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High order regularity of the singularities of the solution of a parabolic equation in a singular domain

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ABSTRACT

We are concerned, in this work, with the parabolic equation:

$$(*) \quad \frac{\partial}{\partial t} u + \frac{\partial^4}{\partial x^4} u = f$$

in the following non convex polygonal domain Ω described by the variables $(t, x) \in \mathbb{R}^2$:

$$\Omega = (-1, 0) \times (-1, 1) \cup [0, 1) \times (0, 1).$$

The boundary conditions are of Cauchy–Dirichlet type while the second member f of the equation lies in the non symmetric Sobolev space $H^{p,4p}(\Omega)$ defined, for $p \in \mathbb{N}$, by:

$$H^{p,4p}(\Omega) = \left\{ u \in L^2(\Omega): \frac{\partial^{j+k}}{\partial t^j \, \partial x^k} u \in L^2(\Omega), \, 4j + k \leqslant 4p \right\}.$$

Here, $L^2(\Omega)$ denotes the well known Lebesgue space.

If f satisfies some compatibility conditions, the "natural" space of solutions, in case of Ω regular enough (e.g., convex domain) is the Sobolev space $H^{p+1,4(p+1)}(\Omega)$. According to some works, we know that the space of solutions may be different. In Sadallah (1996), we dealt with the equation (*) for $p = 0$. Now, we are interested in the optimal regularity of the singularities which appear in the solution when p is any integer.

The main result of this work shows the existence of $2(p+1)$ singularities $(v_{j,k})_{j=0,1;\,k=0,\dots,p}$ such that, for all $f \in H^{p,4p}(\Omega)$ satisfying some compatibility conditions, the solution belongs to the space

$$H^{p+1,4(p+1)}(\Omega) \oplus \sum_{j=0,1;\,k=0,\dots,p} IR v_{j,k}.$$

In addition, the singularities fulfill the optimal regularity condition, for all $j = 0, 1$ and $k = 0, \dots, p$:

$$v_{j,k} \in H^{r_j+k,\,4(r_j+k)}(\Omega) \Leftrightarrow r_j < \frac{5+2j}{8}$$

1. INTRODUCTION

Let Ω be an open polygonal domain in $\mathbb{R}^2$ described by the variables (t, x). We denote by $L^2(\Omega)$ the Lebesgue space of square integrable functions on Ω, and for each p in $\mathbb{N}$, $H^{p,4p}(\Omega)$ is the anisotripic Sobolev space defined by:

$$H^{p,4p}(\Omega) = \{u \in L^2(\Omega) \colon \partial_t^{p_1} \partial_x^{p_2} u \in L^2(\Omega);\ \forall (p_1, p_2) \in \mathbb{N} \times \mathbb{N} \colon 4p_1 + p_2 \leqslant 4p\}$$

where ∂_t and ∂_x are the derivatives with respect to t and x respectively. Clearly, $H^{p,4p}(\Omega) = L^2(\Omega)$ when $p = 0$.

The boundary $\partial\Omega$ of Ω is positively oriented. Let Γ_+ be the set of edges of Ω parallel and in the same sense as the x-axis. Then we look at the following boundary value problem:

$$(\mathbf{P}_1)\begin{cases} \partial_t u + \partial_x^4 u = f \\ u\,|_{\partial\Omega\setminus\Gamma_+} = 0 \\ \partial_x u\,|_{\partial\Omega\setminus\Gamma_+} = 0 \end{cases}.$$

From now on, the parabolic operator $\partial_t + \partial_x^4$ will be denoted by L.

When $f \in H^{p,4p}(\Omega)$ and satisfies some compatibility conditions, the "natural" space for the solutions u of problem $(\mathbf{P}_1)$ is the space $H^{p+1,4(p+1)}(\Omega)$. In Sadallah (1983), we have studied this problem and shown that the solution u is, indeed, in this space when Ω is a convex polygonal. However, if Ω is a non convex polygonal domain, u is not, generally, in the space $H^{p+1,4(p+1)}(\Omega)$. Singularities may appear in the expression of the solution u, and their number depends upon the integer p and the vertices of Ω whose angles are greater than π. In this work, we plan to determine the optimal regularity; i.e., tp characterize the Sobolev spaces containing these singularities.

