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Some characterizations of CR-submanifolds of generalized complex space forms

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ABSTRACT

Some necessary and sufficient conditions for submanifolds of a generalized complex space form to be *CR*-submanifolds have been obtained.

INTRODUCTION

Let $\bar{M}$ be an almost Hermitian manifold with an almost Hermitian structure (J, g). A submanifold M of $\bar{M}$ is called a *CR*-submanifold (Bejancu 1986) if the tangent bundle TM of M can be decomposed as the direct sum of a holomorphic and an anti-invariant distributions.

In the Theorem 6.1 (Blair & Chen 1979) and the Theorem B′ (Hsu 1984) *CR*-submanifolds of a complex space form are characterized in terms of the curvature tensor of the ambient manifold. Analogous to the Theorem 6.1 of Blair & Chen (1979), a characterization of *CR*-submanifold of a generalized complex space form (Barros & Urbano 1979; Vanhecke 1975–76) in terms of the curvature tensor of the ambient manifold is given in Theorem 3.1 of Barros & Urbano (1979).

In this paper we present some more necessary and sufficient conditions for submanifolds of generalized complex space forms (in terms of the curvature tensor of the space form) to be *CR*-submanifolds.

PRELIMINARIES

Let $\bar{M}$ be an almost Hermitian manifold with an almost Hermitian structure (J, g). Then $\bar{M}$ is called an *RK*-manifold (Vanhecke 1975–76) if

$$R(X, Y, Z, W) = R(JX, JY, JZ, JW), \ (X, Y, Z, W \in T\bar{M})$$

where R is the curvature tensor of $\bar{M}$.

An almost Hermitian manifold, which is Kaehler and is of constant holomorphic sectional curvature, is called a complex space form. A complex space form belongs to the class formed by *RK*-manifolds of constant holomorphic sectional curvature μ and constant type a. These *RK*-manifolds are called generalized complex space forms $\bar{M}(\mu, a)$. In this case it is known that

$$4R(X, Y)Z = (\mu + 3\alpha)\,(g(Y, Z)X - g(X, Z)Y) + (\mu - a)(g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)\,JZ \quad (1)$$

where R is the curvature tensor (Vanhecke 1975–76).

A submanifold M of $\overline{M}$ is said to be a *CR-submanifold* (Bejancu 1986) if there is a differentiable distribution $\mathcal{D}$ on M which is holomorphic (or invariant), i.e., $J\mathcal{D} = \mathcal{D}$ and the complementary orthogonal distribution $\mathcal{D}^\perp$ of $\mathcal{D}$ in M is anti-invariant, i.e., $J\mathcal{D}^\perp \subset T^\perp M$. If M is a CR-submanifold of $\overline{M}$ then $T^\perp M = \overline{\mathcal{D}} \oplus \overline{\mathcal{D}}^\perp$ where $\overline{\mathcal{D}}^\perp = J\mathcal{D}^\perp$ is an anti-invariant subbundle of $T^\perp M$, i.e., $J\overline{\mathcal{D}}^\perp \subset TM$ (In fact, $J\overline{\mathcal{D}}^\perp = \mathcal{D}^\perp$) and $\overline{\mathcal{D}}$ is orthogonal complementary subbundle of $\overline{\mathcal{D}}^\perp$ in $T^\perp M$ such that $J\overline{\mathcal{D}} = \overline{\mathcal{D}}$.

A CR-submanifold M of an almost Hermitian manifold $\overline{M}$ is known to be invariant, anti-invariant, anti-holomorphic, and proper if $\mathcal{D}^\perp = \{0\}$, $\mathcal{D} = \{0\}$, $J\mathcal{D}^\perp = T^\perp M$, and $\mathcal{D} \neq \{0\} \neq \mathcal{D}^\perp$ respectively.

We say that CR-submanifold M is vertical proper, non-anti-invariant, and non-anti-holomorphic if $\overline{\mathcal{D}} \neq \{0\} \neq \overline{\mathcal{D}}^\perp$, $\mathcal{D} \neq \{0\}$, and $\overline{\mathcal{D}} \neq \{0\}$ respectively.

