mathrm{erf}(C_3 s)] \tag{2}$$

where $s \geqslant 0$ by definition; *erf* stands for the standard error function; and, C_1, C_2, and C_3 are certain parameters that depend on the ordinary moments of the frequency spectrum of waves (Hamadeh 1989). For waves characterized with narrow-banded spectra, Eqn. 2 is simplified further to the Rayleigh form given by

$$p(s) = \frac{2s}{s_{rms}} \exp\left(-\frac{s^2}{s_{rms}^2}\right) \tag{3}$$

where s_{rms} denotes the root-mean-square (*rms*) value of s. Hamadeh shows that the Rayleigh density corresponds to an approximation previously devised by Overnik & Houmb (1977). He then proceeds to compare Eqn. 2 and its narrow-band limit, Eqn. 3, and the associated statistics such as the mean and *rms* steepness values with wave measurements gathered in the Gulf of Mexico during the passage of a severe hurricane. These comparisons by and large indicate that theoretical approximations in the form of either Eqn. 2 or Eqn. 3 fail to represent the actual data with sufficient quantitative accuracy.

Available field data such as those compiled by Myrhaug & Kjeldsen (1984) and Hamadeh (1989) suggest that observed histograms of wave steepness are typically characterized with a single narrow and sharp mode located at lower steepness values than those predicted by theoretical approximations such as Eqn. 2 or 3. Furthermore, the observed mean and *rms* values are invariably much larger than the respective theoretical values depicted by the same equations. These observations coupled with the general trend of the histograms indicate that the log-normal probability law might provide a quantitatively more accurate approximation of the distribution of wave steepness, as has also been suggested by some results of Myrhaug & Kjeldsen (1984).

The log-normal probability density has relative simplicity and is entirely specified or characterized by two parameters, viz. the mean and the *rms* value of the random variate under consideration. If it is accepted that the observed histograms of wave steepness do actually tend to fit the log-normal law, as the available albeit limited evidence suggests, then its practical utility in engineering applications would depend on how accurately the mean and *rms* wave steepness can be predicted for a given sea state. Thus, the present study quite simply explores first whether these two key statistics can be predicted in a closed form with sufficient quantitative accuracy. Indeed, based on certain approximations and given the frequency spectrum of waves, it shall be shown that these statistics can be expressed as simple functions of the moments of the wave spectrum. These functions and the corresponding log-normal predictions implied are then compared with observational data.

2. NARROW-BAND WAVES

Consider long-crested or unidirectional random waves characterized by narrow-banded spectra $F(\omega)$ over frequency ω in rad/s and $W(k)$ over wave number k in rad/m. The ordinary spectral moments are defined by

$$m_j = \int_0^\infty \omega^j F(\omega)\, d\omega \tag{4}$$

$$\mu_j = \int_0^\infty k^j W(k)\, dk \tag{5}$$

where $j = 0, 1, 2, \ldots$. Since the frequencies and wave numbers are related via the dispersion relation $\omega^2 = gk$, it can be shown that (Tayfun 1986)

$$m_{2j} = g^j \mu_j \tag{6}$$

Finally, a relative measure of spectrum bandwidth is given by (see e.g. Longuet-Higgins 1975)

$$\epsilon^2 = 1 - \frac{m_2^2}{m_0 m_4} \tag{7}$$

In general, $0 \leqslant \epsilon^2 \leqslant 1$: If $\epsilon^2 \ll 1$, then the spectrum, F or W, is said to be narrow-banded. Conversely, if $\epsilon^2 \approx 1$, then the spectrum is considered as wide-banded.

Let the wave elevation or displacement from the mean sea level be denoted by $\eta(x, t)$, where the x axis is coincident with the mean level and t is time. Assume that η is a spatially and temporally homogeneous random function. Thus, whenever it is viewed at a fixed $t = t_0$ as a function of x, there is no loss of generality in setting $t_0 = 0$, in which case η can be expressed as a function of x in the form (Tayfun 1986)

$$\eta = A \cos(\bar{k}x + \phi) \tag{8}$$

where $k = \mu_1/\mu_0 = m_2/gm_0$ corresponds to the spectral "mean" wave number; and, A and ϕ are, respectively, the random wave amplitude and phase functions. The displacement η is Gaussian, provided that A is Rayleigh-distributed, and if ϕ is specified to be uniformly distributed over the interval $(0, 2\pi)$.

