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Characterization of parallel paths in arrangement graphs

KHALED DAY* AND ANAND TRIPATHI**

* *Dept. of Computer Science, Sultan Qaboos University, OMAN*
** *Dept. of Computer Science, University of Minnesota, USA*

ABSTRACT

In this paper we fully characterize node-disjoint (parallel) paths in arrangement graph interconnection networks which have been presented as generalized star graphs for interconnecting large multiprocessor systems. We characterize complete families of parallel paths between any two nodes of an arrangement graph. The length of each path is at most four plus the minimum distance between the two nodes.

1. INTRODUCTION

The *arrangement graph* has been proposed (Day & Tripathi 1992) as an attractive interconnection topology for large multiprocessor systems. An arrangement graph, $A_{n,k}$, of parameters n and k, is regular, has degree $k(n-k)$, has $n!/(n-k)!$ vertices, and has diameter $\lfloor \frac{3}{2}k \rfloor$. $A_{n,k}$ is vertex and edge symmetric, recursively structured, has a simple and optimal distributed routing algorithm and many fault tolerance properties. This topology has been presented as a generalization of the *star graph* (Akers & Krishnamurthy 1986, Akers *et al.* 1987). The star graph topology has drawn a lot of attention recently for analyzing its topological properties and deriving a number of results for the problems of sorting (Menn & Somani 1990, Rajasekaran & Wei 1993), embedding (Nigam *et al.* 1990, Jwo *et al.* 1991b, Ranka *et al.* 1993, Shen *et al.* 1993, Lee & Chang 1994), broadcasting (Graham & Siedel 1988, Bagherzadeh *et al.* 1993, Sheu *et al.* 1993, Latifi & Bagherzadeh 1994), parallel path characterization (Jwo *et al.* 1991a, Day & Tripathi 1994), computing Fourier transforms (Fragopoulou & Akl 1994), fault tolerance (Najjar & Srimani 1991, Rouskov & Srimani 1993, Latifi 1993a), load balancing (Qiu & Akl 1993), routing (Latifi 1993b, Gargano *et al.* 1993), unidirectional and incomplete variants (Day & Tripathi 1993, Latifi & Bagherzadeh 1994), and data exchange algorithms (Coolsaet & Fack 1994).

The *arrangement graph* has been first defined in (Day & Tripathi 1992) as a generalization of the *star graph* in an attempt to bring a solution to a major drawback of the *star graph* topology (related to its poor scalability) while preserving its attractive features. An *n-star graph* denoted by S_n, is an undirected graph consisting of $n!$

* *email*: kday@computer.org
** *email*: tri.pathi@cs.umn.edu

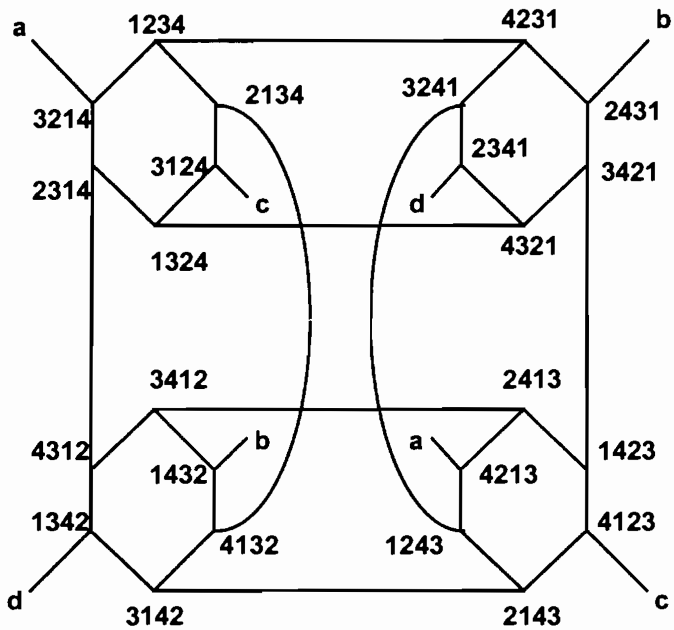

Fig. 1. The 4-Star Graph S_4

vertices labeled with the $n!$ permutations on n symbols (the n symbols $1, 2, \ldots, n$ are used) and such that there is an edge between any two vertices if, and only if, their labels differ only in the first and in one other position. Figure 1 above shows a drawing of S_4.

A major practical difficulty with the *star graph*, however, is related to its number of nodes. Indeed, the number of nodes in an *n-star graph* is equal to $n!$ (Table 1), and since the factorial function increases very rapidly, the set of its values is spread widely over the set of integers. For example 5! is equal to 120, while 11! is about forty million. This distribution of the factorials limits considerably the size choices when designing an interconnection network based on the *star graph* topology.

The arrangement graph brings a solution to the scalability problem from which the star graph suffers, while preserving its desirable features. An *(n, k)-arrangement* graph is specified by the two integers n and k ($1 \leqslant k < n$); its nodes correspond to the arrangements of k elements chosen out of the n elements $1, 2, \ldots, n$. The edges connect nodes which correspond to arrangements differing in exactly one of their k positions. For example, in the (6, 4)*-arrangement graph*, the two nodes 4136 and 4236 are connected since these two arrangements of four elements out of the six elements $1, 2, \ldots, 6$ differ only in their second position (from the left). The *arrangement graph* has a hierarchical structure, is vertex and edge symmetric, has a simple

Table 1. Basic Parameters of S_n

Dimension	# Vertices	Degree	Diameter	Average distance
n	$n!$	$n-1$	$\lfloor \frac{3}{2}(n-1) \rfloor$	$n - 4 + \frac{2}{n} + \sum_{i=1}^{n} 1/i$

shortest path routing algorithm, and possesses attractive fault tolerant properties. The (n, k),-*arrangement graph* is regular of degree $k(n-k)$, number of nodes $n!/(n-k)!$ and diameter $\lfloor \frac{3}{2}k \rfloor$. If k is made equal to $n-1$ then the *n-star graph* will be obtained. The *complete graph* is also a special *arrangement graph*, since if k is equal to 1 then the *complete graph* with n nodes results. The set of possible values for the number $n!/(n-k)!$ of nodes of the (n, k)-*arrangement graph* contains and is much larger than the set of values for the number $n!$ of nodes of the *n-star graph*. This allows more flexibility in selecting the network size. When designing an interconnection network based on the *arrangement graph* topology, we can, by tuning the two parameters n and k, make a more suitable choice for the number of nodes and for the degree/diameter tradeoff. For a comparable number of nodes we can, by adjustng n and k, increase the degree to achieve a decrease in the diameter and vice-versa.

