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Subsets of nearly commuting projections

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ABSTRACT

It is well known that projection operators are typical elements in Boolean algebras, where a number of relevant theorems are proved for commutative projections. In the present paper we propose an extension of the concept of commutativity, which we call near commutativity. We extend to this concept the main theorems on commutative projections, and in various ways we frame the class of nearly commutative projections in Boolean algebras.

1. INTRODUCTION

Let V be a vector space over $\mathbb{C}$. Two linear projections p and q, operating on V, are said to *nearly commute* if

$$pqp = qp \text{ and } qpq = pq. \tag{1}$$

If $f(z_1, \ldots, z_n)$ is an element of the function space, $V = \{f : \mathbb{C}^n \to \mathbb{C}\}$, then projections defined by substituting constants for some or all the arguments $z_1, \ldots, z_n$, in general, nearly commute.

Wilde (1993) displays several properties of nearly commuting, linear projections. The present paper displays various subsets of nearly commuting projections. Some of the subsets are ranges of projection operators while others are not.

Section 2 defines X as a set of pairwise nearly commuting, linear projections on V, closed under three sets of operations. Eqns. 2–9 give eight projection operators on X. Theorem 1 characterizes their ranges in X. Also, equations 10–12 define operations on these operators that form out of them a Boolean algebra.

Section 3 has several theorems giving necessary and sufficient conditions for when ranges of these eight operators 2–9 are contained in each other.

Section 4 discusses conditions when operations on X have distributive, self-distributive, and associative properties. Two error terms occur which vanish on subsets of X for some of the laws to hold.

Section 5 contains theorems which motivate Section 6. In Section 6, two theorems result in necessary and sufficient conditions for the error terms, mentioned before, to vanish. The result is two conditions, labeled (*) and (**). Theorem 8 characterizes projections in X (not necessarily commuting) that satisfy those conditions. Section 7 displays matrices satisfying either (*), (**), or both.

Note that Wilde (1993) gives a matrix representation for *two* nearly commuting, linear projections. A canonical representation for n nearly commuting projections is not known. However, n matrices with a canonical matrix form satisfying (*) and (**) are displayed in Section 7.

Before continuing, note that, by Wilde (1993), if $p_1, \ldots, p_n, q$ are pairwise nearly commuting linear projections on V, then q nearly commutes with $p_1 \ldots p_n$.

2. EIGHT PROJECTION OPERATORS

We display four levels of concepts in this section:

(i) a vector space V over $\mathbb{C}$, which can be a function space over $\mathbb{C}$, in Wilde 1993;
(ii) a set of projections on V;
(iii) operators on those projections; and
(iv) operations on those operators.

We will describe (ii)–(iv) in turn.

To derive the operators in (ii) let $c \in \mathbb{C}$ be a scalar. Let p, q, and r be pairwise nearly commuting, linear projections on V. Let $E = \frac{1}{2}(pq + qp)$ and $N = \frac{1}{2}(pq - qp)$. Then $E^2 = E$ and $N^2 = 0$. Also, $p + cN$, $E + cN$, and $p + q - E + cN$ are projections on V which nearly commute with r. Also, by Theorems 1 and 3 in Wilde (1993),

(i) $Ran(p + cN) = Ran\, p$;
(ii) $Ran(E + cN) = Ran\, pq = Ran\, p \cap Ran\, q$;

and

(iii) $Ran(p + q - E + cN) = Ran(p + q - pq) = Ran\, p + Ran\, q$.

For these reasons, let X be a set of pairwise nearly commuting, linear projections on V, closed under the operations $p + cN$, $E + cN$, and $p + q - E + cN$ for all scalars $c \in \mathbb{C}$. We will be dealing with X throughout the paper.

Letting c equal 1 or -1 and $q = x$, we derive projection operators on X:

$$F_p(x) = x - px + xp; \tag{2}$$

$$H_p(x) = p + px - xp; \tag{3}$$

$$E_p(x) = p + x - xp; \tag{4}$$

$$J_p(x) = p + x - px; \tag{5}$$

$$L_p(x) = px; \tag{6}$$

$$R_p(x) = xp; \tag{7}$$

$$K_p(x) = p; \text{ and} \tag{8}$$

$$I_p(x) = x. \tag{9}$$

We motivate these operators. A set of *commuting* projections form a Boolean algebra with the operations pq (meet), $p + q - pq$ (join), and $I - p$ (complement).