Under the narrow-band assumption, the wave height is given by

$$H \approx 2A \tag{9}$$

with the corresponding wave length

$$L \approx \frac{2\pi}{\phi_x + \bar{k}} \tag{10}$$

in which the subscript denotes partial differentiation with respect to x. With the preceding definitions at hand, the wave steepness can now be expressed in the form

$$s = \frac{H}{L} \approx \frac{1}{\pi} A(\phi_x + \bar{k}) \tag{11}$$

3. MEAN AND RMS STEEPNESS

The joint probability density of A and ϕ_x is of the form (see for example Tayfun 1986; Hamadeh 1989)

$$p(A, \phi_x) = \frac{A^2}{\sqrt{2\pi\mu_0}\Delta} \exp\left[-\frac{\mu_0}{2\Delta^2} A^2 \left(\phi_x^2 + \frac{\Delta^2}{\mu_0^2}\right)\right] \tag{12}$$

in which

$$A \geqslant 0, \qquad -\infty < \phi_x < \infty,$$

and

$$\Delta^2 = \mu_0 \mu_2 - \mu_1^2 = \frac{\epsilon^2}{1-\epsilon^2} \bar{k}^2 m_0^2 \tag{13}$$

The last identity follows from Eqns. 6 & 7.

The density of s defined by Eqn. 11, valid for unidirectional waves, follows from the preceding joint density via a straightforward change of variables. Provided that $0 \leqslant \epsilon < 1$, this density can be shown to be in the form

$$p(s) = A_0 \exp(-A_1 s^2)\{1 + A_2 s \exp(-A_3 s^2)[1 + \mathrm{erf}(A_4 s)]\} \tag{14}$$

where $s \geqslant 0$ as before, and A's are parameters that depend on the spectral moments and ϵ^2.

The preceding density is similar to Eqn. 2, and reduces to the Rayleigh form given by Eqn. 3 for s/s_{rms} as $\epsilon^2 \rightarrow 0$. It is not valid for the extreme wide-band limit $\epsilon^2 = 1$, for which s_{rms} becomes unbounded. Furthermore, it is noted that while Eqns. 2 & 3 relate to s defined in terms of H and T based on observations at a fixed point as a function of time, Eqn. 14 relates to H and L observed as a function of space at a fixed time. Unfortunately, wave observations are invariable gathered via probes fixed in space. Furthermore, the correspondence between the properties L and T is in analogy with the deterministic sinusoidal and unidirectional classical wave concepts. For actual sea waves, no such correspondence really exists. Thus, one is faced with the problem of predicting the nature of s in the manner defined by Eqn. 1, which is valid only the strictly narrow-band limit $\epsilon^2 \rightarrow 0$ and is rather heuristic for random wind seas for which $\epsilon^2 \approx 0.5$–0.6 typically. It is therefore not surprising that various theoretical expressions such as Eqns. 2, 3 & 14 do not do well in representing the observed values of wave steepness.

As a more practical alternative, one can perhaps hope to approximate the observed data by a known functional form involving a few free parameters to be chosen, so as to fit the data in terms of certain key properties, such as the mean and mean-square values. Provided that this approach works, the main object then becomes whether such key properties can be predicted in a closed form with sufficient accuracy. This is in essence the principal motivation of the present study.

Hamadeh (1989) has shown that the mean $\langle s \rangle$ and mean-square $\langle s^2 \rangle$ implied by Eqns. 2 & 3 tend to underestimate the observed values by as much as 20–25%. In contrast, the corresponding statistics to be obtained via Eqn. 13 in what follows shall be shown to do considerably better. To be more specific, the mean and mean-square of s are given by

$$s_m = \langle s \rangle \approx \frac{1}{\pi} \langle A(\phi_x + \bar{k}) \rangle \tag{15}$$

$$s_{rms}^2 = \langle s^2 \rangle \approx \frac{1}{\pi^2} \langle A^2(\phi_x + \bar{k})^2 \rangle \tag{16}$$

The statistical averages required on the right sides of the preceding equations shall follow after some algebra from Eqns. 12 & 13 as

$$\langle A(\phi_x + \bar{k}) \rangle = \frac{\sqrt{\pi}}{2} \alpha \tag{17}$$

$$\langle A^2(\phi_x + \bar{k})^2 \rangle = \left(1 + \frac{1}{2} \frac{\epsilon^2}{1 - \epsilon^2}\right) \alpha^2 \tag{18}$$

where $\epsilon^2 \neq 1$ and

$$\alpha = A_{rms} \bar{k} = \frac{m_2}{g} \sqrt{\frac{2}{m_0}} \tag{19}$$

with $A_{rms} = (2m_0)^{1/2}$ representing the *rms* value of A. These results coupled with the narrow-band assumption can now be used in Eqns. 15 & 16 to show that the mean and *rms* values of s are given by

$$s_m \approx \frac{\alpha}{2\sqrt{\pi}} \tag{20}$$

$$s_{rms} \approx \frac{\alpha}{\pi} (1 + \tfrac{1}{4}\epsilon^2 + \cdots) \approx \frac{\alpha}{\pi} \tag{21}$$

correct to $O(\epsilon)$.