In this paper we contribute further to the study of the *arrangement graph* family of interconnection networks by characterizing maximum size families of node-disjoint paths between any two nodes of $A_{n,k}$. These parallel paths are proven of minimum length plus possibly a small constant (at most four). When applied to the *star graph* which is a special *arrangment graph*, the results of this paper confirm the earlier findings about node-disjoint paths in star graphs (Jwo *et al.* 1991a, Day & Tripathi 1994) even though the construction method of this paper is different and the results are more general. The existence of parallel paths is an important consideration in the design of interconnection networks (Lakshmivarahan *et al.* 1991). These paths can be used to speed up transfer of large amounts of data and to provide alternative routes in cases of failures. A future extension of the work of this paper would be the development of fault-tolerant routing algorithms based on the obtained paths.

The remainder of the paper is organized as follows. In Section II we introduce some notations and we summarize the basic topological properties of the arrangment graphs which are used in the construction of the parallel paths. Complete characterization of parallel paths is then given in Section III, with proofs of their node-disjoint property and of their optimal (or near optimal) lengths. We conclude the paper in Section IV by a recount of the obtained results.

2. NOTATIONS AND PRELIMINARIES

Let n and k be two integers satisfying $1 \leqslant k \leqslant n-1$, and let us denote $\langle n \rangle = \{1, 2, \ldots, n\}$ and $\langle k \rangle = \{1, 2, \ldots, k\}$. Let P_k^n be the set of permutations of the n elements of $\langle n \rangle$ taken k at a time, i.e., the set of arrangements of k elements out of the n elements of $\langle n \rangle$. The k elements of an arrangement p are denoted $p_1, p_2, \ldots, p_k$ ($p = p_1 p_2 \cdots p_k$).

Definition 1. The (n, k)-*arrangement graph* $A_{n,k} = (V, E)$ is given by:

$V = \{p_1 p_2 \ldots p_k \mid p_i \text{ in } \langle n \rangle \text{ and } p_i \neq p_j \text{ for } i \neq j\} = P_k^n$, and
$E = \{(p, q) \mid p \text{ and } q \text{ in } V, \text{ and for some } i \text{ in } \langle k \rangle\ p_i \neq q_i, \text{ and } p_j = q_j \text{ for } j \neq i\}$.

The nodes of $A_{n,k}$ are the arrangements of k elements out of the n elements of $\langle n \rangle$, and the edges of $A_{n,k}$ connect nodes which differ in exactly one of their k positions. For example in $A_{4\,2}$ (Fig. 2) the node $p = 23$ is connected to the nodes 21, 24, 13, and 43.

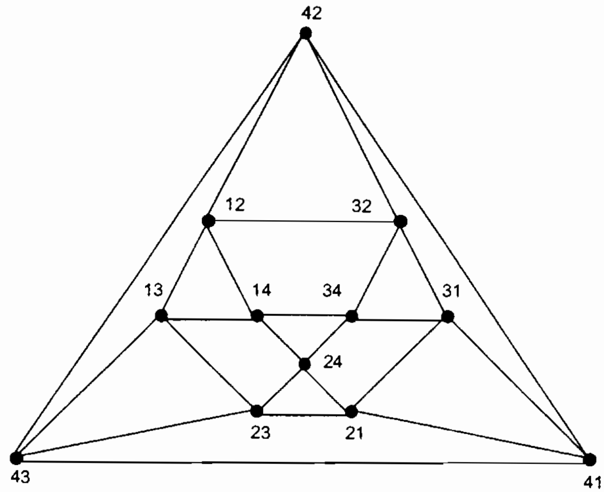

Fig. 2. The $A_{4,2}$ Arrangement Graph

An edge connecting two arrangements p and q which differ only in position i, is called an *i-edge*, p and q are called *i-adjacent*, and q is called the *(q_i, i)-neighbor* of p. For all values of n and k, $A_{n\,k}$ is a regular graph with a number of nodes $n!/(n-k)!$, a degree $k(n-k)$, and a diameter $\lfloor \frac{3}{2}k \rfloor$. It is vertex and edge symmetric. Intuitively, this means that it looks the same when viewed from different vertices and from different edges; refer to (Day & Tripathi 1992) for formal definitions and proofs. Vertex symmetry permits the reduction of the problem of routing between two arbitrary nodes to the routing between an arbitrary node and a fixed node.

The routing in arrangement graphs is based on a cycle representation for the set of arrangements. An arrangement of k elements out of n elements can be represented using a cycle structure similar to the well known cycle structure for the set of permutations (Knuth 1973). To introduce this cycle representation, we first define some notations. Let $p = p_1 p_2 \ldots p_k$ be an arrangement in P_k^n and let $EXT(p) = \langle n \rangle - \{p_1, p_2, \ldots, p_k\}$. We also refer to the elements of $EXT(p)$ as the *external elements* of p. The special node (arrangement) $12 \ldots k$ is called the *identity* node and is denoted I_k. The elements of $EXT(I_k)$ are called the *foreign elements*, they are the elements $k+1$, $k+2, \ldots, n$. A tuple $C = (x_1, x_2, \ldots, x_t)$ is called a *p-cycle* for some node p in $A_{n,k}$, if for any i between 1 and $t-1$, the position x_i in p is held by x_{i+1}, and the position x_t is held by x_1. The cycle C is called an *internal cycle* if all of its elements $x_1, x_2, \ldots$, and x_t are smaller than or equal to k. On the other hand C is called an *external cycle* if it contains one foreign element. Any node p of $A_{n,k}$ can be represented by a set of internal cycles and external cycles such that each external cycle contains exactly one foreign element. This foreign element is always written as the last element in the cycle. Therefore, the first element of an external *p-cycle* is always non-foreign and it is external to p. In other words, if $C = (x_1, x_2, \ldots, x_t)$ is an external *p-cycle*, then $x_t > k$, $x_1 \leqslant k$, and $x_1 \in EXT(p)$.

Example. In $A_{9,7}$ the node $p = 6351792$ can be represented by the two cycles (2, 3, 5, 7) and (4, 1, 6, 9). The cycle (2, 3, 5, 7) is an internal cycle since all its elements are less than or equal to $k = 7$, while the cycle (4, 1, 6, 9) is external since it has the foreign element 9. Using vertex symmetry we reduce the routing between two arbitrary nodes p and q to the routing between an arbitrary node p and the identity node $I_k = 12 \ldots k$. Such routing can be achieved by first writing a cycle representation for p, then 'sorting' the p-cycles one by one. By sorting a cycle, we mean moving each of its misplaced elements to its correct position in I_k. The correct position of a non-foreign element x is position x, while the correct position of a foreign element x is outside the arrangement. An external cycle $C = (x_1, x_2, \ldots, x_t)$ can be sorted by first moving its non-foreign external element x_1 to its correct position (held by x_2), then taking x_2 to its correct position (held by x_3) and so on until x_{t-1} is taken to its correct position (held by x_t). As a result of correcting x_{t-1}, x_t moves outside the arrangement, which is its correct position since x_t is foreign. Sorting an internal p-cycle $C' = (y_1, y_2, \ldots, y_l)$ requires first that one of its elements (say y_1) be exchanged with any external element (say z) then y_1 is taken to its correct position (held by y_2), then y_2 is taken to its correct position (held by y_3) and so on until y_l takes its correct position which makes z external again.