For a submanifold M of an almost Hermitian manifold $\overline{M}$ it is known (Yano & Kon 1983) that

$$-I - P^2 = tF, \quad FP + fF = 0 \quad (2)$$

$$-I - f^2 = Ft, \quad tf + Pt = 0. \quad (3)$$

where P, F, t and f are given by

$$JX = PX + FX,\ (X,\ PX \in TM,\ FX \in T^\perp M),$$

$$JN = tN + fN,\ (tN \in TM,\ N,\ fN \in T^\perp M).$$

where $T^\perp M$ denotes the normal bundle of M. Hence, the following statements are equivalent:

$$FP = 0, \quad fF = 0, \quad tf = 0, \quad Pt = 0.$$

Moreover, M is a CR-submanifold if and only if $FP = 0$ (Yano & Kon 1983).

CHARACTERIZATIONS OF CR-SUBMANIFOLDS

From relations (2) & (3) it follows that

$$-P - P^3 = PtF = -tfF = tFP. \quad (4)$$

$$-F - FtF = FP^2 = -fFP = f^2F. \quad (5)$$

$$-f - f^3 = fFt = -FPt = Ftf. \quad (6)$$

$$-t - tFt = tf^2 = -Ptf = P^2t. \quad (7)$$

Moreover, for a submanifold M of an almost Hermitian manifold $\overline{M}$, one easily gets

$$g(PX, Y) = -g(X, PY),\ g(FX, N) = -g(X, tN),\ g(fN, V) = -g(N, fV)$$

where X, $Y \in TM$ and N, $V \in T^\perp M$. Consequently

$$g(FPX, N) = -g(PX, tN) = g(X, PtN). \quad (8)$$

$$g(fFX, N) = -g(FX, fN) = g(X, tfN). \tag{9}$$

$$g(tFPX, PY) = -g(FPX, FPY) = -g(fFX, fFY) = g(f^2FX, FY). \tag{10}$$

$$g(Ftf\,N, fV) = -g(tf\,N, tfV) = -g(PtN, PtV) = g(P^2tN, tV). \tag{11}$$

First we prove the following

Proposition 1. For a submanifold M of an almost Hermitian manifold $\overline{M}$ the following statements are equivalent:

$(a)\ T_xM = \mathcal{D}_x \oplus \mathcal{D}_x^\perp \quad (b)\ T_x^\perp M = \overline{\mathcal{D}}_x \oplus \overline{\mathcal{D}}_x^\perp, (x \in M) \quad (c)\ FP = 0$

$(d)\ tFP = 0 \quad (e)\ tfF = 0, \quad (f)\ Ptf = 0, \quad (g)\ P^3 + P = 0$

$(h)\ f^2F = 0, \quad (i)\ fFP = 0, \quad (j)\ FP^2 = 0, \quad (k)\ FtF + F = 0$

$(l)\ Ftf = 0, \quad (m)\ FPt = 0, \quad (n)\ fFt = 0, \quad (o)\ f^3 + f = 0$

$(p)\ P^2t = 0, \quad (q)\ Ptf = 0, \quad (r)\ tf^2 = 0, \quad (s)\ tFt + t = 0.$

where $\mathcal{D}_x$ (resp. $\overline{\mathcal{D}}_x$) are the maximal invariant subspaces while $\mathcal{D}_x^\perp$ (resp. $\overline{\mathcal{D}}_x^\perp$) are the maximal anti-invariant subspaces of T_xM (resp. $T_x^\perp M$).

Proof. The statements (a) and (b) are obviously equivalent. In view of equivalence of $FP = 0$, $fF = 0$, $tf = 0$, and $Pt = 0$; and (4)–(11) we get equivalence of the statements (c) to (s).

Now we prove the equivalence of (a) and (c). Since $Ker(FP)_x = \mathcal{D}_x \oplus \mathcal{D}_x^\perp = T_xM$, (a) implies (c). Conversely, let (c) hold. Then for all $X_x \in T_xM$, we get $J(PX_x) = P^2X_x$, and defining $\mathcal{D}_x = P(T_xM)$, it follows that

$$J(\mathcal{D}_x) \subset \mathcal{D}_x.$$

In view of $JX_x = PX_x$ for $X_x \in D_x$, we get

$$-X_x = J^2X_x = JP(X_x), \text{ i.e., } \mathcal{D}_x \subset J(\mathcal{D}_x).$$

Thus

$$J(\mathcal{D}_x) = D_x.$$

Now, let $\mathcal{D}_x^\perp$ denote the orthogonal complement to $\mathcal{D}_x$ in T_xM. Then for $X_x \in \mathcal{D}_x^\perp$ and $Y_x \in T_xM$, we have

$$g(JX_x, Y_x) = -g(X_x, JY_x) = -g(X_x, PY_x) = 0$$

which yields $J(\mathcal{D}_x^\perp) \subset T_x^\perp M$. Hence (c) implies (a). This completes the proof.