In Sadallah (1992), the same problem was treated for the heat operator, and in Sadallah (1996) we have studied problem (P_1) in the case $p = 0$ (i.e., $f \in L^2(\Omega)$. So, the present work is the extension of the two above-mentioned papers. We will need the Sobolev spaces $H^{r,4r}(\mathbb{R}^2)$, for $r \in \mathbb{R}^+$, which we can define in terms of Fourier transform as follows:

$$H^{r,4r}(\mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2) \colon [(1 + \tau^2)^{r/2} + (1 + \xi^2)^{2r}]\hat{u} \in L^2(\mathbb{R}^2)\}.$$

Then, we put

$$H^{r,4r}(\Omega) = \{u_{|\Omega} \colon u \in H^{r,4r}(\mathbb{R}^2)\}.$$

To illustrate our main result, we consider the model domain Ω defined by: $\Omega = (1, 0) \times (-1, 1) \cup (0, 1) \times (0, 1)$. In this domain, we prove that (under mild compatibility conditions on f), there exist $2(p+1)$ singularities $(u_{k,j})_{k=0,1,\dots,p;\,j=0,1}$ such that, for each $f \in H^{p,4p}(\Omega)$, the solution u of problem (P_1) can be represented as a sum $u = u_R + u_S$ of a regular part $u_R \in H^{p,4p}(\Omega)$ and a singular part:

$$u_S = \sum_{k=0,1,\dots,p;\,j=0,1} \alpha_{k,j} u_{k,j}$$

where $(\alpha_{k,j})$ are real numbers. In addition, the optimal regularity of the singularities $(u_{k,j})$ which we will make explicit, is described as follows:

$$u_{k,j} \in H^{r_j+k;\,4(r_j+k)}(\Omega) \Leftrightarrow r_j < \frac{5+2j}{8}.$$

The proof of this result is inspired by the heat operator case (Sadallah 1992) and uses the particular case $p = 0$ (Sadallah 1996). It would be too long to give an exhaustive list of contributions to this kind of problem. We refer the interested reader to the following: Grisvard (1985, 1992); Dauge (1988); Baderko (1978); Azzam (1980); Banasiak & Roach (1991).

2. PRELIMINARY RESULTS

In Sadallah (1983), we dealt with the number of singularities which appear in the solution of some parabolic problems. In particular we proved the following result, where R is the rectangle $(0, T) \times (0, a)$:

Proposition 2.1. Let p be an integer, ϕ be an element of $H^{4p+2}(0, a) \cap H_0^2(0, a)$ and $f \in H^{p,4p}(R)$. Then, the unique solution u of the problem

$$\begin{cases} Lu = f \\ u|_{\{0\}\times(0,\,a)} = \phi \\ u(t, 0) = u(t, a) = 0,\ t \in (0, T) \\ \partial_x u(t, 0) = \partial_x u(t, a) = 0,\ t \in (0, T) \end{cases}$$

belongs to $H^{p+1,4(p+1)}(R)$ iff the functions $(f_k)_{k=0,1,\dots,p}$ defined on $(0, a)$ by:

$$\begin{cases} f_0 = \phi \\ f_k = (-1)^{3k}\phi^{(4k)} + \sum_{i=0}^{k-1} (-1)^{3i}\, \partial_x^{4i}\, \partial_t^{k-i-1} f \\ k = 1, \dots, p \end{cases}$$

are in the space $H_0^2(0, a)$.

Remark 2.2. (a) The symmetric spaces $H^s(\Omega)$ and $H_0^s(\Omega)$ used in this work are those defined in Lions & Magenes (1968).

(b) The conditions ($f_k \in H_0^2(0, a)$ mean here that the functions f_k and f_k' vanish at the points $x = 0$ and $x = a$, because the hypothesis on ϕ, f and the definition of f_k lead to $f_k \in H^2(0, a)$.

Now, let us recall a result due to Grisvard (1967):

Proposition 2.3. For each $\phi \in H^{4p-2}(0, a)$, there exists $u \in H^{4p-2}(R)$ such that:

$$\phi(x) = u(0, x),\ x \in (0, a).$$

In view of fixing some notations, let us denote:

$$R_1 = (-1, 0) \times (-1, 1).$$

$$R_2 = (0, 1) \times (0, 1).$$

$$\gamma = \{0\} \times (0, 1).$$

In Sadallah (1996), we have shown the next result:

Theorem 2.4. Let Ω be the domain $R_1 \cup R_2 \cup \gamma$. For each $f \in L^2(\Omega)$, there exist $u_R \in H^{1,4}(\Omega)$ and two real numbers α_0 and α_1 such that the solution u of problem

(P_1) can be written as:

$$u = u_R + a_0 U_0 + a_1 U_1$$

where $(j = 0, 1)$:

$$U_j \in H^{r_j, 4r_j}(\Omega) \Leftrightarrow r_j < \frac{5 + 2j}{8}.$$

Also, we need the following proposition (see, for example, Grisvard (1967) and Triebel (1983).