Example. The external p-cycle $C = (4, 1, 6, 9)$ for $p = 6351792$ in $A_{9,7}$ is sorted along the path $\pi = 6351792 \rightarrow 635\underline{4}792 \rightarrow \underline{1}354792 \rightarrow 13547\underline{6}2$, while the internal p'-cycle $C' = (2, 3, 5, 7)$ for $p' = 1354762$ in $A_{9,7}$ is corrected along the path:

$$\pi' = 1354762 \rightarrow 1354768 \rightarrow 1\underline{2}54768 \rightarrow 12\underline{3}4768 \rightarrow 1234\underline{5}68 \rightarrow 123456\underline{7}.$$

In the above example we sorted the two p-cycles $C = (4, 1, 6, 9)$ and $C' = (2, 3, 5, 7)$ of the node $p = 6351792$ of $A_{9,7}$ to achieve a routing from p to the identity node I_7 (combining π and π'). From this routing algorithm is derived (see Day & Tripathi 1992) the distance between an arbitrary node p and I_k. Let c denote the total number of cycles in a cycle representation of p, let e denote the number of external cycles among these c cycles, and let m be the total number of elements in all the c cycles. The (shortest) distance $D(p)$ between p and I_k in $A_{n,k}$ is given by: $D(p) = c + m - 2e$. An i-edge (p, q) in $A_{n,k}$ is on a shortest path from p to I_k if, and only if, either:

$$q_i = i, \tag{1}$$

or

$$p_i \text{ belongs to an internal } p\text{-cycle.} \tag{2}$$

3. PARALLEL PATHS

Path correction orders

Constructing node-disjoint (parallel) paths between two arbitrary nodes is reduced to constructing parallel paths between an arbitrary node $p = p_1 p_2 \ldots p_k$ and the special node $I_k = 1, 2 \ldots k$ (identity node). This is possible due to the vertex symmetry of the graph. In this section, we characterize a maximum size family of node-disjoint paths between p and I_k. Our goal is therefore to build a family ϕ of $k(n-k)$ parallel paths between p and I_k such that the length of each of these paths differs by no more than a small constant from the shortest distance $D(p)$ between p and I_k.

To achieve the above goal we should be able to include in ϕ a path $\pi(x, y)$ for each x in $EXT(p)$ and for each y between 1 and k. That is, for each (x, y)-neighbor of p (let q be this neighbor), there must exist a path $\pi(x, y)$ in ϕ that contains the edge (p, q). The edge (p, q) corresponds to an exchange of the external element x with the element at position y in p. The following notations will help in characterizing the parallel paths. Let $C_1, C_2, \ldots, C_c$ the p-cycles; we assume that the first e cycles $C_1, C_2, \ldots, C_e$ $(e \leqslant c)$ are external and the remaining p-cycles are internal. We will also use the notations:

$|C_i| = m_i =$ the number of elements in the i'th p-cycle,
$C_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$, $1 \leqslant i \leqslant c$, is the i'th p-cycle,
$m_E = \sum_{i=1}^{e} m_i$ is the total number of elements in the external p-cycles,
$m_I = \sum_{i=e+1}^{c} m_i$ is the total number of elements in the internal p-cycles,
$m = m_I + m_E$ is the total number of elements in all p-cycles,
$D(p) = c + m - 2e$ is the shortest distance between p and I_k.

Let σ be the sequence of the non-foreign misplaced symbols in p, obtained by listing the non-foreign elements of C_1 followed by the non-foreign elements of C_2 and so on, until listing the non-foreign elements of C_c. Recall that all the elements of an internal cycle are non-foreign and all the elements, except the last element, of an external cycle are non-foreign. Let y be a non-foreign misplaced element in p, we define σ_y to be the sequence of misplaced elements of p obtained by left or right circular shifting the sequence σ until y becomes the last element. For instance, if $n = 7$, $k = 5$, $p = 13256$, then $c = 2$, $C_1 = (4, 5, 6)$, $C_2 = (2, 3)$, and $\sigma = 4523$. The shifted sequences are: $\sigma_2 = 3452$, $\sigma_3 = \sigma$, $\sigma_4 = 5234$, $\sigma_5 = 2345$. Let π be a path from p to I_k, we define the correction order of π in the following.

Definition 2. A sequence $\sigma = a_1, a_2, \ldots, a_s$ of elements is called the *correction order* of the path π if, and only if,

(i) a_i is non-foreign, for each i, $1 \leqslant i \leqslant s$,
(ii) a_i is misplaced in p (i.e., $p_{a_i} \neq a_i$), for each i, $1 \leqslant i \leqslant s$,
(iii) if $1 \leqslant i \leqslant j \leqslant s$, q is a node in path π and $q_{a_j} = a_j$, then $q_{a_i} = a_i$.

The correction order of the path π is the order in which the non-foreign misplaced elements of p are corrected along π. We denote the correction order of π by $ORD(\pi)$. The following lemma will be used to prove the node-disjoint property of the paths of ϕ.

Lemma 1. If π_1 and π_2 are two paths between p and I_k with respective correction orders $ORD(\pi_1) = \sigma_{y_1}$ and $ORD(\pi_2) = \sigma_{y_2}$, for two distinct non-foreign elements y_1 and y_2, then π_1 and π_2 are node-disjoint.

Proof. Let $\sigma = a_1, a_2, \ldots, a_s$, we assume without loss of generality that $y_1 = a_s$ and $y_2 = a_i$, $1 \leqslant i < s$. Therefore $ORD(\pi_1) = \sigma_{a_s} = a_1, a_2, \ldots, a_s = \sigma$, and $ORD(\pi_2) = \sigma_{a_i} = a_{i+1}, a_{i+2}, \ldots, a_s, a_1, a_2, \ldots, a_i$. Assume π_1 and π_2 have a common node q other than the source p and the destination I_k. Let α (resp. β) be the proper subsequence of $ORD(\pi_1)$ (resp. $ORD(\pi_2)$), which corresponds to the correction order of the path from p to q along π_1 (resp. along π_2). α and β must contain the same elements if π_1 and π_2 are to join at q. Let r be the length of α and β. Thus $\alpha = a_1, a_2, \ldots, a_r$ and $\beta = a_{i+1}, a_{i+2}, \ldots, a_{i+r}$, if $i + r \leqslant s$, or $\beta = a_{i+1}, a_{i+2}, \ldots, a_s, a_1, a_2, \ldots, a_{i+r-s}$, if $i + r > s$. If $r < i + 1$ then a_{i+1} appears in β but not in α, and if $r \geqslant i + 1$ then a_i appears in α but not in β. Therefore α and β do not contain the same elements (contradiction). Q.E.D.

