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Anti-invariant submanifolds of a Kenmotsu manifold

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ABSTRACT

Many subjects for anti-invariant submanifolds of a Sasakian manifold have been investigated extensively by Yano & Kon (1976) and others. Kenmotsu (1972) introduced a new class of almost contact metric structure known as 'Kenmotsu structure, which is closely related to the warped product of two Riemannian manifolds. The purpose of this note is to study anti-invariant submanifolds of Kenmotsu manifold. We study the theory of such submanifold from two different point of view, namely, one is the case where the anti-invariant submanifolds are tangent to the structure vector field, and the other is the case where these are normal to the structure vector field.

INTRODUCTION

Let $\bar{M}$ be an odd-dimensional differentiable manifold and ϕ, ξ, η be a tensor field of type (1.1), a vector field, a 1-form on $\bar{M}$ respectively such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0 \text{ or } \eta o \phi = 0 \text{ or } \eta(\xi) = 1 \tag{1.1}$$

where I denotes the identity transformation on $T\bar{M}$. Now we suppose that $\bar{M}$ is given by a Riemannian metric ϕ which satisfies the condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.2}$$

Putting ξ and Y in (1.2) and using (1.1), we get

$$\eta(X) = g(X, \xi)$$

for any $X, Y \in T\bar{M}$. Then $\bar{M}(\phi, \xi, n, g)$ is called an almost contact metric manifold.

An almost contact metric manifold is a Kenmotsu manifold if, and only if, we have (Kenmotsu 1972)

$$(\bar{\nabla}_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{1.3}$$

This equation implies

$$\bar{\nabla}_X \xi = X - \eta(X)\xi \tag{1.4}$$

for any vector field X, Y tangent to $\bar{M}$, where ∇ denotes the Riemannian connection on $\bar{M}$.

This structure is closely related to the warped product of two Riemannian manifolds. One of the typical examples of Kenmotsu manifold is the hyperbolic space $\bar{M}(-1)$.

Now let M be an m-dimensional Riemannian manifold, isometrically immersed in a Kenmotsu manifold $\bar{M}$, and by T_xM, $T_x^{\perp}M$, the tangent and normal bundle of M at $x\in M$. For the submanifold M of Kenmotsu manifold $\bar{M}$, the Riemannian connection $\bar{\nabla}$ of $\bar{M}$ induced Riemannian connection ∇ on M and connection $\nabla^{\perp}$ in the normal bundle $T^{\perp}M$ satisfying the condition

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y); \quad \bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{1.5}$$

for any X, $Y\in TM$, $N\in T^{\perp}M$ where h and A_N are the second fundamental form, Weingarten map satisfying $g(h(X, Y), N) = g(A_N X, Y)$.

A submanifold M of Kenmotsu manifold $\bar{M}$ is called anti-invariant if $\phi T_xM \subset T^{\perp}{}_xM$ for all $x\in M$.

Now suppose that M^m is an m-dimensional anti-invariant submanifold of a Kenmotsu manifold $\bar{M}^{2n+1}$. Then for any X of $\bar{M}^{2n+1}$ at a point of M^m, we put

$$X = X_t + X_n \tag{1.6}$$

where X_t (resp. X_n) is the tangential (resp. normal) component of X. Define homomorphisms P and Q of the normal bundle into the tangent and normal bundle of M^m, respectively by

$$PN = (\phi N)_t, \text{ and } QN = (\phi N)_n \tag{1.7}$$

for every normal vector field N of M^m.

If X is a vector field on an anti-invariant submanifold M^m, then QX is a vector field in the normal bundle of M^m, and $m>1$, as any 1-dimensional submanifold is anti-invariant submanifold.

Now pre-multiplying ϕX, ϕN and ξ by ϕ and comparing tangential and normal components we get the following

$$\left.\begin{array}{ll} -X+\eta(X)\xi_t = P\phi X, & \eta(X)\xi_n = Q\phi X \\ \eta(N)\xi_t = PQN, & -N+\eta(N)\xi_n = \phi PN + Q^2N \\ P\xi_n = 0, & P\xi_t + Q\xi_n = 0 \end{array}\right\} \tag{1.8}$$

for any $X\in TM$ and $N\in T^{\perp}M$.

A Kenmotsu manifold with constant ϕ-holomorphic sectional structure c is called Kenmotsu space form $\bar{M}(c)$. The curvature tensor of $\bar{M}(c)$ is given by Kenmotsu (1972)

$$\begin{aligned} \bar{R}(X, Y, Z, W) = & \frac{(c-3)}{4}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & + \frac{(c+1)}{4}[\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ & + \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z) \\ & + g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) \\ & + 2g(X, \phi Y)g(\phi Z, W)] \end{aligned} \tag{1.9}$$

for any $X, Y, Z, W \in TM$.