Proposition 2.5. Let Ω be the domain $R_1 \cup R_2 \cup \gamma$. One has:

(1) For each $j \in \mathbb{N}$:

$$H^{r+j, 4(r+j)}(\Omega) = \{u \in L^2(\Omega),\ \partial_t^{p_1} \partial_x^{p_2} u \in H^{r, 4r}(\Omega),\ \forall (p_1, p_2) \in \mathbb{N}^2 : 4p_1 + p_2 \leqslant 4j\}$$

(2) For each $r > 0$:

$$\left.\begin{array}{l} u \in L^2(\Omega) \\ \partial_t u \in H^{r, 4r}(\Omega) \\ \partial_x^4 u \in H^{r, 4r}(\Omega) \end{array}\right\} \Rightarrow u \in H^{r+1, 4(r+1)}(\Omega).$$

Remark 2.6. The "horn" property of Besov (1967), which is similar to the cone one, allows the continuation of the anisotropic Sobolev spaces. This property holds in the domains R_2 and $R_1 \cup R_2 \cup \gamma$ for the spaces $H^{r, 4r}(\Omega)$ when $r > 0$.

3. STUDY OF THE SINGULAR PART

In this section, we shall exhibit the singular part of the solution of problem (P_1), when $\Omega = R_2$ or $\Omega = R_1 \cup R_2 \cup \gamma$. Let us recall the functions P_0 and P_1 introduced in Sadallah (1996) defined on (0, 1) by

$$P_0(x) = (1 - x)^2(1 + 2x) \tag{1}$$

$$P_1(x) = x(1 - x)^2. \tag{2}$$

On the other hand, we proved that the singularities U_0 and U_1 given in Theorem 2.4 satisfy the conditions: $U_{0|\gamma} = P_0$ and $U_{1|\gamma} = P_1$. If we consider an orthonormed basis (ϕ_n) of $L^2(0, 1)$ contained in $H^4(0, 1) \cap H_0^2(0, 1)$ such that each ϕ_n is an eigenfunction of the operator A defined by $Au = u^{(4)}$ (=the 4th derivative of u on (0, 1)), we may write the functions P_0 and P_1 as a Fourier series:

$$P_j(x) = \sum_{n \geqslant 1} a_{n_j} \phi_n(x),\ j = 0, 1 \tag{3}$$

Now, for all $k \in IN$ and $j = 0, 1$, we put

$$u_{k,j}(t, x) = (-1)^k \sum_{n \geqslant 1} \frac{a_{n_j}}{\lambda_n^k} e^{-\lambda_n t} \phi_n(x) \tag{4}$$

Also, for $k \in IN$ and $j = 0, 1$, we put:

(5) $$u_{k,j}(0, x) = \psi_{k,j}(x),\ x \in (0, 1).$$

Observe that

(6) $$u_{0,j}(0, x) = \sum_{n \geq 1} a_{n_j} \phi(x) = P_j(x).$$

In the next proposition, we shall list some properties concerning the functions $u_{k,j}$ and $\psi_{k,j}$.

Proposition 3.1. One has:

(1) $$\partial_t u_{k,j} = -\partial_x^4 u_{k,j} = u_{k-1,j}$$
$$\psi_{k,j}^{(4)} = -\psi_{k-1,j},\ \forall k \in \mathbb{N}^*$$

(2) $$Lu_{k,j} = 0\ \forall k \in \mathbb{N}$$

(3) $$\psi_{k,j}^{(1)}(0) = \psi_{k,j}^{(1)}(1) = 0\ \forall k \in IN^*\ 1 = 0, 1$$
$$\psi_{0,j}(0) = P_j(0) = \delta_{1,j}\quad (\delta_{i,j} \text{ is the Kronecker symbol})$$
$$\psi_{0,j}(1) = P_j(1) = 0$$

(4) For each $k \in \mathbb{N}$ and $j = 0, 1$, the function $\psi_{k,j}$ is a polynomial of $4k + 3$ degree, consequently: $\psi_{k,j} \in H^{4p+2}(\gamma)$.