A path π in $A_{n,k}$ is characterized by a source node p and a list $T(x_1, y_1)\ T(x_2, y_2) \ldots T(x_t, y_t)$ of transitions. A transition $T(x, y)$ corresponds to an edge along which the external element x is substituted to the element at position y. We write $\pi = \langle p, T(x_1, y_1)\ T(x_2, y_2) \ldots T(x_t, y_t)\rangle$, and we denote by $|\pi|$ the length of π. For example, the path $\pi = 3452 \to 1452 \to 1432 \to 1436 \to 1236 \to 1234$ from node 3452 to node 1234 in $A_{6,4}$ is uniquely identified by the source node 3452 and the list of transitions: $T(1, 1)$, $T(3, 3)$, $T(6, 4)$, $T(2, 2)$, and $T(4, 4)$. This is written as: $\pi = \langle 3452, T(1, 1)\ T(3, 3)\ T(6, 4)\ T(2, 2)\ T(4, 4)\rangle$. The length of this path is $|\pi| = 5$.

Transition lists for optimal cycle sorting

We characterize transition lists for sorting internal and external p-cycles in a minimum number of steps. Sorting a cycle consists of moving each of its non-foreign elements to its correct position (i.e., element x is moved to position x)

Definition 3. An *optimal cycle-sorting transition list* for the p-cycle $C_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$, denoted T_i^*, is a list of transitions such that the path $\pi_i^* = \langle p, T_i^*\rangle$ and its destination q satisfy:

(i) if C_i is external then $|\pi_i^*| = |C_i| - 1$, $q_{a_{ij}} = a_{ij}$, $1 \leqslant j \leqslant m_i - 1$, and $q_{a_{im_i}} \in EXT(q)$,

(ii) if C_i is internal then $|\pi_i^*| = |C_i| + 1$, $q_{a_{ij}} = a_{ij}$ for $1 \leqslant j \leqslant m_i$.

π_i^* is a minimum length path that corrects all the non-foreign elements of C_i.

Lemma 2. If $C_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$ is an external p-cycle then the list: $T_i^* = T(a_{i1}, a_{i1})\ T(a_{i2}, a_{i2}) \ldots T(a_{im_i-1}, a_{im_i-1})$ is the only optimal cycle-sorting transition list for C_i.

Proof. Let $\pi_i^* = \langle p, T_i^*\rangle$; we have $|\pi_i^*| = m_i - 1$ since there are $m_i - 1$ transitions in T_i^*. On the other hand, after executing T_i^*, every $a_{ij}(1 \leqslant j \leqslant m_i - 1)$ is taken to position a_{ij}, and a_{im_i} is made external by the last transition $T(a_{im_i-1}, a_{im_i-1})$. Therefore T_i^* is an optimal cycle-sorting transition list for C_i. The uniqueness of T_i^* is derived from the optimal routing rules (1) and (2) given at the end of Section II. These rules imply that to sort an external cycle optimally, it is necessary to correct its elements in the other of appearance in the cycle. Q.E.D.

Lemma 3. If $C_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$ is an internal p-cycle, then for every z in $EXT(p)$ and for every j, $1 \leqslant j \leqslant m_i$, the transition list: $T_i^j(z) = T(z, a_{ij})\ T(a_{ij+1}, a_{ij+1}) \ldots T(a_{im_i}, a_{im_i})\ T(a_{i1}, a_{i1})\ T(a_{i2}, a_{i2}) \ldots T(a_{ij}, a_{ij})$ is an optimal cycle-sorting transition list for C_i.

Proof. Let $\pi_i^j(z) = \langle p, T_i^j(z)\rangle$; we have $|\pi_i^j(z)| = m_i + 1$ since there are $m_i + 1$ transitions in T_i^*. On the other hand, after executing T_i^*, every $a_{ij}(1 \leqslant j \leqslant m_i)$ is taken to position a_{ij}. Q.E.D.

Example. We illustrate these concepts using the following example. Let $p = 57123$ be a node in $A_{7,5}$; the p-cycles are $C_1 = (4, 2, 7)$ and $C_2 = (3, 1, 5)$. C_1 is external and C_2 is internal. The transition list $T_1^* = T(4, 4)\ T(2, 2)$ is the unique optimal cycle-sorting transition list for C_1. The path $\pi_1^* = \langle 57123,\ T(4, 4)\ T(2, 2)\rangle = 57123 \to 57143 \to 52143$ sorts the cycle C_1 and is of optimal length. However, there are six optimal cycle-sorting transition lists for C_2. There is a different list for each choice of an element z from $EXT(p) = \{4, 6\}$ and for each choice of an element a_{2j}, $1 \leqslant j \leqslant 3$, from cycle $C_2 = (3, 1, 5)$. For instance, if $z = 4$ and $j = 1$ ($a_{2j} = 3$), we obtain the optimal transition list $T_2^1(4) = T(4, 3)\ T(1, 1)\ T(5, 5)\ T(3, 3)$, and the path $\pi_2^1(4) = \langle 57123,\ T(4, 3)\ T(1, 1)\ T(5, 5)\ T(3, 3)\rangle = 57123 \to 57423 \to 17423 \to 17425 \to 17325$, which sorts the cycle $C_2 = (3, 1, 5)$ optimally. If instead $z = 6$ and $j = 2$ ($a_{2j} = 1$), we obtain the optimal transition list $T_2^2(6) = T(6, 1)\ T(5, 5)\ T(3, 3)\ T(1, 1)$, and the path $\pi_2^2(6) = \langle 57123,\ T(6, 1)\ T(5, 5)\ T(3, 3)\ T(1, 1)\rangle = 57123 \to 67123 \to 67125 \to 67325 \to 17325$, which also sorts C_2 optimally.

Definition 4. The *generalized optimal cycle-sorting transition list* of a p-cycle C_i is denoted $\tau_i^*(z)$ and is given by:

(i) if C_i is external: $\tau_i^*(z) = T_i^* = T(a_{i1}, a_{i1})\ T(a_{i2}, a_{i2}) \ldots T(a_{im_i-1}, a_{im_i-1})$, and
(ii) if C_i is internal: $\tau_i^*(z) = T_i^{m_i}(z) = T(z, a_{im_i})\ T(a_{i1}, a_{i1})\ T(a_{i2}, a_{i2}) \ldots T(a_{im_i}, a_{im_i})$.

If C_i is internal, the argument z in $\tau_i^*(z)$ specifies the element of $EXT(p)$ that is exchanged with an internal element of p during the first transition of $\tau_i^*(z)$. We call such a z the *initiator* of the sorting of C. However, if C is external, then the argument z in $\tau_i^*(z)$ is irrelevant, and the initiator is the first element a_{i1} of C. Notice that along the path $\langle p, \tau_i^*(z)\rangle$ the elements of C_i are corrected in the order $a_{i1}, a_{i2}, \ldots, a_{im_i}$ (i.e., in the order of their appearance in the cycle C_i).

