Kuwait J. Sci. & Eng. **23**(2), 1996

Nearly trans-sasakian manifolds

ANJALI SHUKLA

B-71, Nirala Nagar, Lucknow 22602, India.

ABSTRACT

In an earlier paper, a trans-Sasakian manifold was defined and studied by the author. The object of this paper is to define a nearly trans-Sasakian manifold and study some of its properties.

INTRODUCTION

An almost contact metric manifold is an odd-dimensional differentiable manifold V_{2m+1}, on which there are defined a tensor field f of the type (1,1), a vector field V, an 1-form v and a metric tensor field h, satisfying for arbitrary vector fields $X, Y, Z, \ldots \varepsilon V_{2m+1}$ (Sasaki 1960, Hatakeyama *et al.* 1963, Mishra 1984):

$$a)\quad f^2 = -I_{2m+1} + v \otimes V, \qquad b)\quad fV = 0. \tag{1.1}$$

$$'f(X, Y) = h(fX, Y) = -'f(Y, X) \Leftrightarrow h(fX, fY) = h(X, Y) - v(X)v(Y). \tag{1.2}$$

Let D be the Riemannian connection on an almost contact metric manifold. Then

$$(D_X\, 'f)(fY, fZ) + (D_X'f)(Y, Z) = v(Y)(D_Xv)(fZ) - v(Z)(D_Xv)(fY). \tag{1.3}$$

This equation implies

$$(D_X'f)(V, Z) = (D_Xv)(fZ). \tag{1.4}$$

An almost contact metric manifold is said to be a nearly co-symplectic manifold (Goldberg 1963, Mishra 1991), if

$$a)\quad (D_X'f)(Y, Z) = (D_Y'f)(Z, X), \qquad b)\quad (D_Xv)(Y) + (D_Yv)(X) = 0 \tag{1.5}$$

and is said to be a generalised nearly co-symplectic manifold (Mishra 1991), if

$$(D_X'f)(Y, Z) - v(Y)(D_Xv)(fZ) + \{(D_Xv)(fY) + (D_Yv)(fX)\}v(Z)$$
$$= (D_Y\, 'f)(Z, X) + v(X)(D_Yv)(fZ). \tag{1.6}$$

The generalised nearly co-symplectic manifold becomes a nearly nqs (normal quasi-Sasakian) manifold, provided (Mishra 1991).

$$(D_Xv)(fY) = -(D_{fX}v)(Y) = (D_Yv)(fX). \tag{1.7}$$

The equation of a generalised quasi-Sasakian manifold is

$$(D_X{'f})(Y, Z) + (D_Y{'f})(Z, X) + (D_Z{'f})(X, Y)$$
$$= v(X)\{(D_Z v)(fY) - (D_Y v)(fZ)\} + v(Y)\{(D_X v)(fZ) - (D_Z v)(fX)\} +$$
$$v(Z)\{(D_Y v)(fX) - (D_X v)(fY)\}. \tag{1.8}$$

In a trans K contact Riemannian manifold (Shukla 1994)

$$(D_X v)(Y) = h(fX, fY) \Leftrightarrow D_X V = -f^2 X. \tag{1.9}$$

Therefore, in this manifold, (1.8) assumes the form

$$(D_X{'f})(Y,Z) + (D_Y{'f})(Z, X) + (D_Z{'f})(X, Y)$$
$$= 2v(X){'f}(Y, Z) + 2v(Y){'f}(Z, X) + 2v(Z){'f}(X, Y). \tag{1.10}$$

An almost contact metric manifold is said to be normal (Mishra 1991), provided

$$(D_{fX}{'f})(fY,Z) = (D_X{'f})(Y,Z) - v(Y)(D_X v)(fZ). \tag{1.11}$$

and pseudo-normal (Mishra 1991), provided

$$(D_{fX}{'f})(fY,Z) + (D_X{'f})(Y,Z) - v(Y)(D_X v)(fZ) = 0. \tag{1.12}$$

In a normal manifold

$$a)\quad (D_{fX} v)(fY) = (D_X v)(y) \Leftrightarrow (D_{fX} v)(Y) + (D_X v)(fY) = 0,$$
$$b)\quad D_V f = 0. \tag{1.13}$$

and in a pseudo-normal manifold

$$a)\quad (D_{fX} v)(fY) + (D_X v)(Y) = 0 \Leftrightarrow (D_{fX} v)(Y) = (D_X v)(fY),$$
$$b)\quad D_V f = 0. \tag{1.14}$$