(5) For each $k \in \mathbb{N}$ and $j = 0, 1$, one has:

$$u_{k,j} \in H^{r_j+k,\,4(r_j+k)}(R_2) \Leftrightarrow r_j < \frac{5 + 2j}{8}.$$

(6) For each $j = 0, 1$ one has:
If $k \in \mathbb{N}$: $\psi_{k,j}^{(4)}(1) = 0$.
If $k \geq 2$: $\psi_{k,j}^{(4)}(0) = 0$.

$$\psi_{1,j}^{(4)}(0) = -\psi_{0,j}(0) = -\delta_{1,j}.$$

Proof. (i) The result can be deduced, by differentiation, from Relation (4).

(ii) The equality is a consequence of the relations given in 1).

(iii) The proof may be obtained from the Relations (4), (5) and (6).

(iv) One has $\psi_{0,j} = P_j$. Hence $\psi_{0,j}$ is a polynomial of third degree. By induction, it is easy to prove that $\psi_{k,j}$ is a polynomial of degree $4k + 3$ because $\psi_{k,j}^{(4)} = -\psi_{k-1,j}$. Then, $\psi_{0,j} \in H^{4p+2}(\gamma)$.

(v) According to Theorem 2.4, one has the equivalence:

$$u_{0,j} \in H^{r_j,\,4r_j}(R_2) \Leftrightarrow r_j < \frac{5 + 2j}{8}.$$

On the other hand the preceding property (1) yields:

$$\partial_t u_{1,j} = u_{0,j} \in H^{r_j,\,4r_j}(R_2)$$
$$\partial_x^4 u_{1,j} = -u_{0,j} \in H^{r_j,\,4r_j}(R_2)$$

Then property (4) implies: $u_{1,j} \in H^{r_j+1,\, 4(r_j+1)}(R_2)$, and we continue the proof by induction using properties (1) and (4).

(vi) This result is a consequence of the last properties (1) and (3).

For each integer k, let us consider the extension $\tilde{\psi}_{k,j}$ to the segment $[-1, 1]$ of $\psi_{k,j}$ which satisfies the conditions:

$$(7)\qquad \begin{cases} \tilde{\psi}_{k,j} \in H^{4p+2}(-1, 1), \forall p \in IN \\ \tilde{\psi}_{k,j}(x) = 0, \forall x \in V(-1) = \text{neighborhood } 6 \text{ of } x = -1. \end{cases}$$

Observe that $\tilde{\psi}_{k,j}$ exists because $\psi_{k,j}$ is a polynomial by virtue of the previous proposition.

Let us consider the following problem in the domain $R = (0, 1) \times (-1, 1)$ (p and k being integers and $j = 0, 1$):

$$(P_2)\begin{cases} Lv_{k,j} = 0 \\ v_{k,j}(0, .) = \tilde{\psi}_{k,j} \in H^{4p+2}(\gamma) \\ v_{k,j}(t, -1) = v_{k,j}(t, 1) = 0, \qquad \forall t \in (0, 1) \\ \partial_x v_{k,j}(t, -1) = \partial_x v_{k,j}(t, 1) = 0 \quad \forall t \in (0, 1) \end{cases}$$

We know (see Proposition 2.1) that this problem admits a unique solution $v_{k,j} \in H^{p+1,\, 4(p+1)}(R)$ iff:

$$(8)\qquad \tilde{\psi}^{(4l)}_{k,j} \in H^2_0(-1, 1), \forall l = 0, \ldots, p.$$

But the relationship (8) holds. Indeed:
For $x = -1$: $\tilde{\psi}_{k,j}$ vanishes in a neighborhood of -1 according to (7).
For $x = 1$: Proposition 3.1, (i) yields:

$$\tilde{\psi}^{(4l)}_{k,j}(x) = \psi^{(4l)}_{k,j}(x) = \psi^{(4(l-k))}_{0,j}(x) = P^{(4(l-k))}_j(x)$$

So, we distinguish three cases:

(a) $l = k$: one has $\psi^{(4l)}_{k,j} = P_j$ and $P_j(1) = P'_j(1) = 0$.

(b) $l > k$: one has $4(l - k) \geqslant 4$, but P_j is of third degree. Consequently, $P^{(4(l-k))}_j = 0$.