Parallel path characterization

Recall that our goal is to build a family ϕ of $k(n-k)$ parallel paths from p to I_k of optimal or near optimal length. ϕ must contain a path $\pi(x, y)$ for each x in $EXT(p)$ and for each y, $1 \leqslant y \leqslant k$. The first edge of $\pi(x, y)$ would then correspond to the transition $T(x, y)$. For clarity of presentation we build the family ϕ in stages by characterizing a number of disjoint sub-families $\phi_1, \phi_2, \ldots \phi_7$ of parallel paths for different (x, y) choices. The union of the ϕ_i sub-families forms the desired family ϕ. The following propositions characterize one by one the ϕ_i sub-families and prove that all their paths are node-disjoint. We show that their union contains $k(n-k)$ paths which forms a complete set of node-disjoint paths.

Proposition 1. There is a family ϕ of $(n-k-e)m_I$ node-disjoint paths between p and I_k, each of length $D(p)$.

Proof. Let x be an external foreign element. Let y, $1 \leqslant y \leqslant k$, be such that $y = a_{ij}$ is the j'th element of an internal cycle $C_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$. We define $\pi(x, y) = \pi(x, a_{ij}) = \langle p,\ T(x, a_{ij})\ T(a_{ij+1}, a_{ij+1}) \ldots T(a_{im_i}, a_{im_i})\ \tau_{i+1}^*(a_{i1})\ \tau_{i+2}^*(a_{i1}) \ldots \tau_c^*(a_{i1})\ \tau_1^*(a_{i1})\ \tau_2^*(a_{i1}) \ldots \tau_{i-1}^*(a_{i1})\ T(a_{i1}, a_{i1})\ T(a_{i2}, a_{i2}) \ldots T(a_{ij}, a_{ij})\rangle$. The following observations about $\pi(x, a_{ij})$ can be easily verified.

(a) If q is an intermediate node in $\pi(x, a_{ij})$ then $q_{a_{ij}} = x$. In other words, the foreign element x occupies position a_{ij} all along the path $\pi(x, a_{ij})$.

(b) All misplaced elements of p are corrected along $\pi(x, a_{ij})$ in the order $\sigma_{a_{ij}}$.

(c) No external foreign element other than x is internal to any node of $\pi(x, a_{ij})$.

(d) The length of $\pi(x, a_{ij})$ is $D(p)$ because optimal cycle-sorting transition lists have been used to sort all the cycles other than C_i and $(m_i + 1)$ transitions have been used to sort C_i optimally.

From (b) we derive that $\pi(x, a_{ij})$ is a path between p and I_k. From (a) and (c) we derive that for different choices of (x, y) we obtain node-disjoint paths since if two choices differ in x then a different external foreign element is made constantly internal in each path (without touching other external foreign elements); and if the two choices agree in the x values and differ in the y values, then a different position is constantly occupied by the same foreign element in each path. Since there are $(n - k - e)$ external foreign elements and m_I misplaced elements belonging to internal p-cycles, we conclude that there are $(n - k - e)m_I$ parallel paths of length $D(p)$. Q.E.D.

Example 1. Let p be the node $p = 34165$ in $A_{7,5}$. There are two p-cycles: $C_1 = (2, 4, 6)$ and $C_2 = (1, 3)$; C_1 is external and C_2 is internal. We also have: $c = 2$, $e = 1$, $m_I = 2$, $m_E = 3$, $D(p) = 5$ and $EXT(p) = \{2, 7\}$. The family ϕ_1 of parallel paths between the node p and the node $I_5 = 12345$ contains $(n - k - e)m_I = (7 - 5 - 1)2 = 2$ paths as shown below:

$$\pi(7, 1) = 34165 \rightarrow \underline{7}4165 \rightarrow \underline{7}4365 \rightarrow \underline{7}2365 \rightarrow \underline{7}2345 \rightarrow 12345$$

$$\pi(7, 3) = 34165 \rightarrow 34\underline{7}65 \rightarrow 32\underline{7}65 \rightarrow 32\underline{7}45 \rightarrow 12\underline{7}45 \rightarrow 12345.$$

Proposition 2. There is a family ϕ_2 of $(n - k - e)(m_E - e)$ parallel paths between p and I_k, each of length $D(p) + 1$. These paths are node-disjoint with those of ϕ_1.

Proof. Let x be an external foreign element. Let y, $1 \leqslant y \leqslant k$, be such that $y = a_{ij}$ is the j'th element of an external cycle $C_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$. a_{ij} cannot be the last element a_{im_i} of C_i since a_{ij} is non-foreign while a_{im_i} is foreign. We define $\pi(x, y) = \pi(x, a_{ij}) = \langle p, T(x, a_{ij})\ T(a_{ij+1}, a_{ij+1}) \ldots T(a_{im_i-1}, a_{im_i-1})\ \tau^*_{i+1}(a_{im_i})\ \tau^*_{i+2}(a_{im_i}) \ldots \tau^*_c(a_{im_i})\ \tau^*_1(a_{im_i})\ \tau^*_2(a_{im_i}) \ldots \tau^*_{i-1}(a_{im_i})\ T(a_{i1}, a_{i1})\ T(a_{i2}, a_{i2}) \ldots T(a_{ij}, a_{ij})\rangle$. The previous observations (a), (b) and (c) can be verified in this case too. This allows in a similar manner to show that, for different choices of the pair (x, y), we obtain node-disjoint paths. However, the length of $\pi(x, a_{ij})$ is $D(p) + 1$ in this case, that is because optimal cycle-sorting transition lists have been used for all p-cycles except for C_i. To sort C_i, m_i transitions have been used, which is 1 plus the optimal number of transitions needed to sort C_i. Since there are $(n - k - e)$ external foreign elements and $(m_E - e)$ non-foreign misplaced elements belonging to external p-cycles (an external p-cycle C_j has $m_j - 1$ non-foreign elements and one foreign element), therefore there are $(n - k - e)(m_E - e)$ parallel paths of length $D(p) + 1$; we group these paths in a family ϕ_2. The paths of ϕ_1 and those of ϕ_2 are node-disjoint and this is implied by observations (a) and (c) which are satisfied for both ϕ_1 and ϕ_2. Q.E.D.