Since $p, x \in X$ do not necessarily commute, there are two meets (6) and (7), and two joins (4) and (5). Operators (8) and (9) are trivial. Operators (2) and (3) are projections in X plus or minus the commutator. They are new and do not arise immediately from the commutative situation.

As operators on $x \in X$, each operator has a range on X, characterized as follows.

Theorem 1. (a) $x \in Ran\, F_p$ if and only if $px = xp$;
(b) $x \in Ran\, H_p$ if, and only if, $px = x$ and $xp = p$;
(c) $x \in Ran\, E_p$ if, and only if, $xp = p$;
(d) $x \in Ran\, J_p$ if, and only if, $px = xp = p$;
(e) $x \in Ran\, L_p$ if, and only if, $px = x$; and
(f) $x \in Ran\, R_p$ if, and only if, $px = xp = x$.

Let us define three operations on these operators. Let f and g denote any two of them. Define *meet* by

$$(f^\wedge g)(x) = f(g(x)); \tag{10}$$

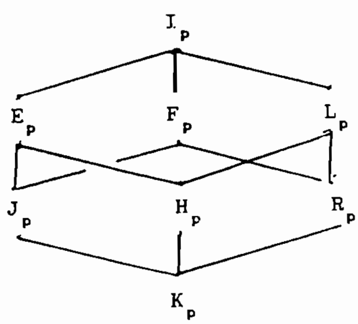
define *join* by

$$(f_\vee g)(x) = f(x) + g(x) - f(g(x)); \tag{11}$$

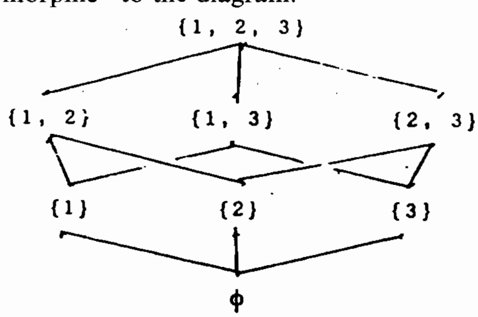
and define *complement* by

$$f'(x) = p + x - f(x). \tag{12}$$

On these operators, meet and join are commutative. Indeed, the operators (2)–(9) form a Boolean algebra with this lattice diagram:



This diagram is "isomorphic" to the diagram:



with union, intersection, and complement.

Among the eight operators, H_p is special. Wilde (1993) proves that the relation on X, defined by the condition "$pq = q$ and $qp = p$", is an equivalence relation. The equivalence classes are the sets $Ran\,H_p$ in X. (See Theorem 1(b).) Thus $Ran\,H_p = Ran\,H_q$ if, and only if, $q \in Ran\,H_p$. From now on, we say that $p, q \in X$ are equivalent if $pq = q$ and $qp = p$.

Note, too, that if $p, q, r \in X$, then pq and $p + q - pq$ are associative, while $p + pq - qp$ is not, in general.

3. CONTAINMENT THEOREMS

We state theorems about when certain ranges are contained in each other. In these theorems, let p, q, x be elements of X.

Theorem 2. The following statements are equivalent:

(1) $q \in Ran\,H_p$;
(2) $Ran\,E_p = Ran\,E_q$;
(3) $Ran\,L_p = Ran\,L_q$.

Proof. (1) $\Rightarrow$ (2): Firstly, $q \in Ran\,H_p$ implies $pq = q$ and $qp = p$. If $x \in Ran\,E_p$, then $xp = p$. So $xq = x(pq) = (xp)q = pq = q$, implying $x \in Ran\,E_q$. Similarly, $x \in Ran\,E_q$ implies $x \in Ran\,E_p$.