A trans-Sasakian manifold has already been defined by (Kenmotsu 1972 and Shukla 1994) as an almost contact metric manifold, satisfying

$$a)\quad (D_X{'f})(Y, Z) = v(Y){'f}(Z, X) + v(Z){'f}(X, Y)$$
$$b)\quad (D_X f)Y = -v(Y)fX + {'f}(X, Y)V,$$
$$c)\quad f(D_X f)Y + v(Y)f^2 X = 0,$$
$$d)\quad f^2(D_X f)Y = v(Y)fX \Leftrightarrow (D_X f)Y + (D_X v)(fY)V + v(Y)fX = 0. \tag{1.15}$$

The equations (1.15) are also equivalent to

$$a)\quad (D_X f)fY = h(fX, fY)V,$$
$$b)\quad (D_X f)f^2 Y + {'f}(X, Y)V = 0 \Leftrightarrow (D_X f)Y + v(Y)fD_X V + {'f}(X, Y)V = 0. \tag{1.16}$$

A nearly trans-Sasakian manifold will now be defined as an almost contact metric manifold, satisfying

$$(D_X{'f})(Y,Z) - v(Y){'f}(Z, X) - v(Z){'f}(X, y)$$
$$= (D_Y{'f})(Z, X) - v(Z){'f}(X, Y) - v(X){'f}(Y, Z) = (D_Z{'f})(X, Y) -$$

$$v(X)'f(Y, Z) - v(Y)'f(Z, X) . \tag{1.17}$$

In the next section we will study some of the properties of a nearly trans-Sasakian manifold.

EQUATIONS ANALYSED

The equations (1.17) are equivalent to

$$a) \quad (D_X f)Y + (D_Y f)X + v(Y)fX + v(X)fY = 0 ,$$

$$b) \quad (D_X f)f\,Y + (D_{f\,Y} f)X + v(X)f^2 Y = 0 ,$$

$$c) \quad (D_X f)f^2 Y + (D_{f^2 Y} f)X = v(X)fY$$

$$d) \quad (D_X f)Y + (D_Y f)X + v(Y)\{fD_X V - (D_V f)X\} + v(X)fY = 0 \tag{2.1}$$

Comparing (2.1) a and (2.1) d, we have

$$(D_V f)X = fD_X V - fX . \tag{2.2}$$

This equation can be obtained directly from (2.1)a.
Also from (2.1)a, we obtain

$$a) \quad (D_X v)(fY) + (D_Y v)(fX) = 0, \qquad b)\ D_V = 0 . \tag{2.3}$$

Putting V for X in (1.17), we get

$$(D_V'f)(Y, Z) = -(D_Y v)(fZ) - 'f(Y, Z) = (D_Z v)(fY) - 'f(Y, Z) . \tag{2.4}$$

Pre-multiplying Y and Z by f in (2.4) and using (2.3)b and (1.3), we obtain

$$(D_V'f)(Y, Z) = -(D_{fY} v)(Z) + 'f(Y, Z) = (D_{fZ} v)(Y) + 'f(Y, Z) . \tag{2.5}$$

From (2.4) and (2.5), we get

$$(D_{fY} v)(Z) + (D_Z v)(fY) = 2'f(Y, Z) . \tag{2.6}$$

Pre-multiplying Y by f in the above equation and the use of (2.3)b yields

$$(D_X v)(Y) + (D_Y v)(X) = 2h(fX, fY) . \tag{2.7}$$

From (2.4) and (2.5), we also get

$$a) \quad (D_{fY} v)(Z) - (D_Y v)(fZ) = 2'f(Y, Z) \Leftrightarrow$$

$$b) \quad (D_{fY} v)(fZ) + (D_Y v)(Z) = 2h(fY, fZ) . \tag{2.8}$$

Also (2.1)b yields

$$(D_X v)(Y) + (D_{fY} v)(fX) = 0 . \tag{2.9}$$

Contracting (2.1)a and (2.7), we conclude that on a nearly trans-Sasakian manifold, the following hold

$$a) \quad (div\, f)Y = 0, \qquad b) \quad (div\, f)V = 0 . \tag{2.10}$$

$$div\ V = 2m. \tag{2.11}$$

It may be noted that on a trans-Sasakian manifold, the same equations, (2.10) and (2.11), also hold.