Example 2. We construct the family ϕ_2 for the same data for example 1: $n = 7$, $k = 5$, $p = 34165$, $C_1 = (2, 4, 6)$, $C_2 = (1, 3)$, $c = 2$, $e = 1$, $m_I = 2$, $m_E = 3$, $D(p) = 5$

and $EXT(p) = \{2, 7\}$. The size of ϕ_2 is $|\phi_2| = (n - k - e)(m_E - e) = (7 - 5 - 1)(3 - 1) = 2$. The two paths of ϕ_2 are:

$$\pi(7, 2) = 34165 \rightarrow 3\underline{7}165 \rightarrow 3\underline{7}145 \rightarrow 3\underline{7}645 \rightarrow 1\underline{7}645 \rightarrow 1\underline{7}345 \rightarrow 12345$$

$$\pi(7, 4) = 34165 \rightarrow 341\underline{7}5 \rightarrow 346\underline{7}5 \rightarrow 146\underline{7}5 \rightarrow 143\underline{7}5 \rightarrow 123\underline{7}5 \rightarrow 12345.$$

It can be observed from the above construction of ϕ_1 and ϕ_2, that our approach in building the sub-families of parallel paths is based on the following steps. First, select an external element x which can be foreign or non-foreign. Second, select a position y which can be one of the following: an element of an internal p-cycle, the first element of an external p-cycle, a non-first element of an external p-cycle, or a non misplaced element of p. Third, characterize a path $\pi(x, y)$ using transitions and optimal cycle sorting transition lists. Fourth, based on Lemma 1, Lemma 2, Lemma 3, and the order of transitions in $\pi(x, y)$, show that for different selections of x and y we obtain a sub family of optimal or near-optimal paths which are node-disjoint among each other as well as with respect to previously obtained paths. For brevity, we will skip the fourth step in the proofs of the remaining propositions. These can be fairly easily obtained using a similar reasoning to that used in the proofs of Proposition 1 and Proposition 2.

Proposition 3. There is a family ϕ_3 of em_I parallel paths between p and I_k, each of length $D(p) + 2$. These paths are node-disjoint with those of $\phi_1 \cup \phi_2$.

Proof. Let x be an external non-foreign element; x is then the first element of an external p-cycle C_r (i.e., $x = a_{r1}$). Let y, $1 \leqslant y \leqslant k$, be such that $y = a_{ij}$ is the j'th element of an internal cycle $C_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$. We define $\pi(x, y) = \pi(a_{r1}, a_{ij}) = \langle p,\ T(a_{r1}, a_{ij})\ T(a_{ij+1}, a_{ij+1}) \ldots T(a_{im_i}, a_{im_i})\ T(a_{i1}, a_{rm_r-1})\ T(a_{rm_r}, a_{ij})\ \tau^*_{i+1}(a_{r1})\ \tau^*_{i+2}(a_{r1}) \ldots \tau^*_r(a_{r1})\ \tau^*_{r+1}(a_{i1})\ \tau^*_{r+2}(a_{i1}) \ldots \tau^*_c(a_{i1})\ \tau^*_1(a_{i1})\ \tau^*_2(a_{i1}) \ldots \tau^*_{i-1}(a_{i1})\ T(a_{i1}, a_{i1})\ T(a_{i2}, a_{i2}) \ldots T(a_{ij}, a_{ij})\rangle$. Let ϕ_3 be the set of such paths. There are m_I possible values for a_{ij} and e possible values for a_{r1}, hence there are em_I parallel paths in ϕ_3 of length $D(p) + 2$ which are node-disjoint with the paths of $\phi_1 \cup \phi_2$. Q.E.D.

Example 3. We continue with the same data $n = 7$, $k = 5$, $p = 34165$, $C_1 = (2, 4, 6)$, $C_2 = (1, 3)$, $c = 2$, $e = 1$, $m_I = 2$, $m_E = 3$, $D(p) = 5$ and $EXT(p) = \{2, 7\}$. The family ϕ_3 in this case contains $em_I = 2$ paths described below:

$$\pi(2, 1) = 34165 \rightarrow \underline{2}4165 \rightarrow \underline{2}4365 \rightarrow \underline{2}4315 \rightarrow \underline{6}4315 \rightarrow \underline{6}2315 \rightarrow \underline{6}2345 \rightarrow 12345$$

$$\pi(2, 3) = 34165 \rightarrow 34\underline{2}65 \rightarrow 34\underline{2}15 \rightarrow 34\underline{6}15 \rightarrow 32\underline{6}15 \rightarrow 32\underline{6}45 \rightarrow 12\underline{6}45 \rightarrow 12345.$$

Proposition 4. There is a family ϕ_4 of $m_E - e$ parallel paths between p and I_k, each of length $D(p) + 2$. These paths are node-disjoint with those of $\phi_1 \cup \phi_2 \cup \phi_3$.

Proof. Let x be an external non-foreign element; x is then the first element of an external p-cycle $C_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$, that is $x = a_{i1}$. Let $y = a_{ij}$ be the j'th element of C_i, $j \neq m_i$. For each a_{ij}, we construct a path $\pi(x, y) = \pi(a_{i1}, a_{ij})$.
Case 1 ($j \neq m_i - 1$, $j \neq 1$): $\pi(a_{i1}, a_{ij}) = \langle p,\ T(a_{i1}, a_{ij})\ T(a_{ij+1}, a_{ij+1}) \ldots T(a_{im_i-1}, a_{im_i-1})\ T(a_{im_i}, a_{ij})\ \tau^*_{i+1}(a_{i1})\ \tau^*_{i+2}(a_{i1}) \ldots \tau^*_c(a_{i1})\ \tau^*_1(a_{i1})\ \tau^*_2(a_{i1}) \ldots \tau^*_{i-1}(a_{i1})\ T(a_{i1}, a_{i1}) \ldots T(a_{ij}, a_{ij})\rangle$,

Case 2 ($j = m_i - 1$): $\pi(a_{i1}, a_{im_i-1}) = \langle p, T(a_{i1}, a_{im_i-1})\ \tau^*_{i+1}(a_{im_i})\ \tau^*_{i+2}(a_{im_i}) \ldots \tau^*_c(a_{im_i})$ $\tau^*_1(a_{im_i})\ \tau^*_2(a_{im_i}) \ldots \tau^*_{i-1}(a_{im_i})\ T(a_{im_i}, a_{im_i-1})\ T(a_{i1}, a_{i1})\ T(a_{i2}, a_{i2}) \ldots T(a_{im_i-1}, a_{im_i-1})\rangle$,
Case 3 ($j = 1$): $\pi(a_{i1}, a_{i1}) = \langle p, T(a_{i1}, a_{i1})\ T(a_{i2}, a_{i2}) \ldots T(a_{im_i-1}, a_{im_i-1})\ T(a_{im_i}, a_{i1})$ $\tau^*_{i+1}(a_{i1})\ \tau^*_{i+2}(a_{i1}) \ldots \tau^*_c(a_{i1})\ \tau^*_1(a_{i1})\ \tau^*_2(a_{i1}) \ldots \tau^*_{i-1}(a_{i1})\ T(a_{i1}, a_{i1})\rangle$.
Let ϕ_4 be the set of such paths. There are $m_E - e$ elements a_{ij} in external cycles which are not tails of cycles. Therefore, there are $(m_E - e)$ node-disjoint paths in ϕ_4 each of length $D(p) + 2$ and which are node-disjoint with the paths of $\phi_1 \cup \phi_2 \cup \phi_3$. Q.E.D.