(2) $\Rightarrow$ (1): Suppose $Ran\,E_p = Ran\,E_q$. Then $p \in Ran\,E_p$ implies $p \in Ran\,E_q$, so $pq = q$. Also, $q \in Ran\,E_q$ implies $q \in Ran\,E_p$, so $qp = p$. Thus $q \in Ran\,H_p$.

(1) $\Rightarrow$ (3): Suppose $q \in Ran\,H_p$. Then $pq = q$ and $qp = p$. If $x \in Ran\,L_p$, then $px = x$. So $qx = q(px) = (qp)x = px = x$, implying $x \in Ran\,L_q$. Similarly, $x \in Ran\,L_q$ implies $x \in Ran\,L_p$.

(3) $\Rightarrow$ (1): Suppose $Ran\,L_p = Ran\,L_q$. Then $p \in Ran\,L_p$ implies $p \in Ran\,L_q$, so $qp = p$. Also, $q \in Ran\,L_q$ implies $q \in Ran\,L_p$, so $pq = q$. Thus $q \in Ran\,H_p$. Q.E.D.

The following theorem is an inclusion instead of an equality.

Theorem 3. The following statements are equivalent:

(1) $q \in Ran\,E_p$;
(2) $Ran\,E_q \subset Ran\,E_p$;
(3) $Ran\,L_p \subset Ran\,L_p$; and
(4) $p \in Ran\,L_q$.

Proof. (1) $\Rightarrow$ (2): Suppose $q \in Ran\,E_p$. Then $qp = p$. Let $x \in Ran\,E_q$. Then $xq = q$, so $xp = x(qp) = (xq)p = qp = p$, and $x \in Ran\,E_p$.

(2) $\Rightarrow$ (1): Suppose $Ran\,E_q \subset Ran\,E_p$. Since $q \in Ran\,E_q$, $q \in Ran\,E_p$.

(1) $\Rightarrow$ (3): Suppose $q \in Ran\,E_p$. Then $qp = p$. Let $x \in Ran\,L_p$. Then $px = x$. So $qx = q(px) = (qp)x = px = x$, and $x \in Ran\,L_q$.

(3) $\Rightarrow$ (1): Suppose $Ran\,L_p \subset Ran\,L_q$. Since $p \in Ran\,L_p$, then $p \in Ran\,L_q$. So $qp = p$. Thus $q \in Ran\,E_p$.

(1) $\Leftrightarrow$ (4): $q \in Ran\, E_p$ if, and only if, $qp = p$. This holds if, and only if, $p \in Ran\, L_q$. Q.E.D.

The properties of L_p and E_p in Theorem 3 also hold of R_p and J_p except that $Ran\, R_p = Ran\, R_q$ (and $Ran\, J_p = Ran\, J_q$) if, and only if, $p = q$. $Ran\, F_p = Ran\, F_q$ has a different equivalent condition.

Theorem 4. $Ran\, F_p = Ran\, F_q$ if and only if $F_p(x) = F_q(x)$ for all $x \in X$.

Proof of ($\Rightarrow$): Suppose $Ran\, F_p = Ran\, F_q$. Since $q \in Ran\, F_q$, $q \in Ran\, F_p$. So $pq = qp$. By Wilde 1993, $pq = qp$ implies $F_p F_q = F_q F_p$. But, since $Ran\, F_p = Ran\, F_q$, $F_p F_q = F_q$ and $F_q F_p = F_p$. So $F_p = F_q$, i.e., $F_p(x) = F_q(x)$ for all $x \in X$.

Proof of ($\Leftarrow$): Easy. Q.E.D.

4. DISTRIBUTIVE AND ASSOCIATIVE PROPERTIES

Three operations on X have distributive and associative properties for p, q, r, in all of X or in subsets of X. Let us define the operations: $p^\wedge q = pq$ (meet), $p_\vee q = p + q - qp = E_p(q)$ (join), and $p \circ q = p + pq - qp$ (circle). If $p,\ q,\ r \in X$, then the following equations hold:

$$(p_\vee q) \wedge r = (p^\wedge r) \vee (q^\wedge r); \tag{13}$$

$$p \wedge (q_\vee r) = (p^\wedge q) \vee (p^\wedge r) + (rp - pr)q; \tag{14}$$

$$(p^\wedge q) \vee r = (p_\vee r) \wedge (q_\vee r); \tag{15}$$

$$p \vee (q^\wedge r) = (p_\vee q) \wedge (p_\vee r) + (I - q)(rp - pr); \tag{16}$$

$$(p \circ q) \wedge r = (p^\wedge r) \circ (q^\wedge r); \tag{17}$$

$$p \wedge (q \circ r) = (p^\wedge q) \circ (p^\wedge r) + (pq - qp)r + (rp - pr)q; \tag{18}$$

$$(p \circ q) \vee r = (p_\vee r) \circ (q_\vee r); \tag{19}$$

$$p \vee (q \circ r) = (p_\vee q) \circ (p_\vee r) + (I - r)(pq - qp) + (I - q)(rp - pr); \tag{20}$$

$$(p^\wedge q) \circ r = (p \circ r) \wedge (q \circ r); \tag{21}$$

$$p \circ (q^\wedge r) = (p \circ q) \wedge (p \circ r) + (pq - qp)r + (I - q)(rp - pr); \tag{22}$$

$$(p_\vee q) \circ r = (p \circ r) \vee (q \circ r); \tag{23}$$

$$p \circ (q^\wedge r) = (p \circ q) \vee (p \circ r) + (I - r)(pq - qp) + (rp - pr)q. \tag{24}$$

The three operations were chosen so that all right distributive laws hold. Note that the operations are not commutative in all of X. The left distributive laws have "error" terms like $(pq - qp)r$ and $(I - q)(rp - pr)$. These error terms must vanish in order for those distributive laws to hold. In Section 6, we prove two theorems, stating on what subsets of X these terms vanish; and later we display matrices p, q, r on which these terms vanish.

In the meantime, we note self-distributive properties:

$$(p^{\wedge}q) \wedge r = (p^{\wedge}r) \wedge (q^{\wedge}r); \tag{25}$$

$$p \wedge (q_{\vee}r) = (p^{\wedge}q) \wedge (p^{\wedge}r) + (pq - qp)r; \tag{26}$$

$$(p_{\vee}q) \vee r = (p_{\vee}r) \vee (q_{\vee}r); \tag{27}$$

$$p \vee (q_{\vee}r) = (p_{\vee}q) \vee (p_{\vee}r) + (I - r)(pq - qp); \tag{28}$$

$$(p \circ q) \circ r = (p \circ r) \circ (q \circ r); \tag{29}$$

$$p \circ (q \circ r) = (p \circ q) \circ (p \circ r) + (qr - rq)p + (I - r)(pq - qp) + (I - q)(rp - pr). \tag{30}$$

Note that intersection and union are self-distributive in set theory because they are commutative and idempotent. Meet, join, and circle are idempotent, but not commutative. Also, the error terms occur here, too.

Now, if $p, q, r \in X$, then

$$p \wedge (q \wedge r) = (p \wedge q) \wedge r; \tag{31}$$

$$p \vee (q \vee r) = (p \vee q) \vee r; \tag{32}$$

$$p \circ (q \circ p) = (p \circ q) \circ p = p; \text{ and} \tag{33}$$

$$p \circ (q \circ r) = (p \circ q) \circ r + (rp - pr)q + (I - q)(rp - pr). \tag{34}$$

In Eqn. 34, the error terms occur again. Eqn. 33 is similar to the conditions of a Moore-Penrose inverse.

5. MISCELLANEOUS THEOREMS

We shall prove three theorems that will be used later. First, we prove the following for $p, q, x \in X$.

Theorem 5. $F_p = F_q$ if and only if $pq = qp$ and $(p + q - 2pq)x = x(p + q - 2pq)$ for all $x \in X$.

Proof of ($\Rightarrow$). *Suppose* $F_p = F_q$. *Then*

(i) $(p - q)x = x(p - q)$

for all values of x. Let $x = q$. Then

(ii) $pq = qp$.

Multiplying both sides of (i) on the left by $p - q$, we obtain

(iii′) $(p - q)^2 x = (p - q)x(p - q)$.