It is interesting to note that on a Sasakian and a nearly Sasakian manifold

$$a) \quad (div\, f)\ V = 2m, \qquad b) \quad div\ V = 0\,. \tag{2.12}$$

INTERSECTION

Let a nearly trans-Sasakian manifold be a nearly co-symplectic manifold. Then from (1.5) and (1.17)

$$v(Y)'f(Z,X) = v(X)'f(Y,Z) \Rightarrow 'f(Z,X) = 0$$

Hence, nearly trans-Sasakian manifold cannot be a nearly cosymplectic manifold.

Comparing (1.6) and (1.17), we get

$$v(Y)\{(D_X v)(fZ) + 'f(X,Z)\} + v(X)\{(D_Y v)(fZ) - 'f(Z,Y)\}$$
$$= v(Z)\{(D_X v)(fY) + (D_Y v)(fX)\}.$$

Putting V for X or Y in the above equation and using (2.3)b, we get

$$(D_X v)(fZ) = 'f(Z,X) \Leftrightarrow (D_X v)(Z) = h(fX, fZ) \Leftrightarrow D_X V = -f^2 X\,. \tag{3.1}$$

Hence, the necessary and sufficient condition that a generalised, nearly co-symplectic manifold is a nearly trans-Sasakian manifold is (3.1).

$$(D_X v)(fZ) = 'f(Z,X) = -(D_{fX} v)(Z) = -(D_Z v)(fX)\,. \qquad \text{From (3.1)}$$

Comparing these equations with (1.7), we see that

A nearly nqs manifold cannot be a nearly trans-Sasakian manifold and vice-versa. As a particular case, a nearly Sasakian manifold cannot be a nearly trans-Sasakian manifold and vice-versa.

Substituting the values of $(D_Y 'f)(Z, X)$ and $(D_Z 'f(X,Y)$, from (1.17) in (1.8), we obtain

$$3(D_X 'f)(Y,Z) = v(Y)\{'f(Z,X) + (D_X v)(fZ) - (D_Z v)(fX)\} +$$
$$v(Z)\{'f(X,Y) + (D_Y v)(fX) - (D_X v)(fY)\} -$$
$$v(X)\{2'f(Y,Z) + (D_Y v)(fZ) - (D_Z v)(fY)\} = 0\,. \tag{3.2}$$

Hence, the intersection of a nearly trans-Sasakian manifold and a generalised quasi-Sasakian manifold is given by (3.2).

Let the intersection be a generalised co-symplectic manifold. Then

$$(D_X v)(fZ) - (D_Z v)(fX) = 2'f(Z,X)\,. \tag{3.3}$$

In that case (3.2) assumes the form

$$(D_Z 'f)(X,Y) = v(X)'f(Y,Z) + v(Y)'f(Z,X)\,, \tag{3.4}$$

which is the equation of a trans-Sasakian manifold.

Hence, the intersection of a nearly trans-Sasakian manifold and a generalised quasi-Sasakian manifold is a trans-Sasakian manifold, provided (3.3) is satisfied.

The equation (3.3) is satisfied when

$$(D_X v)\,(fY) = -\,'f\,(X,\ Y) \Rightarrow (D_X v)\,(Y) = h(fX, fY) \tag{3.5}$$

that is, when the manifold is trans-K-contact Riemannian (Shukla 1994).
Hence, the intersection of a nearly trans-Sasakian manifold and a generalised quasi-Sasakian manifold in a trans-K-contact Riemannian manifold is a trans-Sasakian manifold.

NORMALITY AND INCLUSION

Let the nearly trans-Sasakian manifold be normal. Then substituting from

$$(D_X{'f})\,(Y, Z) - v(Z)'f\,(X,\ Y) = (D_Z{'f})(X,\ Y) - v(X)'f\,(Y, Z)\ ,$$

in (1.11), we get

$$2(D_Z{'f})\,(X,\ Y) = v(X)\{(D_Z v)\,(fY) + {'f}\,(Y, Z)\} -$$
$$v(Y)\{(D_Z v)\,(fX) - (D_X v)\,(fZ)\}. \tag{4.1}$$