From the definition of CR-submanifold it is clear that

(a) $T_xM = D_x \oplus \mathcal{D}_x^\perp$, $x \in M$, where
$\mathcal{D}_x$ (resp. $\mathcal{D}_x^\perp$) are the maximal invariant (resp. anti-invariant) subspaces of T_xM,

(b) the dimensions of $\mathcal{D}_x$ and $\mathcal{D}_x^\perp$ are independent of $x \in M$, and thus $\mathcal{D}_x$ and $\mathcal{D}_x^\perp$ define invariant distribution $\mathcal{D} = \cup_{x \in M}\mathcal{D}_x$ and anti-invariant distribution $\mathcal{D}^\perp = \cup_{x \in M}\mathcal{D}_x^\perp$ respectively such that $TM = \mathcal{D} \oplus \mathcal{D}^\perp$, and

(c) the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are differentiable.

Remark 2. The condition (a) is sufficient for a submanifold of an almost Hermitian manifold to be a CR-submanifold.

Proof. Let M be a submanifold of an almost Hermitian manifold $\bar{M}$, and let the condition (a) hold.

Then from Proposition 1, we get $P^3 + P = 0$ on M, which implies that $Dim(\mathcal{D}_x) = Rank(P_x)$ is independent of $x \in M$ (Stong 1977) (Thus P becomes an f-structure (Yano & Kon 1984) on M). Therefore dimension of $\mathcal{D}_x^\perp = Ker(P_x)$ is also independent of $x \in M$, and we get condition (b).

Now, if $\mathcal{D}^\perp = \{0\}$ or $\mathcal{D} = \{0\}$ then the condition (c) is trivial. So let us assume that $\mathcal{D}^\perp \neq \{0\} \neq \mathcal{D}$. Since $\mathcal{D} = Ker(P^2 + I)$, $D^\perp = Ker(P^2)$, and P^2 is a differentiable symmetric operator, i.e., $g(P^2X, Y) = g(X, P^2Y)$, from the study of Nomizu (1973) it follows that $\mathcal{D}$ and $\mathcal{D}^\perp$ are differentiable, which is (c). Hence the proof.

The following theorems characterize CR-submanifolds of generalized complex space forms. First, analogous to Theorem B′ of Hsu (1984) we prove the following:

Theorem 3. Let M be a submanifold of a generalized complex space form $\bar{M}(\mu, a)$ with $\mu \neq a$. Then M is a non-anti-invariant non-anti-holomorphic CR-submanifold of $\bar{M}(\mu, a)$ if and only if the maximal anti-invariant subspaces $\mathcal{D}_x^\perp$ of T_xM, such that $J\mathcal{D}_x^\perp \subset T_x^\perp M$, $x \in M$, define a distribution $\mathcal{D}^\perp$ on M such that

$$R(\mathcal{D}, J\mathcal{D}, \bar{\mathcal{D}}, \mathcal{D}) = 0 \tag{12}$$

where $\mathcal{D}$ denotes the non-trivial orthogonal complementary distribution of $\mathcal{D}^\perp$ in M and $\bar{\mathcal{D}}$ denotes the non-trivial orthogonal complementary subbundle of $J\mathcal{D}^\perp$ in $T^\perp M$.

Proof. Let M be a non-anti-invariant non-anti-holomorphic CR-submanifold of $\bar{M}(\mu, a)$. From (1) using $J\mathcal{D} = \mathcal{D}$ and $J\bar{\mathcal{D}} = \bar{\mathcal{D}}$, we get

$$2R(X, JY)N = (a - \mu)g(X, Y)JN, \ (X, Y \in \mathcal{D}, N \in \bar{\mathcal{D}}).$$

Since $JN \in T^\perp M$, from above equation, for each $Z \in \mathcal{D}$ we obtain (12).