(c) $l < k$: According to Proposition 3.1, (1), one has:

$$\tilde{\psi}^{(4l)}_{k,j} = \psi_{k-1,j}$$

on the other hand, the first equality of Proposition 3.1, (3) gives:

$$\psi_{k-1,j}(1) = \psi'_{k-1,j}(1) = 0.$$

Thus, we may conclude that condition (viii) is fulfilled. Hence, problem (P_2) admits a unique solution $v_{k,j} \in H^{p+1,\, 4(p+1)}(R)$, and the following result will be derived when using the symmetry with respect to the x-axis:

Theorem 3.2. Let p and k be two positive integers and $j = 0, 1$. There exists a unique solution $v_{k,j} \in H^{p+1,\, 4(p+1)}(R_1)$ of the problem:

$$\begin{cases} Lv_{k,j} = 0 \\ v_{k,j}(0, .) = \tilde{\psi}_{k,j} \\ v_{k,j}(0, .) = v_{k,j}(., 1) = 0 \\ \partial_x v_{k,j}(., -1) = \partial_x v_{k,j}(., 1) = 0 \end{cases}$$

From this theorem, Proposition 2.5 and Proposition 3.1 we deduce the following:

Theorem 3.3. For each $k \in \mathbb{N}$ and $j = 0, 1$, the function $\tilde{u}_{k,j}$ defined by:

$$\tilde{u}_{k,j} = \begin{cases} v_{k,j} & \text{in } R_1 \\ u_{k,j} & \text{in } R_2 \end{cases}$$

lies in

$$H^{r_j+k,\,4(r_j+k)}(R_1 \cup R_2 \cup \gamma) \text{ iff: } r_j < \frac{5+2j}{8}.$$

[The functions $v_{k,j}$ and $u_{k,j}$ are respectively defined in Theorem 3.2 and in Relationship (4)].

Proof. It is clear that $Lu_{k,j} = 0$. In addition. Theorem 3.2 gives:

$$v_{k,j} \in H^{p+1,\,4(p+1)}(R_1),\ \forall p \in IN.$$

Therefore, $\tilde{u}_{k,j|R,1} = v_{k,j} \in H^{r_j+k,\,4(r_j+k)}(R_1)$ because $H^{p+1,\,4(p+1)}(R_1) 1 H^{r_j+k,\,4(r_j+k)}(R_1)$ when $p + 1 \geqslant r_j + k$. On the other hand, Proposition 3.1, (v) leads to:

$$\tilde{u}_{k,j|R,2} = u_{k,j} \in H^{r_j+k,\,4(r_j+k)}(R_2).$$

Furthermore,

(9) $$u_{k,j|\gamma} = v_{k,j|\gamma} = \psi_{k,j}$$

The function $\tilde{u}_{k,j}$ lies in the space $H^{r_j+k,\,4(r_j+k)}(R_1 \cup R_2 \cup \gamma)$ if the equality:

(10) $$\partial_t^l u_{k,j|\gamma} = \partial_t^l v_{k,j|\gamma} \quad l = 1, \ldots, k \text{ holds.}$$

Obviously, Relationship (9) implies

(11) $$\partial_x^l u_{k,j|\gamma} = \partial_x^l v_{k,j|\gamma} \quad l = 1, \ldots, 4k.$$

In addition, $Lu_{k,j} = Lv_{k,j} = 0$. Hence, according to (11), Relationship (10) holds for $p = 1$. Then, as:

$$L\partial_t u_{k,j} = L\partial_t v_{k,j|\gamma} \quad \text{in } R_2,$$

one has: $L\partial_t u_{k,j|\gamma} = L\partial_t v_{k,j|\gamma}$.
Consequently, $\partial_t \partial_x^4 u_{k,j|\gamma} = \partial_t \partial_x^4 v_{k,j|\gamma}$; finally, we obtain Relationship (10) in the case $l = 2$. The same arguments lead to Relationship (10) for each $l \leqslant k$.

4. THE CASE $\Omega = R_2$

Let us prove the next result, by induction from $p \in \mathbb{N}$:

Theorem 4.1. Let p be a positive integer, $f \in H^{p,\,4p}(R_2)$ and $\phi \in H^{4p+2}(\gamma)$. The functions $(f_k)_{k=0,\ldots,p}$ are defined on (0, 1) as in Proposition 2.1. Assume that $f_k(1) = f_k'(1) = 0$ for each $k = 0, \ldots, p$. Hence, the problem:

$$(P_3)\begin{cases} Lu = f \\ u_{|\gamma} = \phi \\ u(t, 0) = u(t, 1) = 0 & t \in (0,1) \\ \partial_x u(t, 0) = \partial_x u(t, 1) = 0 & t \in (0, 1) \end{cases}$$

admits a unique solution u which can be represented in the form:

$$u = u_R + \sum_{k=0}^{p} (f_k(0)u_{k,0} + f_k'(0)u_{k,1})$$

where $u_R \in H^{p+1,4(p+1)}(R_2)$ and the functions $(u_{k,j})$ are the singularities defined by Relationship (4), which satisfy (see Propositions 3.1, (5):

$$u_{k,j} \in H^{r_j+k,4(r_j+k)}(R_2) \Leftrightarrow r_j < \frac{5+2j}{8}.$$

Proof. The case $p = 0$ has been treated in Sadallah (1996). So, suppose that the result is true for the order p and let us prove it for order $p+1$. For this purpose, we assume that $f \in H^{p+1,4(p+1)}(R_2)$ and $\phi \in H^{4p+6}(\gamma)$. According to the case $p = 0$, problem (P_3) admits a unique solution u which can be represented as follows:

(12) $$u = u_R + \alpha_0 u_{0,0} + \alpha_1 u_{1,0}.$$

where, $u_R \in H^{1,4}(R_2)$, $\alpha_{0,0} = f(0)$ and $\alpha_{0,1} = \phi^{(4)}(0)$.

Then, we put

(13) $$v = \partial_t u_R \in L^2(R_2).$$

Hence, $v_{|\gamma} = -\phi^{(4)} + f_{|\gamma}$. Indeed,

$$u_{|\gamma} = u_{R|\gamma} + f_0(0)u_{0,0|\gamma} + f_0(0)u_{0,1|\gamma}.$$

Thus, by differentiation with respect to x, we get: $\phi^{(4)} = \partial_x^4 u_{R|\gamma}$, since $\partial_x^4 u_{0,j|\gamma} = P_j^{(4)} = 0, j = 0, 1$.

Therefore:

$$v_{\|\gamma} = -\partial_x^4 u_{R\|\gamma} + f_{\|\gamma} = -\phi^{(4)} + f_{\|\gamma}.$$

So, the function v satisfies the following conditions:

$$(P_4)\begin{cases} Lv = \partial_t f \\ v_{|\gamma} = -\phi^{(4)} + f_{|\gamma} \\ v(t,0) = v(t,1) = 0 & t \in (0,1) \\ \partial_x v(t,0) = \partial_x v(t,0) = 0 & t \in (0,1). \end{cases}$$

Now, we may set:

$$\begin{cases} \psi = v_{|\gamma} = -\phi^{(4)} + f_{|\gamma}, \\ g = \partial_t f,\ g_0 = \psi, \\ g_k = (-1)^{3k}\psi^{(4k)} + \sum_{i=0}^{k-1} (-1)^{3i}\, \partial_x^{4i}\, \partial_t^{k-i-1} g, \\ \text{for each } k = 1, \ldots, p. \end{cases}$$

On the other hand, we know that $f \in H^{p+1,4(p+1)}(R_2)$ and $\phi \in H^{4p+6}(R_2)$. Then, using Proposition 2.3 and the definition of the Sobolev spaces $H^{p,4p}(R_2)$ we obtain:

(14) $$\begin{cases} g \in H^{p,4p}(R_2), \\ \psi \in H^{4p+2}(\gamma) \end{cases}$$

Also, we have: $g_0 = f_1$, and

$$\begin{aligned} g_k &= (-1)^{3k}(-\phi^{(4)} + f_{|\gamma})^{(4k)} + \sum_{i=0}^{k-1}(-1)^{3i}\,\partial_x^{4i}\,\partial_t^{k-i-1}g \\ &= (-1)^{3(k+1)}\phi^{(4(k+1))} + \sum_{i=0}^{k}(-1)^{3i}\,\partial_x^{4i}\,\partial_t^{k-i}f \\ &= f_{k+1}, \text{ for each } k = 1, \dots, p. \end{aligned}$$

Observe that f_{p+1} exists since $f \in H^{p+1,\,4(p+1)}(R_2)$. So, problem (P_4) can be represented as follows:

$$(P'_4)\begin{cases} Lv = g \\ v_{|\gamma} = \psi \\ v(t, 0) = v(t, 1) = 0, & t \in (0, 1) \\ \partial_x v(t, 0) = \partial_x v(t, 1) = 0, & t \in (0, 1). \end{cases}$$

Since the functions $(f_k)_{k=1,\dots,p+1}$ satisfy the hypothesis $f_k(1) = f'_k(1) = 0$, the functions $(g_k)_{k=0,\dots,p}$ verify the same condition: $g_k(1) = g'_k(1) = 0$. Consequently, the Relationship (4) and the induction hypothesis show that problem (P'_4) admits a unique solution v which can be written in the following form:

$$\partial_t u_R = v = v_R + \sum_{k=0}^{p}(g_k(0)u_{k,0} + g'_k(0)u_{k,1}) \tag{15}$$

where $v_R \in H^{p+1,\,4(p+1)}(R_2)$.