Multiplying both sides of (i) on the right by $p - q$, we obtain

(iii″) $(p - q)x(p - q) = x(p - q)^2$.

So

(iv) $(p - q)^2 x = x(p - q)^2$.

But $(p-q)^2 = p+q-2pq$. Thus

(v) $(p+q-2pq)x = x(p+q-2pq)$.

Proof of ($\Leftarrow$). Suppose (ii) and (iv) hold. Note that $(p-q)^3 = p-q$. Multiplying both sides of (iv) on the left by $p-q$, we obtain

(vi) $(p-q)^3x = (p-q)x(p-q)^2$.

Multiplying both sides of (iv) on the right by $p-q$, we obtain

(vii) $(p-q)^2x(p-q) = x(p-q)^3$.

But

(viii) $$\begin{cases} (p-q)^2x(p-q) = (p-q)x(p-q)^2 \\ = xp - xq + pxq - qxp. \end{cases}$$

So

(ix) $(p-q)^3x = x(p-q)^3$,

or

(x) $(p-q)x = x(p-q)$.

Thus $F_p = F_q$. Q.E.D.

If $p, q \in X$, then $(p-q)^2 = p+q-pq-qp$ in general. Using a method of proof, similar to "Proof of ($\Rightarrow$)" in Theorem 5, we can show the following.

Theorem 6. Let $p, q, x \in X$. If $F_p(x) = F_q(x)$, then $x \in Ran\, F_{p+q-pq-qp}$.

6. SUBSETS OF X

Note that if $p, q, x \in X$, x does not always nearly commute with $p+q-pq-qp$, which appeared in Theorem 6. The next theorem says when it does.

Theorem 7. The following statements are equivalent:

(1) x nearly commutes with $p+q-pq-qp$;
(2) $pxq + qxp = xpq + xqp$;
(3) $pxq = xpq$ and $qxp = xqp$.

Proof (1) $\Leftrightarrow$ (2). Calculation.

Proof (2) $\Rightarrow$ (3). Multiply both sides of (2) on the right by q. Then $pxq + xpq = xpq + xpq$, or $pxq = xpq$. The other equation is immediate.

Proof (3) $\Rightarrow$ (2). Easy. Q.E.D.

For the rest of this paper, let $pVq = p+q-pq$.

By Theorem 7, if $p, q, r \in X$, then p nearly commutes with $q+r-qr-rq$; q, with

$p+r-pr-rp$; and r, with $p+q-pq-qp$ if and only if the following condition holds:

$$(*)\qquad \begin{cases} pqr = qpr, \\ rpq = prq, \text{ and} \\ qrp = rqp. \end{cases}$$

By induction, if $p_1, \ldots, p_n, q \in X$ and they triplewise satisfy (*), then all products with any permutation of $p_1, \ldots, p_n$ and with q on the right are equal.

Similarly, if p, q, $r \in X$, then p nearly commutes with $I-q-r+qr+rq$; q, with $I-p-r+pr+rp$; and r, with $I-p-q+pq+qp$ if and only if these conditions hold:

$$(**)\qquad \begin{cases} pVqVr = pVrVq, \\ qVrVp = qVpVr, \text{ and} \\ rVpVq = rVqVp. \end{cases}$$

Equivalent conditions are

$$(**')\qquad \begin{cases} (I-r)(pq-qp) = 0; \\ (I-p)(qr-rq) = 0; \text{ and} \\ (I-q)(rp-pr) = 0. \end{cases}$$

Before proving a major theorem, we prove few lemmas.

Lemma 1. If p, q, r, $s \in X$ triplewise satisfy (*), then p and q with any of rs, $r+s-rs$, $r+rs-sr$, and $r+s-rs-sr$ satisfy (*).

Proof of lemma. Calculation.

Lemma 2. Let $pVq = p+q-pq$ and $p \wedge q = pq$. Then

(1) $pV(q^\wedge r) = (pVq)^\wedge(pVr)$;
(2) $(pVq)^\wedge r = (p^\wedge r)V(q^\wedge r)$;
(3) $p^\wedge(qVr) = (p^\wedge r)V(p^\wedge r) + (qp-pq)r$;

and

(4) $(p^\wedge q)Vr = (pVr)^\wedge(qVr) + (I-p)(qr-rq)$.