Example 4. Using $n = 7$, $k = 5$, $p = 34165$, $C_1 = (2, 4, 6)$, $C_2 = (1, 3)$, $c = 2$, $e = 1$, $m_I = 2$, $m_E = 3$, $D(p) = 5$ and $EXT(p) = \{2, 7\}$ of examples 1, 2 and 3, we construct the paths of the family ϕ_4 for this case. There are $m_E - e = 3 - 1 = 2$ paths in ϕ_4 given by:

$$\pi(2, 2) = 34165 \to 3\underline{2}165 \to 3\underline{2}145 \to 3\underline{6}145 \to 3\underline{6}245 \to 1\underline{6}245 \to 1\underline{6}345 \to 12345$$

$$\pi(2, 4) = 34165 \to 341\underline{2}5 \to 346\underline{2}5 \to 146\underline{2}5 \to 143\underline{2}5 \to 143\underline{6}5 \to 123\underline{6}5 \to 12345.$$

Proposition 5. There is a family ϕ_5 of $(m_E - e)(e\text{-}1)$ parallel paths between p and I_k, of length $D(p) + 3$. These paths are node-disjoint with those of $\phi_1 \cup \phi_2 \cup \phi_3 \cup \phi_4$.

Proof. Let x be an external non-foreign element; x is the first element of an external p-cycle; let C_r be this cycle. Let y, $1 \leqslant y \leqslant k$, be such that $y = a_{ij}$ is the j'th element of an external cycle $C_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$ and $i \neq r$. a_{ij} can not be the last element a_{im_i} of C_i since a_{ij} is non-foreign while a_{im_i} is foreign. We define $\pi(x, y) = \pi(a_{r1}, a_{ij}) = \langle p, T(a_{r1}, a_{ij})\ T(a_{ij+1}, a_{ij+1}) \ldots T(a_{im_i-1}, a_{im_i-1})\ T(a_{im_i}, a_{rm_r-1})\ T(a_{rm_r}, a_{ij})$ $\tau^*_{i+1}(a_{r1})\ \tau^*_{i+2}(a_{r1}) \ldots \tau^*_r(a_{r1})\ \tau^*_{r+1}(a_{i1})\ \tau^*_{r+2}(a_{i1}) \ldots \tau^*_c(a_{i1})\ \tau^*_1(a_{i1})\ \tau^*_2(a_{i1}) \ldots \tau^*_{i-1}(a_{i1})$ $T(a_{i1}, a_{i1})\ T(a_{i2}, a_{i2}) \ldots T(a_{ij}, a_{ij})\rangle$. Let ϕ_5 be the set of such paths. There are (m_E − e) elements a_{ij} of external cycles which are not tails of these cycles. For each such a_{ij} there are $(e - 1)$ elements a_{r1} which are heads of external cycles other than C_i. Hence ϕ_5 contains $(m_E - e)(e - 1)$ node-disjoint paths of length $D(p) + 2$ which are node-disjoint with the previous paths of $\phi_1 \cup \phi_2 \cup \phi_3 \cup \phi_4$. Q.E.D.

Example 5. The parameters of examples 1, 2, 3 and 4 correspond to an empty set for the family ϕ_5 (see example 8 for a case with a non-empty ϕ_5).

Proposition 6. There is a family ϕ_6 of $(n - k - e)(k - m_I - m_E + e)$ parallel paths from p to I_k, of length $D(p) + 2$. These are node-disjoint with those of $\phi_1 \cup \phi_2 \cup \phi_3 \cup \phi_4 \cup \phi_5$.

Proof. Let x be an external foreign element ($x \in EXT(p)$ and $x > k$) and let y, $1 \leqslant y \leqslant k$, be such that p_y is in its correct position ($p_y = y$). We define $\pi(x, y) = \langle p, T(x, y)\ \tau^*_1(y)\ \tau^*_2(y) \ldots \tau^*_c(y)\ T(y, y)\rangle$. Let ϕ_6 be the set of such paths. There are (n $- k - e$) external foreign elements and $(k - m_I - m_E + e)$ correctly positioned elements in p, therefore ϕ_6 contains (n $- k - e)(k - m_I - m_E + e)$ parallel paths. All

p-cycles are optimally sorted along $\pi(x, y)$ and two extra transitions are used (to misplace then correct the element y), so the length of $\pi(x, y)$ is $D(p) + 2$. Q.E.D.

Example 6. Using the same data as before (i.e., $n = 7$, $k = 5$, $p = 34{,}165$, $C_I = (2, 4, 6)$, $C_2 = (1, 3)$, $c = 2$, $e = 1$, $m_I = 2$, $m_E = 3$, $D(p) = 5$ and $EXT(p) = \{2, 7\}$) yields a family ϕ_6 containing $(n - k - e)(k - m_I - m_E + e) = 1$ path of length $D(p) + 2 = 5 + 2 = 7$ given by:

$$\pi(7, 5) = 34165 \rightarrow 3416\underline{7} \rightarrow 3216\underline{7} \rightarrow 3214\underline{7} \rightarrow 3254\underline{7} \rightarrow 1254\underline{7} \rightarrow 1234\underline{7} \rightarrow 12345.$$

Proposition 7. There is a family ϕ_7 of $e(k - m_I - m_E + e)$ parallel paths between p and I_k of length $D(p) + 4$. These are parallel to the paths of $\phi_1 \cup \phi_2 \cup \phi_3 \cup \phi_4 \cup \phi_5 \cup \phi_6$.

Proof. Let x be an external non-foreign element and $1 \leqslant x \leqslant k$; x must be the first element of an external p-cycle; let C_r be this cycle (i.e., $x = a_{r1}$). Let y, $1 \leqslant y \leqslant k$, be such that p_y is in its correct position (i.e., $p_y = y$). We define $\pi(x, y) = \pi(a_{r1}, y) = \langle p, T(a_{r1}, y)\ T(y, a_{rm_r-1})\ T(a_{rm_r}, y)\ \tau_1^*(a_{r1})\ \tau_2^*(a_{r1}) \ldots \tau_r^*(a_{r1})\ \tau_{r+1}^*(y)\ \tau_{r+2}^*(y) \ldots \tau_c^*(y)\ T(y, y)\rangle$. Let ϕ_7 be the set of such paths. Since there are e external non-foreign elements and $(k - m_I - m_E + e)$ correctly positioned elements in p, then ϕ_7 contains $e(k - m_I - m_E + e)$ paths. All p-cycles are optimally sorted along each path π of ϕ_7 and four extra transitions are used, so the length of these paths is $D(p) + 4$. Q.E.D.