Integrating the members of (15) with respect to the variable t, we get:

$$u_R = w_R + \sum_{k=1}^{p+1}(f_k(0)u_{k,0} + f'_k(0)u_{k,1}) \tag{16}$$

with

$$\begin{aligned} w_R(t, x) = u_R(0, x) + \int_0^1 v_R(s, x)\,dx \\ - \sum_{k=0}^{p}[g_k(0)u_{k+1,0}(0, x) + g'_k(0)u_{k+1,1}(0, x)]. \end{aligned} \tag{17}$$

The preceding relationship implies:

$$\partial_t w_R = v_R \in H^{p+1,\,4(p+1)}(R_2). \tag{18}$$

In addition, since $Lu_{k,j} = 0$, we deduce: $f = Lu = Lu_R = Lw_R$. Thus, according to (18), we get:

$$\partial_x^4 w_R = -\partial_t w_R + f \in H^{p+1,\,4(p+1)}(R_2). \tag{19}$$

Furthermore, it is clear that $w_R \in L^2(R_2)$. So, the Relationships (18), (19) and Proposition 2.5 lead to:

$$w_R \in H^{p+2,\,4(p+2)}(R_2). \tag{20}$$

Then, the Relationship (16) shows that:

$$u = u_R + f_0(0)u_{0,0} + f'_0(0)u_{0,1}$$
$$= w_R + \sum_{k=0}^{p+1} [f_k(0)u_{k,0} + f'_k(0)u_{k,1}]$$

(here, w_R verifies the condition (20)).

5. THE CASE $\Omega = R_1 \cup R_2 \cup \gamma$

In this section, we consider problem (P_1) in the domain $\Omega = R_1 \cup R_2 \cup \gamma$. Let f be in $H^{p,4p}(\Omega)$. Thus, $f \in L^2(\Omega)$ and problem (P_1) admits, according to Theorem 2.4, a unique solution represented as follows:

$$u = v_R + \alpha_0 U_0 + \alpha_1 U_1, \tag{21}$$

where $v_R \in H^{1,4}(\Omega)$.

More precisely, we have shown in Sadallah (1996) that the singularities U_0 and U_1 are equal, respectively, to $\tilde{u}_{0,0}$ and $\tilde{u}_{0,1}$ introduced in Theorem 3.3. Our aim is now to extend Relationship (21) to the case where $f \in H^{p,4p}(\Omega)$.

Let us denote by A, B, C and D the vertices $(-1, 1)$, $(-1, -1)$, $(0, -1)$ and $(0, 1)$ of the rectangle R_1. Then we introduce the functions (f_k) defined on the side AB by:

$$\begin{cases} f_0 = u_{|AB} \ (u \text{ is given by } (21)) \\ f_k = \sum_{i=0}^{k-1} (-1)^{3i}\, \partial_x^{4i}\, \partial_t^{k-i-1} f,\ k = 1, \ldots, p \end{cases} \tag{22}$$

The main result of this work is the following:

Theorem 5.1. Let p be a positive integer. Assume that f is an element of $H^{p,4p}(\Omega)$ such that the functions (f_k) given by (22) are in $H_0^2(AB)$. Then, the unique solution u of problem (P_1) can be written in the form:

$$u = u_R + \sum_{k=0}^{p} [h_k(0)\tilde{u}_{k,0} + h'_k(0)\tilde{u}_{k,1}]$$

where

(1) $u_R \in H^{p+1,4(p+1)}(\Omega)$.

(2) $$\begin{cases} h_0 = u_{|\gamma} (u \text{ is given by } (21)) \\ h_k = (-1)^{3k} h_0^{(4k)} + \sum_{i=0}^{k-1} (-1)^{3i}\, \partial_x^{4i}\, \partial_t^{k-i-1} f,\ k = 1, \ldots, p. \end{cases}$$

Proof. Sadallah (1983) have shown that if the functions (f_k) defined in (22) are in $H_0^2(AB)$, then, there exists a unique solution $w_1 \in H^{p+1,4(p+1)}(R_1)$ for the problem:

$$\begin{cases} Lw_1 = f_{|R_1} \\ w_{1|\partial R,1\setminus CD} = 0 \\ \partial_x w_{1|\partial R,1\setminus CD} = 0. \end{cases}$$