But from (1.13) b and (2.4)

$$(D_Z v)\,(fY) = {'f}\,(Y, Z). \tag{4.2}$$

In consequence of (4.2), the equation (4.1) becomes

$$(D_Z{'f})\,(X,\ Y) = v(X)'f\,(Y, Z) + v(Y)'f\,(Z, X). \tag{4.3}$$

Hence, if a nearly trans-Sasakian manifold is normal, it is a trans-Sasakian manifold.
Now, substituting from (1.17) in (1.12), we get

$$(D_{fY}{'f})\,(fX, Z) + (D_Y{'f})\,(X, Z) + v(X)\,(D_Y v)\,(fZ)$$
$$= v(Y)\{'f\,(Z, X) - (D_X v)\,(fZ)\} - v(X)\{'f\,(Y, Z) + (D_Y v)\,(fZ)\}. \tag{4.4}$$

But from (1.14)b and (2.1), we again have (4.2).

In consequence of (1.12) and (4.2), the equation (4.4) is identically satisfied.

Hence, a nearly trans-Sasakian manifold is contained in an almost = contact metric pseudo-normal manifold.

We will now consider generalised almost = contact metric normal manifolds and generalised almost contact metric pseudo-normal manifolds, whose equations are given by Mishra (1991).

$$(D_{fX}{'f})\,(fY, Z) - (D_X{'f})\,(Y, Z) - v(Z)\{(D_{fX} v)\,(Y) + (D_X v)\,(fY)\} +$$
$$v(Y)\,(D_X v)\,(fZ) = 0\ . \tag{4.5}$$

$$(D_{fX}{'f})\,(fY, Z) + (D_X{'f})\,(Y, Z) - v(Z)\{(D_{fX} v)\,(Y) - (D_X v)\,(fY)\} -$$
$$v(Y)\,(D_X v)\,(fZ) = 0\ . \tag{4.6}$$

In both cases

$$(D_V{'f})\,(Y, Z) = v(Y)\,(D_V v)\,(fZ) - v(Z)\,(D_V v)\,(fY).$$

Comparing this equation with (2.1), we get

$$v(Y)\,(D_V v)\,(fZ) - v(Z)\,(D_V v)\,(fY) = -\,(D_Y v)\,(fZ) - 'f(Y, Z),$$

which implies

$$D_V V = 0\ .$$

Consequently, when the nearly trans-Sasakian manifold is generalised normal or generalised pseudo-normal, we have (4.2).

Substituting from the equation of a nearly trans-Sasakian manifold in (4.5) and using (4.2), we obtain the equation of a trans-Sasakian manifold.

Hence if a nearly trans-Sasakian manifold is generalised normal, it is trans-Sasakian.

In the case of a generalised pseudo-normal manifold if we proceed in the same way as in the case of a pseudo-normal manifold, we see that a nearly trans-Sasakian manifold cannot be contained in a generalised pseudo-normal manifold.

REFERENCES

Goldberg, S.I. 1963. Totally geodesic hypersurfaces of Kahler manifolds Pacific. Journal of Mathematics **27:** 275–281.

Hatakeyama, Y., Ogawa, Y. & Tanno, S. 1963. Some properties of manifolds with contact metric structure. Tôhoku Mathematical Journal. **15:** 42–48.

Kenmotsu, K. 1972. A class of almost contact Riemannian manifolds. Tôhoku Mathematical Journal **24:** 93–103.

Mishra, R.S. 1984. Structures on a differentiable manifold and their applications, Chandrama Prakashan, 50A, Rai Ram Charan Das Road, Allahabad-211002.

Mishra, R.S. 1994. Almost contact metric manifolds. Monograph No. 1. Tensor Society of India, 21, Chandganj Gardens, Lucknow – 226020.

Sasaki, S. 1960. On differentiable manifolds with certain structures which are closely related to almost contact structure. Tôhoku Mathematical Journal **12:** 456–476.

Shukla, A. 1994. Generalised quasi-Sasakian and trans-Sasakian manifolds. Journal of Tensor Society of India.

(Accepted 30 July 1995)

منطويات حول ساساكية تقريبا

انجلي شكلا

71 - B نرالانجر، لكنو 22602 ، الهند

خلاصة

في بحث سابق قمنا بتعريف ودراسة «منطو حول ساساكي». وهدفنا في هذا البحث هو تعريف «منطو حول ساساكي تقريبا» ودراسة بعض خواصه.