Proof of lemma. Calculation.

Lemma 3. If p, q, r, $s \in X$ triplewise satisfy (**), then p and q with any one of $r \wedge s$, rVs, $r+rs-sr$, and $I-r-s+rs+sr$ satisfy (**).

Proof. More calculation. We will do the case $r \wedge s$. First,

$$\begin{aligned} pV(qV(r\wedge s)) &= pV((qVr)\wedge(qVs)) \\ &= (pV(qVr))\wedge(pV(qVs)) \\ &= (pV(rVq))\wedge(pV(sVq)) \\ &= pV((rVq)\wedge(sVq)) \\ &= pV((r\wedge s)Vq). \end{aligned}$$

Similarly, $qV(pV(r\wedge s)) = qV((r\wedge s)Vp)$. Also,

$$\begin{aligned} (r\wedge s)V(pVq) &= (rV(pVq))\wedge(sV(pVq)) \\ &= (rV(qVp))\wedge(sV(qVp)) \\ &= (r\wedge s)V(qVp). \end{aligned}$$

Q.E.D.

Now we prove the main theorem.

Theorem 8. Let $p_1,\dots,p_n \in X$ triplewise satisfy conditions (*) and (**′). Then $p_1,\dots,p_n$ can be decomposed into sums $p_i = q_i + r_i$ for $i = 1,\dots,n$ such that

(1) $q_1,\dots,q_n$, $r_1,\dots,r_n$ are linear projections on V;
(2) $q_1,\dots,q_n$ are pairwise equivalent;
(3) $r_1,\dots,r_n$ pairwise commute; and
(4) all other pairs of $q_1,\dots,q_n$, $r_1,\dots,r_n$ are orthogonal.

(Note that projections p and q are orthogonal if $pq = qp = 0$.)

Proof. Since $p_1,\dots,p_n$ pairwise nearly commute, all ordered products of $p_1,\dots,p_n$ are equivalent. Let $p_1,\dots,p_n$ triplewise satisfy condition (*). The products of $p_1,\dots,p_{i-1}$, $p_{i+1},\dots,p_n$ in any order times p_i (on the right) are equal. So let $q_1 = p_2p_3\cdots p_np_1$; $q_i = p_1\cdots p_{i-1}p_{i+1}\cdots p_np_i$ for $i = 2,\dots,n-1$; $q_n = p_1\cdots p_n$; and $r_i = p_i - q_i$ for $i = 1,\dots,n$. Then $q_1^2 = (p_2\cdots p_np_1)(p_2\cdots p_np_1) = p_2\cdots p_np_1 = q_1$ and $r_1^2 = (p_1 - p_2\cdots p_np_1)(p_1 - p_2\cdots p_np_1) = p_1 - p_2\cdots p_np_1 - p_2\cdots p_np_1 + p_2\cdots p_np_1 = p_1 - p_2\cdots p_np_1 = r_1$. By symmetry, (1) follows. By the first sentence of this proof, (2) follows. Also, $q_1r_1 = p_2\cdots p_np_1(p_1 - p_2\cdots p_np_1) = 0$ and $r_1q_1 = (p_1 - p_2\cdots p_np_1)p_2\cdots p_np_1 = 0$. Now $q_1r_2 = p_2\cdots p_np_1(p_2 - p_1p_3\cdots p_np_2) = p_3\cdots p_np_1p_2 - p_1p_3\cdots p_np_2 = 0$ by condition (*); and $r_2q_1 = (p_2 - p_1p_3\cdots p_np_2)p_2\cdots p_np_1 = p_2\cdots p_np_1 - p_2\cdots p_np_1 = 0$. So all cases of (4) follow, since we are free to permute $p_1,\dots,p_n$ in the products under certain conditions.