Example 7. We use the same data as before (i.e., $n = 7$, $k = 5$, $p = 34165$, $C_1 = (2, 4, 6)$, $C_2 = (1, 3)$, $c = 2$, $e = 1$, $m_I = 2$, $m_E = 3$, $D(p) = 5$ and $EXT(p) = \{2, 7\}$). The family ϕ_7 for this case has $|\phi_7| = e(k - m_I - m_E + e) = 1$ path of length $D(p) + 4 = 9$ shown below.

$$\pi(2, 5) = 34165 \rightarrow 3416\underline{2} \rightarrow 3415\underline{2} \rightarrow 3415\underline{6} \rightarrow 215\underline{6} \rightarrow$$
$$3214\underline{6} \rightarrow 3254\underline{6} \rightarrow 1254\underline{6} \rightarrow 1234\underline{6} \rightarrow 12345.$$

Proposition 8. There are $k(n - k)$ node-disjoint paths between any two nodes of $A_{n,k}$. Each path is of length at most the distance between the two nodes plus four.

Proof. From propositions 1 to 7 we derive that there are $k(n - k)$ parallel paths between p and I_k. By vertex symmetry this is generalized to an arbitrary pair of nodes. Q.E.D.

Example 8. For the data used in the examples 1 through 7, we have built a complete family of $k(n - k) = 10$ parallel paths between $p = 34165$ and $I_5 = 12345$ in $A_{7,5}$. Now we construct a complete family of parallel paths for $p = 34765$ in $A_{7,5}$. There are two p-cycles $C_1 = (1, 3, 7)$, $C_2 = (2, 4, 6)$; C_1 and C_2 are both external. We have $c = 2$, $e = 2$, $m_I = 0$, $m_E = 6$, $D(p) = 4$ and $EXT(p) = \{1, 2\}$. According to the previous propositions there is a family $\phi = \phi_1 \cup \phi_2 \cup \phi_3 \cup \phi_4 \cup \phi_5 \cup \phi_6 \cup \phi_7$ of $k(n - k) = 10$ node-disjoint paths between p and I_k such that: $|\phi_1| = 0$, $|\phi_2| = 0$,

$|\phi_3| = 0$, $|\phi_4| = 4$, $|\phi_5| = 4$, $|\phi_6| = 0$ and $|\phi_7| = 2$. These 10 paths are listed below:

$$\phi_4\colon\ \pi(1, 1) = 34765 \rightarrow \underline{1}4765 \rightarrow \underline{1}4365 \rightarrow \underline{7}4365 \rightarrow \underline{7}2365 \rightarrow \underline{7}2345 \rightarrow 12345,$$

$$\pi(1, 3) = 34765 \rightarrow 34\underline{1}65 \rightarrow 32\underline{1}65 \rightarrow 32\underline{1}45 \rightarrow 32\underline{7}45 \rightarrow 12\underline{7}45 \rightarrow 12345,$$

$$\pi(2, 2) = 34765 \rightarrow 3\underline{2}765 \rightarrow 3\underline{2}745 \rightarrow 3\underline{6}745 \rightarrow 1\underline{6}745 \rightarrow 1\underline{6}345 \rightarrow 12345,$$

$$\pi(2, 4) = 34765 \rightarrow 347\underline{2}5 \rightarrow 147\underline{2}5 \rightarrow 143\underline{2}5 \rightarrow 143\underline{6}5 \rightarrow 123\underline{6}5 \rightarrow 12345.$$

$$\phi_5\colon\ \pi(1, 2) = 34765 \rightarrow 3\underline{1}765 \rightarrow 3\underline{1}745 \rightarrow 3\underline{1}645 \rightarrow 3\underline{7}645 \rightarrow 1\underline{7}645 \rightarrow 1\underline{7}345 \rightarrow 12345,$$

$$\pi(1, 4) = 34765 \rightarrow 347\underline{1}5 \rightarrow 346\underline{1}5 \rightarrow 346\underline{7}5 \rightarrow 146\underline{7}5 \rightarrow 143\underline{7}5 \rightarrow 123\underline{7}5 \rightarrow 12345,$$

$$\pi(2, 1) = 34765 \rightarrow \underline{2}4765 \rightarrow \underline{2}4365 \rightarrow \underline{2}4375 \rightarrow \underline{6}4375 \rightarrow \underline{6}2375 \rightarrow \underline{6}2345 \rightarrow 12345,$$

$$\pi(2, 3) = 34765 \rightarrow 34\underline{2}65 \rightarrow 34\underline{2}75 \rightarrow 34\underline{6}75 \rightarrow 32\underline{6}45 \rightarrow 12\underline{6}45 \rightarrow 12345.$$

$$\phi_7\colon\ \pi(1, 5) = 34765 \rightarrow 3476\underline{1} \rightarrow 3456\underline{1} \rightarrow 3456\underline{7} \rightarrow 1456\underline{7} \rightarrow 1436\underline{7} \rightarrow 1236\underline{7} \rightarrow 1234\underline{7} \rightarrow 12345,$$

$$\pi(2, 5) = 34765 \rightarrow 3476\underline{2} \rightarrow 3475\underline{2} \rightarrow 3475\underline{6} \rightarrow 1475\underline{6} \rightarrow 1435\underline{6} \rightarrow 1235\underline{6} \rightarrow 1234\underline{6} \rightarrow 12345.$$

Corollary: There are $n - 1$ node-disjoint paths between any two nodes of an n-star graph. Each path is of minimum distance plus at most four.

4. CONCLUSION

We have characterized complete families of parallel paths between nodes of the arrangement graph. Using vertex symmetry, we have reduced the problem of constructing parallel paths between two arbitrary nodes to the problem of constructing parallel paths between an arbitrary node and the identity node I_k. We have shown the existence of a family of $k(n - k)$ parallel paths between any node p and I_k. Since $A_{n,k}$ is regular of degree $k(n - k)$, such a family is of maximum size. Furthermore, all the paths are of length at most four plus the optimal length. This applies to the special arrangement graph—the star graph—yielding $n - 1$ parallel paths between any two nodes of the n-star each of length at most four plus the optimal length. An extension of this work is to develop a fault-tolerant routing algorithm using the constructed paths.

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تحديد المسالك المتوازية في شبكات التراتيب

خالد داي
جامعة السلطان قابوس ، مسقط ، سلطنة عمان
آنند تريباتي
جامعة مينسوتا ، مينيابوليس، مينسوتا، الولايات المتحدة الأمريكية

خلاصة

نقوم في هذا البحث بتحديد كامل للمسالك المتوازية داخل شبكات التراتيب. وقد تم مؤخرا تعريف ودراسة هذه الشبكات كطريقة فعالة لتوصيل نقاط المعالجة في الحواسيب المتوازية. وفي هذا البحث نبرهن على أنه من الممكن دائما توصيل أي نقطتي معالجة داخل شبكة تراتيب بمجموعة كاملة من المسالك المتوازية بأطوال أقصاها أربع توصيلات زائدة عن أقصر مسافة بين النقطتين. تفيد هذه المسالك في النقل السريع لكميات كبيرة من البيانات عبر المسالك المتوازية كما تتيح مسارات بديلة عند وجود أعطال في بعض نقاط المعالجة.