Let $\bar{R}$ (resp. R) be the curvature tensor of $\bar{M}$ (resp. M). Then the equation of Gauss is given by

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(Y, W), h(X, Z)) \tag{1.10}$$

The submanifold M^m is called totally umbilical if $h(X, Y) = g(X, Y)H$ for some normal vector field H. If $h = 0$, then M^m is called totally geodesic.

THE CASE IN WHICH ξ IS TANGENT TO M^m

In this section, we assume that ξ is tangent to M^m. So $\xi_n = 0$ Thus (1.8) gives

$$-X + \eta(X)\xi = P\phi X, \qquad Q\phi X = 0 \tag{2.1}$$

$$PQN = 0, \qquad -N = \phi PN + Q^2 N \tag{2.2}$$

for any $X \varepsilon TM$ and $N \varepsilon T^{\perp}M$.

Assume that M is an anti-invariant submanifold of a Kenmotsu manifold $\bar{M}$. Using (1.5), (1.7) in (1.3) and the fact that $\phi Y \in T^{\perp}M$ for all $Y \in TM$
we get

$$-A_{\phi X}Y + \nabla_Y \phi X - \phi \nabla_Y X - Ph(X, Y) - Qh(X, Y) = -\eta(X)\phi Y$$

Now, comparing the tangential and normal components, we get

$$-A_{\phi X}Y = Ph(X, Y) \tag{2.3}$$

$$\nabla_Y^{?} \phi X = \phi \nabla_Y X + Qh(X, Y) - \eta(X)\phi Y \tag{2.4}$$

for all $X, Y \in TM$.

Similarly for any $X \in TM$ and $N \in T^{\perp}M$, we get

$$P\nabla_X^{\perp} N = \bar{\nabla}_X PN - A_{QN}X \tag{2.5}$$

$$-\phi A_N X + Q\nabla_X^{\perp} N = h(X, PN) + \nabla_X^{\perp} QN \tag{2.6}$$

Also using (1.5) in (1.4) and comparing tangential and normal components, we get

$$\nabla_X \xi = X - \eta(X)\xi, \quad h(X, \xi) = 0 \tag{2.7}$$

for any $Y \in TM$.

From (2.5) we have

$$\bar{\nabla}_\xi PN - P\nabla_\xi^{\perp} N = A_{QN}\xi$$

Now $g(A_{QN}\xi, X) = g(h(X, \xi), QN) = 0$ as $h(X, \xi) = 0$ from (2.7).

Hence $(\bar{\nabla}_\xi P)(N) = \bar{\nabla}_\xi PN - P\nabla_\xi^{\perp} N = 0$.

Similarly $(\bar{\nabla}_\xi Q)(N) = \nabla_\xi^{\perp} QN - Q\nabla_\xi^{\perp} N = 0$

Thus from the above considerations, we have the following

Proposition 2.1. Let M^m be an anti-invariant submanifold of the Kenmotsu manifold $\bar{M}^{2n+1}$ such that ξ is tangent to M. Then

(a) P and Q are parallel along ξ.
(b) the normal curvature $h(\xi, \xi)$ vanishes in the direction of ξ.
(c) the vector field ξ restricted to M is parallel.

If $m = n + 1$ and $\bar{R}$ (resp. $R^{\perp}$) is the curvature on M^{n+1} (resp. on the normal bundle). Then we prove

Proposition 2.2. Let M^{n+1} be an anti-invariant submanifold of the Kenmotsu manifold M^{2n+1} such that ξ is tangent to M^{n+1}. Then $\bar{R} \equiv 0$ if and only if $\bar{R}^{\perp} \equiv 0$.

Proof. Since $m = n + 1$ so obviously $Q = 0$ and by virtue of (2.5) we have

$$P\nabla^{\perp}_{X}N = \bar{\nabla}_{X}PN.$$

Therefore, we easily get

$$\bar{R}(X, Y)PN + PR^{\perp}(X, Y)N$$

for any

$$X, Y \in TM, N \in T^{\perp}M.$$

Thus $\bar{R} \equiv 0$ implies $R^{\perp} \equiv 0$ on the other hand if $R^{\perp} = 0$, then $\bar{R}(X, Y)PN = 0$ and also $\bar{R}(X, Y) \in = 0$ and hence $\bar{R}(X, Y) \equiv 0$.