4. LOG-NORMAL APPROXIMATION

The probability density of s shall be approximated by

$$p(s) = \frac{1}{\sqrt{2\pi}\sigma s} \exp\left[-\frac{1}{2}\left(\frac{\ln s - \mu}{\sigma}\right)^2\right] \tag{22}$$

where $s \geqslant 0$, and

$$\sigma^2 = \text{var}\,(\ln s) \tag{23}$$

$$\mu = \langle \ln s \rangle \tag{24}$$

designate, respectively, the variance and mean of $\ln(s)$, which can be rewritten in terms of s_m and s_{rms} as

$$\sigma^2 = 2 \ln\left(\frac{s_{rms}}{s_m}\right) \tag{25}$$

$$\mu = \ln s_m - \tfrac{1}{2}\sigma^2 \tag{26}$$

In view of Eqns. 20 & 21, these assume the following forms:

$$\sigma^2 \approx \ln\left(\frac{4\pi}{\pi}\right) \tag{27}$$

$$\mu \approx \ln\left(\frac{\alpha}{4}\right) \tag{28}$$

As a final step, let

$$s' = \frac{s}{s_{rms}} = \frac{\pi}{\alpha} s \tag{29}$$

The preceding change of variable together with Eqns. 27 & 28 substituted in Eqn. 22 shall give the density sought as

$$p(s') \approx \frac{1}{\sqrt{2\pi}\sigma' s'} \exp\left[-\frac{1}{2}\left(\frac{\ln s' - \mu'}{\sigma'}\right)^2\right] \tag{30}$$

where $s' \geqslant 0$ obviously, and

$$\sigma'^2 = \text{var}\,(\ln s') = -\mu' \tag{31}$$

$$\mu' = \langle \ln s' \rangle = \ln\left(\frac{\pi}{4}\right) \tag{32}$$

It is noticed that whereas the log-normal density approximation to $p(s)$ is dependent on α, the density $p(s')$ of the scaled steepness s' has an invariant log-normal form free of any wave field parameters.

5. COMPARISONS

In what follows, the relative validity of Eqns. 19 & 20, and the suggested log-normal density approximation in the invariant form of Eqn. 30 shall be checked with observational data and also with the previous theoretical approximations, viz. Eqns. 2 & 3.

The data used in the present comparisons were collected as time series of water surface elevations during the Ocean Data Gathering Program in the Gulf of Mexico in 1969 with a wave staff mounted within the legs of an oil rig (station 1—South Pass 62A). The particular segment of interest here was recorded during the passage of hurricane Camille, lasting for a period of 15.5 hours until the failure of the wave staff due to wave action. The hurricane was a small but intense storm, generating waves of 20–22 m in height as it approached the measurement site. The same data have previously been analyzed by Hamadeh (1989), dividing it into fifteen slightly overlapping segments. Each of these comprises $2^{12} = 4096$ data points at a uniform spacing of 1 s, and starts at 1-hour intervals coincident with the beginning of 15 hourly files entitled camwave.nn, where $nn = 01, 02, \ldots, 15$. Every data segment was spectrally analyzed (Hamadeh 1989), and the spectral moments m_0, m_2, m_4, and thereby the parameters ϵ^2 and α required in Eqns. 27 & 28 were calculated. These and the wave count n_i, i.e. the number of zero upcrossing waves counted within each data segment are tabulated in Table 1.

Notice first that the observed ϵ^2 values indicate that the narrow-band assumption $\epsilon^2 \ll 1$ is not really valid for any data segment. Further note that m_0, which is proportional to the mean total wave energy per unit horizontal area, increases nearly

Table 1. Observed properties of Camille wave data.