Let us put $\phi = w_{1|\gamma}$. As $w_1 \in H^{p+1,4(p+1)}(R_1)$ we get: $\phi \in H^{4p+2}(\gamma)$. Then we consider the following problem in R_2 :

$$\begin{cases} Lw_1 = f_{|R_1} \\ w_{1|\partial R,1\backslash CD} = 0 \\ \partial_x w_{1|\partial R,1\backslash CD} = 0. \end{cases}$$

Observe that the function (h_k) defined in Theorem 5.1 may also be given by:

$$\begin{cases} h_0 = \phi \\ h_k(-1)^{3k}\phi^{(4k)} + \sum_{i=0}^{k-1}(-1)^{3i}\, \partial_x^{4i}\, \partial_t^{k-i-1}f,\ k = 1, \ldots, p. \end{cases}$$

Furthermore, it is not difficult to verify that $h_k(1) = h_k'(1) = 0$, $k = 0, \ldots, p$. Consequently, Theorem 4.1 implies that problem (P_5) admits a unique solution w_2 which may be put in the form:

$$w_2 = w_R + \sum_{k=0}^{p} h_k(0)u_{k,0} + h_k'(0)u_{k,1}$$

where $w_R \in H^{p+1,4(p+1)}(R_2)$.

On the other hand, we put:

$$u_R = \begin{cases} w_1 - \sum_{k=0}^{p} h_k(0)v_{k,0} + h_k'(0)v_{k,1}, & \text{in } R_1 \\ w_R & , \ \text{in } R_2 \end{cases} \tag{23}$$

where $(v_{k,j})$ denote the functions introduced in Theorem 3.2. So, the solution u of problem (P_1), whose existence is proved by the Relationship (21), can be represented as follows:

$$u = u_R + \sum_{k=0}^{p} [h_k(0)\tilde{u}_{k,0} + h_k'(0)\tilde{u}_{k,1}] \tag{24}$$

where $(\tilde{u}_{k,j})$ are the functions defined in Theorem 3.3.

Lemma 5.2. One has:

(1) $\partial_x^j u_{R|\gamma} = \partial_x^j w_{R|\gamma}, \quad j \leqslant 4p$

(2) $\partial_t^i u_{R|\gamma} = \partial_t^i w_{R|\gamma}, \quad i \leqslant p$

Proof. (1) The Relationship (23) implies:

$$u_{R|\gamma} = w_{R|\gamma} \in H^{4p+2}(\gamma).$$

Hence, $\partial_x^j u_{R|\gamma} = \partial_x^j w_{R|\gamma}, j \leqslant 4p$.

(2) We have: $Lu_{|\gamma} = Lu_{R|\gamma} = f_{|\gamma}$, then

$$\partial_t^j u_{R|\gamma} = -\partial_x^4\, \partial_t^{j-1} u_{R|\gamma} + \partial_t^{j-1} f_{|\gamma}$$

$$\partial_t^j w_{R|\gamma} = -\partial_x^4\, \partial_t^{j-1} w_{R|\gamma} + \partial_t^{j-1} f_{|\gamma}.$$

So, the desired result is obtained by induction from j.

Now, we can end the proof of Theorem 5.1: Lemma 5.2 leads to

$$\partial_t^{p_1} \partial_x^{p_2} u_{R|\gamma} = \partial_t^{p_1} \partial_x^{p_2} w_{R|\gamma}$$

for each (p_1, p_2) such that $4p_1 + p_2 \leqslant 4p$. Therefore, $u_R \in H^{p+1, 4(p+1)}(\Omega)$ since:

$$u_{R|R,1} \in H^{p+1, 4(p+1)}(R_1) \quad \text{and} \quad u_{R|R,2} = w_R \in H^{p+1, 4(p+1)}(R_2).$$

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الصقالة العليا للأجزاء الشاذة في حل معادلة مكافئية داخل ساحة شاذة

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خلاصة

ندرس   ندرس في هذا العمل المعادلة المكافئية:

$$\frac{\partial}{\partial t}u + \frac{\partial^4}{\partial x^4}u = f$$

في المضلع Ω غير المحدب التالي الذي يصفه المتغيران $(t,x)\in IR^2$:

$$\Omega = (-1,0) \times (-1,1) \cup [0,1) \times (0,1).$$

وذلك باعتبار شروط كوشي – ديرخليت Cauchy-Dirichlet الحدية وباتخاذ f في أحد فضاءات سوبولاف Sobolev غير المتناظرة.

تنص النتيجة الرئيسة على وجود دوال "شاذة" singular في صيغة حل المسألة وتقدم درجة "صقالة" regularity هذه الدوال.