Next we assume that M^{m} is an anti-invariant submanifold of Kenmotsu space form $\bar{M}(c)$ with ξ tangent to M. Then from (1.9) and using (1.10) we get

$$\begin{aligned} R(X, Y, Z, W) = {} & \frac{(c-3)}{4}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & + \left(\frac{(c+1)}{4}\right)[\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ & + \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z)] \\ & + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \end{aligned} \tag{2.8}$$

Now, if $A_t A_s = A_s A_t$ for all t and s, then second fundamental form of M is said to be commutative.

If the second fundamental form of M is commutative, then we have (equation 3.8 in Yano & Kon (1977).

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) - (g(X, Z)g(Y, W) - g(X, W)g(Y, Z)) \tag{2.9}$$

Now assume that $\bar{M}^{2n+1}(c)$ is kenmotsu space form and the second fundamental form of M^{n+1} is commutative, then using (2.8) in (2.9), we get

$$\begin{aligned} \bar{R}(X, Y, Z, W) = {} & \frac{(c+1)}{4}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ & -\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ & + \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z)] \end{aligned} \tag{2.10}$$

for all $X, Y, Z, W \in TM$.

From this we have

Proposition 2.3. Let M^{n+1} ($n \geq 2$) be an anti-invariant submanifold of a Kenmotsu space form $\overline{M}^{2n+1}(c)$ with commutative second fundamental form such that ξ is tangent to M. Then M is flat if and only if $\overline{M}$ is of constant curvature -1 i.e. $c = -1$.

Since ξ is parallel with respect to the induced connection on M (proposition 2.1). Thus using (2.4) and a theorem by Yano (1953) we have:

Theorem 2.4. Let M^{n+1} be an anti-invariant submanifold of a Kenmotsu space form $\overline{M}^{2n+1}(c)$. If the second fundamental form of M is commutative such that ξ is tangent to M, then M is locally a Riemannian direct product $M^n \times R^1$, where M^n is a hypersurface of M^{n+1} of constant curvature $\frac{1}{4}(c+1)$ and is totally geodesic in M^{n+1}

THE CASE IN WHICH ξ IS NORMAL TO M^m

In this section, we assume that ξ is normal to M^m so $\xi_t = 0$and (1.8) gives

$$-X = P\phi X, \quad Q\phi X = 0, \tag{3.1}$$

$$PQN = 0, \quad -N + \eta(N)\xi = \phi PN + Q^2 N \tag{3.2}$$

for any $X \in TM$, $N \in T^{\perp}M$

Let $\overline{M}^{2n+1}$ be a Kenmotsu manifold. Using (1.5) & (1.6) in (1.3) & (1.4) and comparing tangential and normal components, we get

$$-A_{\phi Y}X = \phi h(X, Y), \nabla^{\perp}_X \phi Y = \phi(\overline{\nabla}_X Y) + Qh(X, Y), \tag{3.3}$$

$$\overline{\nabla}_X PN - A_{QN}X = P\nabla^{\perp}_X N \tag{3.4}$$

$$\phi\nabla^{\perp}_X N - \phi A_N X = h(X, PN) + \nabla^{\perp}_X QN + \eta(N)QX + g(X, \phi N)\xi \tag{3.5}$$

$$A_{\xi}X = -X, \quad \nabla^{\perp}_X \xi = -\eta(X)\xi = 0 \tag{3.6}$$

for any X, $Y \in TM$, $N \in T^{\perp}M$.

If $m = n$, then obviously $Q = 0$ (3.3) and (3.6) give

$$-A_{\phi Y}X = \phi h(X, Y), \quad \nabla^{\perp}_X \phi Y = \phi\overline{\nabla}_X Y) \tag{3.7}$$

$$\overline{\nabla}_X PN - P\nabla^{\perp}_X N = 0 \tag{3.8}$$

$$A_{\xi}X = -X, \quad \nabla^{\perp}_X \xi = -\eta(X)\xi = 0 \tag{3.9}$$

for X, $Y \in TM$ and $N \in T^{\perp}M$.

Proposition 3.1. Let M be an n-dimensional ($n > 1$) anti-invariant submanifold of a Kenmotsu manifold $\overline{M}^{2n+1}$ such that ξ is normal to M. If M is totally umbilical, then M is totally geodesic.

Proof. For X, $Y \in TM$, (3.2) gives

$$-g(A_{\phi Y}X, Z) = g(\phi h(X, Y), Z)$$

i.e. $$-g(X, Z)g(H, \phi Y) = g(X, Y)g(\phi H, Z)$$

Putting $X = Z$ and $Y = \phi H$, then the above equation reduces to

$$g(X, X)g(\phi H, \phi H) = g(X, \phi H)^2.$$

Since $n > 1$ so we have $\phi H = 0$ and thus $H = 0$ which proves our assertion.