Data segment (camwave.nn)	Wave count n_i	m_0 (m^2)	m_2 (m^2/s^2)	m_4 (m^2/s^4)	ϵ^2	α
1	615	0.686	0.679	1.448	0.536	0.118
2	636	0.706	0.744	1.564	0.498	0.128
3	601	0.840	0.803	1.596	0.518	0.126
4	582	0.920	0.848	1.676	0.534	0.128
5	580	1.038	0.899	1.650	0.527	0.127
6	567	1.234	1.004	1.746	0.531	0.130
7	556	1.516	1.125	1.775	0.530	0.132
8	503	1.898	1.213	1.765	0.560	0.127
9	485	2.601	1.523	2.025	0.560	0.136
10	458	3.463	1.881	2.350	0.566	0.146
11	430	5.663	2.521	2.655	0.561	0.153
12	421	6.111	2.622	2.750	0.590	0.153
13	396	7.283	3.064	3.201	0.596	0.164
14	434	7.634	3.602	3.948	0.569	0.118
15	419	10.128	4.502	4.766	0.578	0.204

15 fold from a relatively small value (0.686 m^2) for data segment 1 to a very large value (10.128 m^2) for the segment 15 coincident with the approach of Camille toward the site. Though not shown in Table 1, the corresponding mean wave periods (given by $4096/n_i$ very nearly) also increase from 6–7 s to 9–10 s. All this implies first that the wave field around the measurement site had progressively built up, reaching an extreme prior to the failure of the wave staff. More significantly, the observed mean-period range suggests that the sampling rate employed (i.e. 1 sample per second) may not be sufficiently large for determining the wave heights and periods with precision. Inadequate sampling causes wave heights to be underestimated, and the attendant aliasing tend to make observed periods appear larger than they really are (Tayfun 1993). The ensuing errors are approximately proportional to the square of the ratio of the sampling interval employed to the mean wave period of the data segment. In the present cases, this ratio ranges from 1/6 for the initial data segments to 1/10 for the data segments 11–15. Thus, the latter segments are of central importance here not only because their properties are likely to be more representative of extreme waves of engineering interest, but also because they are relatively less affected by the sampling-rate aliasing errors. Since s is directly proportional to wave heights and also inversely proportional to wave periods squared, the ultimate effect of such errors would be to make the observed s values appear less than the actual ones, particularly, within the leading data segments.

The observed mean and *rms* values of s and their theoretical predictions computed by way of Eqns. 27 & 28 are given in Table 2. Also shown in the same table are the observed values of the ratio s_m/s_{rms}. Theoretically, this ratio follows from Eqns. 31 & 32 as a constant given by $s_m/s_{rms} = \pi^{1/2}/2 = 0.886$. It is seen that the observed values in column 4 lie within ±2.5% of this theoretical ratio. Similarly, the observed and predicted s_m and s_{rms} values compare quite favorably, within ±5% in all cases for s_m and within ±7% for s_{rms}.

To provide a typical comparison between the theoretical approximations of Hamadeh (1989) and the present study relative to the observed data, the density functions implied are plotted in Figs. 1 & 2 together with the observed steepness

Table 2. Observed steepness statistics versus theoretical predictions.

Data segment (camwave.nn)	Observed values			Theoretical values	
	s_m ($\times 10^4$)	s_{rms} ($\times 10^4$)	s_m/s_{rms}	s_m ($\times 10^4$)	s_{rms} ($\times 10^4$)
01	341	403	0.846	333	376
02	367	425	0.864	361	407
03	360	416	0.865	355	401
04	351	405	0.866	361	407
05	359	414	0.867	358	404
06	366	418	0.876	366	414
07	377	425	0.887	372	420
08	358	410	0.873	358	404
09	405	466	0.869	384	433
10	412	466	0.884	412	465
11	443	501	0.884	432	487
12	447	508	0.880	432	487
13	465	537	0.866	462	522
14	534	599	0.891	530	598
15	551	612	0.900	575	649