Now $r_1r_2 = (p_1 - p_2\cdots p_np_1)(p_2 - p_1p_3\cdots p_np_2) = p_1p_2 - p_1p_3\cdots p_np_2 - p_3\cdots p_np_1p_2 + p_1p_3\cdots p_np_2 = p_1p_2 - p_3\cdots p_np_1p_2 = (I - p_3\cdots p_n)p_1p_2$. Also, $r_2r_1 = (p_2 - p_1p_3\cdots p_np_2)(p_1 - p_2\cdots p_np_1) = p_2p_1 - p_2\cdots p_np_1 - p_3\cdots p_np_2p_1 + p_2\cdots p_np_1 = p_2p_1 - p_3\cdots p_np_2p_1 = (I - p_3\cdots p_n)p_2p_1$. Now assume that $p_1,\dots,p_n$ triplewise satisfy (**′). Then, by Lemma 3, p_1, p_2, and $p_3\cdots p_n$ satisfy (**′). So $(I - p_3\cdots p_n)(p_1p_2 - p_2p_1) = 0$, i.e., $(I - p_3\cdots p_n)p_1p_2 = (I - p_3\cdots p_n)p_2p_1$; therefore, $r_1r_2 = r_2r_1$. All cases of (3) follow. Q.E.D.

7. A MATRIX REPRESENTATION

We find a matrix representation for the projections $p_1, \ldots, p_n$ in Theorem 8. Let E_{ij} denote the $(n+2)x(n+2)$ matrix with a 1 as the (i, j) entry and 0's elsewhere. Let $a_1, \ldots, a_m \in \mathbb{C}$ be scalars; let $P_i = E_{11} + a_i E_{12}$ for $i = 1, \ldots, m$; and let $Q_j = E_{2+j,2+j}$ for $j = 1, \ldots, n$. Then

(i) $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ are linear projections;
(ii) $P_1, \ldots, P_n$ are pairwise equivalent;
(iii) $Q_i Q_j = Q_j Q_i = 0$ for $i \neq j$;
(iv) $P_i Q_j = Q_j P_i = 0$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Let $m \geq n$, and let $A_h \subset \{1, \ldots, n\}$ for $h = 1, \ldots, n$. Let $Q_{A_h} = \sum_{j \in A_h} Q_j$ for $h = 1, \ldots, n$. Now let $p_i = P_i + Q_{A_i}$ for $i = 1, \ldots, n$. Then $q_i = P_1''' P_{i-1} P_{i+1}'''$ $p_n p_i = P_i + Q_{A_1 \cap \ldots \cap A_n}$; and $r_i = Q_{A_i} - Q_{A_1 \cap \ldots \cap A_n}$. So $r_1, \ldots, r_n$ do not have any A_j's in common.

Note that the set of projections P_i and $P_i + Q_{A_h}$ for all $i = 1, \ldots, m$ and all $A_h \subset \{1, \ldots, n\}$ satisfies both (*) and (**). The set is closed under the operations pq, $p + q - pq$, and $p + pq - qp$.

The set of projections O, Q_{A_h}, P_i, and $P_i + Q_{A_h}$ for all $i = 1, \ldots, m$ and all $A_h \subset \{1, \ldots, n\}$ satisfies (*). The set is closed under the operations pq, $p + q - pq$, $p + pq - qp$, and $p + q - pq - qp$.

Finally, the set of projections I, $I - Q_{A_h}$, P_i, and $P_i + Q_{A_h}$ for all $i = 1, \ldots, m$ and $A_h \subset \{1, \ldots, n\}$ satisfies (**) (and thus (**′)). The set is closed under the operations pq, $p + q - pq$, $p + pq - qp$, and $I - p - q + pq + qp$.

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إسقاطات تبديلية تقريبا

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خلاصة

من المعروف جيدا أن مؤثرات الإسقاط هي عناصر نموذجية في الجبور البئولية، حيث تم إثبات عدد من المبرهنات المناسبة من أجل الإسقاطات التبديلية. وفي هذا البحث نقترح توسيعا لمفهوم التبديلية، نسمية التبديلية تقريبا. كما نوسع وفق هذا المفهوم المبرهنات الرئيسية طول الاسقاطات التبديلية؛ وفي مختلف الطرق سنقوم بتحديد صف الاسقاطات التبديلية في الجبور البئولية.