Let M be an n-dimensional anti-invariant submanifold of a Kenmotsu space form $\overline{M}^{2n+1}(c)$. Now by virtue of the fact that ξ is normal to M, equation (1.9) and (1.10) imply that

$$\overline{R}(X, Y, Z, W) - \frac{(c-3)}{4}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$$

$$+ g(h(X, W), h(Y, Z)) - g(h(Y, W), h(X, Z)) \tag{3.10}$$

for any $X, Y, Z, W \in TM$.

Thus we have

Proposition 3.2. Let M be an n-dimensional anti-invariant submanifold of a Kenmotsu space form $\overline{M}^{2n+1}(c)$ such that ξ is normal to M. If M is totally geodesic then M is of constant curvature $\left(\frac{c-3}{4}\right)$.

From this and proposition 3.1, we get

Proposition 3.3. Let M be an n-dimensional ($n > 1$) anti-invariant and totally umbilical submanifold of a Kenmotsu space form $M^{2n+1}(c)$ such that ξ is normal to M, then M is of constant curvature $\left(\frac{c-3}{4}\right)$.

Next from (3.10), we easily have

Proposition 3.4. Let M be an n-dimensional anti-invariant submanifold of a Kenmotsu space form $\overline{M}^{2n+1}(c)$ such that ξ is normal to M. Then M is of constant curvature $\left(\frac{c-3}{4}\right)$ if and only if M has the commutative second fundamental form.

Now, we prove

Proposition 3.5. Let M^m be an anti-invariant submanifold of the Kenmotsu manifold $\overline{M}^{2n+1}$ such that ξ is normal to M^m. Then the curvature tensor of the normal bundle annihilates ξ.

Proof. From (3.6) for any $X, Y \in TM$, we have

$$\nabla^{\perp}_{Y}\nabla^{\perp}_{X}\xi = 0 \text{ as } \nabla^{\perp}_{X}\xi = 0.$$

Thus, we have

$$R^{\perp}(X, Y)\xi = \nabla^{\perp}_{X}\nabla^{\perp}_{Y}\xi - \nabla^{\perp}_{Y}\nabla^{\perp}_{X}\xi - \nabla[X, Y]DD^{\perp}\xi = 0$$

which completes the proof.

Finally we prove

Proposition 3.6. Let M^n be an anti-invariant submanifold of a kenmotsu manifold $\overline{M}^{2n+1}$ such that ξ is normal to M^n. Then the connection in the normal bundle is trivial if and only if M^n is of constant curvature zero.

Proof. For any $X, Y, Z \in TH$ and using of (3.7) we have

$$R^{\perp}(X, Y)\phi Z = \nabla^{\perp}_{X}(\nabla^{\perp}_{Y}\phi Z) - \nabla^{\perp}_{Y}(\nabla^{\perp}_{X}\phi Z) - \nabla[X, Y]^{\perp}\phi Z$$

$$= \nabla^{\perp}_{X}(\phi\overline{\nabla}_{Y}Z) - \nabla^{\perp}_{Y}(\phi\overline{\nabla}_{X}Z) - \phi\overline{\nabla}[X, Y]Z \tag{3.11}$$

also $\phi\bar{R}(X, Y)Z = \phi(\bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z)$

$$= \nabla^{\perp}_X(\phi\bar{\nabla}_Y Z) - \nabla^{\perp}_Y(\phi\bar{\nabla}_X Z) - \phi\bar{\nabla}_{[X,Y]}Z \quad (3.12)$$

from (3.11) and (3.12) we see that

$$R^{\perp}(X, Y)\phi Z = \phi\bar{R}(X, Y)Z \quad (3.13)$$

Thus if the connection of the normal bundle is trivial i.e. $R^{\perp} \equiv 0$, then M^n is of constant curvature zero.

Conversely, if M^n is of constant curvature zero, then from (3.13) we have $R^{\perp}(X, Y)\phi Z = 0$. Moreover, from proposition 3.5, we have $R^{\perp}(X, Y)\xi = 0$, which completes the proof.

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منطويات جزئية لا متغيرة تخالفياً لمنطوي كينموتسو

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خلاصة

يهدف هذا البحث إلى دراسة منطويات جزئية لمنطوي كينموتسو لا متغيرة تخالفياً. وتتم الدراسة النظرية لمثل هذه المنطويات من وجهتي نظر مختلفتين، إحداهما في الحالة التي تكون فيها المنطويات الجزئية اللامتغيرة تخالفياً مماسة لحقل متجه البنية، والثانية هي الحالة التي تكون فيها هذه المنطويات الجزئية متعامدة مع حقل متجه البنية.