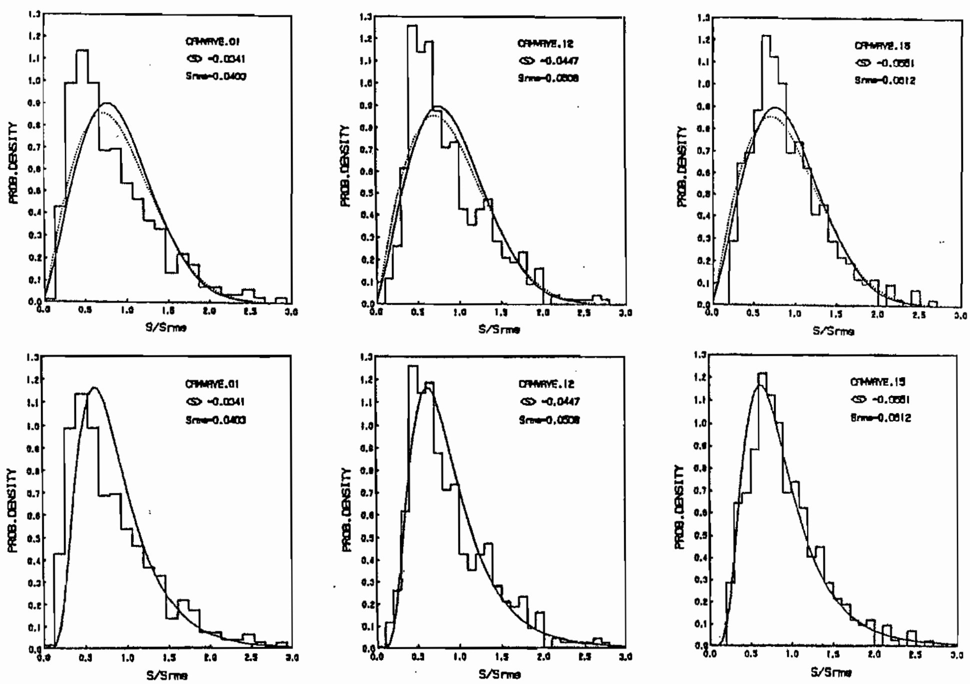

Fig. 1. The density functions against wave steepness ratio for data segments of 1, 12, 15 of Hurricane Camille. Upper parts: show comparison between observed histograms, theoretical prediction (solid curve), and the Rayleigh density (dotted curve).

histograms appropriate to data segments 1, 12, and 15. The top portion of the figure illustrates the comparison between the observed histograms and the theoretical predictions that follow from Eqn. 2 shown as continuous curves, and also the Rayleigh density (Eqn. 3) indicated by a dotted curve. The lower part of the figure

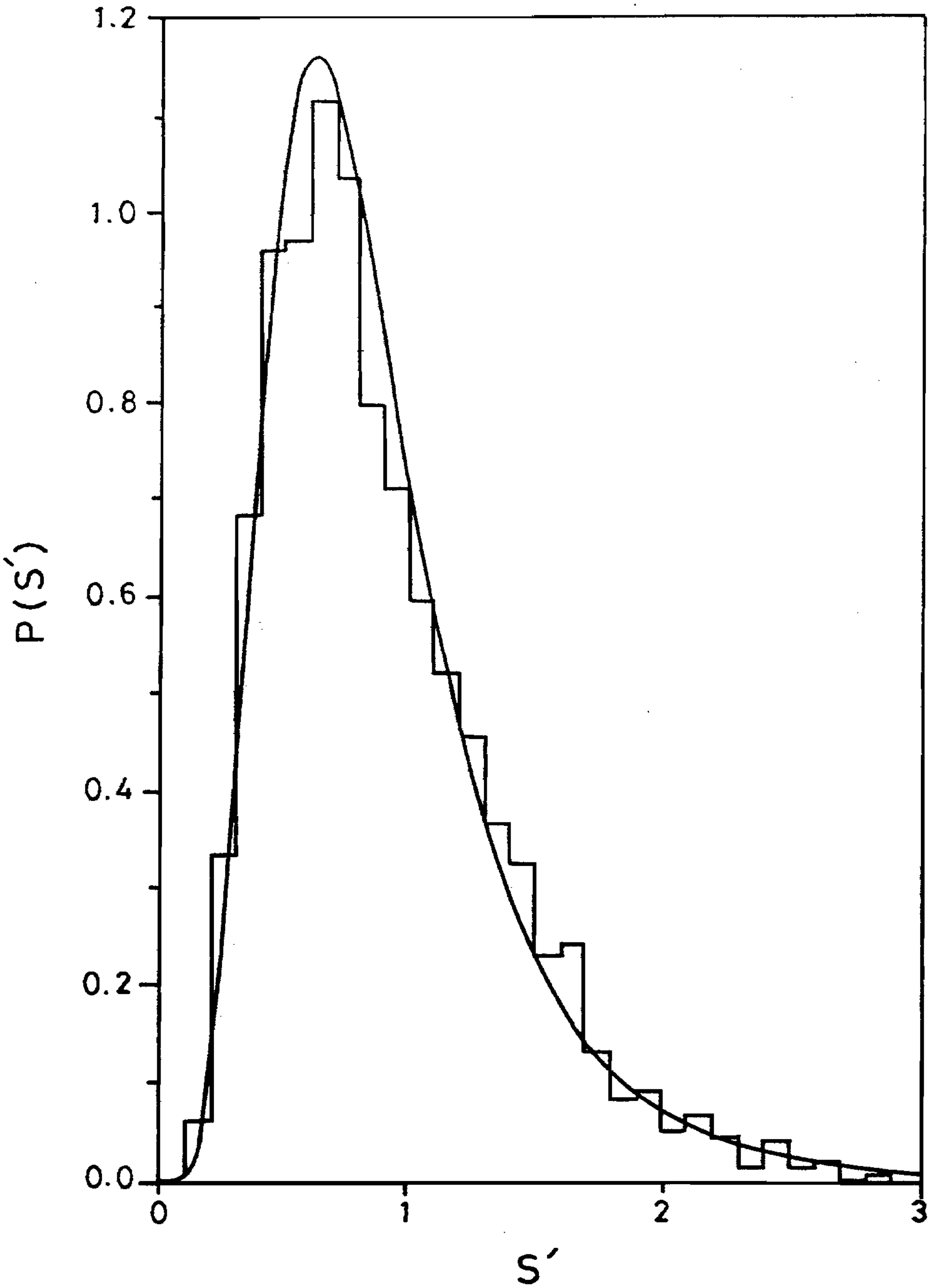

Fig. 2. Composite histograms and log-normal (continuous curve) of the probability density functions vs. central steepness values.

gives the same data in comparison with Eqn. 30 of the present study. On the whole, it appears that the present approximation represents the observed data far better than either of the two previous theoretical approximations implied by Eqns. 2 & 3.

Finally, the invariant form of Eqn. 30 implies that it can also be compared with composite data consisting of different data segments, provided that sample points are scaled with respect to their own segmental *rms* value. For a given class interval characterized by the central steepness value s', the histogram ordinate of the composite data comprising N data segments must then be generated as a weighted average of the segmental histograms in the form

$$p(s') = \frac{\sum_{i=1}^{N} n_i p_i(s')}{\sum_{i=1}^{N} n_i} \tag{33}$$

where p_i stands for the histogram ordinate appropriate to the ith data segment, and n_i is the corresponding wave count. Assuming that the constituent data segments are statistically independent, then this approach serves to increase the underlying sampling population N times, thereby improving the stability of the resulting histogram ordinate $p(s')$ by a factor of $N^{1/2}$. As a specific case of interest here and also, to illustrate the preceding arguments more explicitly, the histograms of the last five data segments 11 through 15 were utilized in conjunction with Eqn. 33 to generate the composite histogram shown in Fig. 2. The log-normal approximation is also shown in the same figure as a continuous curve. It is observed that the stability of histogram ordinates, and the correspondence between them and the log-normal approximation have improved noticeably.

6. CONCLUSIONS

Existing theoretical approximations describing the statistical distribution of wave steepness do not compare well with the empirical data. The observed histograms of wave steepness tend to follow the log-normal probability law. The present analysis and results reinforce this conclusion. However, the practical utility of the log-normal approximation in engineering design requires that the mean and *rms* values of wave steepness be predicted with reasonable accuracy. Based on the narrow-band assumption, the present study developed two theoretical expression, viz. Eqns. 20 & 21, for describing these key statistics in terms of the simple moments m_0 and m_2 of the wave spectrum. These appear to be able to represent the empirical data quite well, particularly, in view of the extremely severe and relatively wide-banded nature of the storm-generated waves considered.

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توزيع الإنحدار الحاد للموجة

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خلاصة

إن جميع الصيغ النظرية المتوفرة والتي تصف توزيع الإنحدار الحاد للموجة لا تقارن تفضيلا مع القراءات العملية. فملاحظة القراءات الهيستوغرامية لإنحدار الموجة تميل إلى التناسب مع قانون الإحتمال العادي ـ اللوغاريثمي. والإستخدام العلمي لهذه التنمية في التصميم الهندسي يتطلب حسابات دقيقة للمعدل المتوسط وجذر المتوسط التربيعي لقيم الإنحدار الحاد للموجة. وقد تم في هذه الورقة إشتقاق حسابات تقريبية تصف هاتين المعادلتين الإحصائية، ومقارنة نتائج التوزيع العادي ـ اللوغاريثمي مع القراءات العملية والتي كانت متطابقة بشكل جيد.