Conversely, if the maximal anti-invariant subspaces $\mathcal{D}_x^\perp$ of T_xM, such that $JD_x^\perp \subset T_x^\perp M$, $x \in M$, define a distribution $\mathcal{D}^\perp$ on M such that (12) holds then for $X \in D$, $N \in \bar{\mathrm{D}}$ we get

$$0 = 2R(X, JX, N, X) = (\mu - a)g(X, X)g(JX, N).$$

Since $\mu \neq a$, above equation implies that $J\mathcal{D}$ is orthogonal to $\bar{\mathcal{D}}$. Moreover, it is easy to see that $J\mathcal{D}$ is orthogonal to $\mathcal{D}^\perp$ and $J\mathcal{D}^\perp$. Thus $\mathcal{D}$ is invariant by J and M becomes a non-anti-invariant non-anti-holomorphic CR-submanifold of $\bar{M}(\mu, a)$.

Theorem 4. Let M be a submanifold of a generalized complex space form $\bar{M}(\mu, a)$ with $\mu \neq a$. Then M is a vertical proper CR-submanifold if and only if the maximal invariant subspaces $\bar{\mathcal{D}}_x = T_x^\perp M \cap J(T_x^\perp M)$, $x \in M$, *of* $T_x^\perp M$ define a non trivial subbundle $\mathcal{D}$ of $T^\perp M$ such that

$$R(\bar{\mathcal{D}}, \bar{\mathcal{D}}, \bar{\mathcal{D}}^\perp, \bar{\mathcal{D}}^\perp) = 0 \tag{13}$$

where $\bar{\mathcal{D}}^\perp$ denotes the non-trivial orthogonal complementary subbundle of $\bar{\mathcal{D}}$ in $T^\perp M$.

Proof. Let M be a vertical proper CR-submanifold of $\bar{M}(\mu, a)$. Then

$T^\perp M = \bar{\mathcal{D}} \oplus \bar{\mathcal{D}}^\perp$ where

$\bar{\mathcal{D}}^\perp = J\mathcal{D}^\perp$ is an anti-invariant subbundle of $T^\perp M$, i.e., $J\bar{\mathcal{D}}^\perp \subset TM$, and $\bar{\mathcal{D}}$ is orthogonal complementary subbundle of $\bar{\mathcal{D}}^\perp$ in $T^\perp M$ such that $J\bar{\mathcal{D}} = \bar{\mathcal{D}}$. For $N, L \in \bar{\mathcal{D}}$ and $U \in \bar{\mathcal{D}}^\perp$, (1) gives

$$2R(N, L)U = (\mu - a)g(N, JL)JU.$$

Since $JU \in TM$, for $V \in \bar{\mathcal{D}}^{\perp}$ we obtain (13).

Conversely, if the maximal invariant subspaces $\bar{\mathcal{D}}_x = T_x^{\perp}M \cap J(T_x^{\perp}M)$, $x \in M$, of $T_x^{\perp}M$ define a non-trivial subbundle $\bar{\mathcal{D}}$ of $T^{\perp}M$ such that (13) holds then for $N \in \bar{\mathcal{D}}$ and $U, V \in \bar{\mathcal{D}}^{\perp}$, we get

$$0 = 2R(N, JN, U, V) = (a - \mu)g(N, N)g(JU, V).$$

Since $\mu \neq a$, above equation implies that $J\bar{\mathcal{D}}^{\perp}$ is orthogonal to $\bar{\mathcal{D}}^{\perp}$. Moreover, $J\mathcal{D}^{\perp}$ is also orthogonal to $\mathcal{D}$. Therefore $J\bar{\mathcal{D}}^{\perp} \subset TM$ and in view of Remark 2 and equivalence of statements (a) and (b) in Proposition 1, M becomes a vertical proper CR-submanifold of $\bar{M}(\mu, a)$.

Theorem 5. Let M be a submanifold of a generalized complex space form $\bar{M}\,(\mu, a)$ with $\mu \neq a$. Then M is a non-anti-invariant non-anti-holomorphic CR-submanifold of $\bar{M}(\mu, a)$ if and only if the maximal anti-invariant subspaces $\bar{\mathcal{D}}_x^{\perp}$ of $T_x^{\perp}M$, such that $J\bar{\mathcal{D}}_x^{\perp} \subset T_x M$, $x \in M$, define a subbundle $\bar{\mathcal{D}}^{\perp}$ of $T^{\perp}M$ such that

$$R(\bar{\mathcal{D}}, J\bar{\mathcal{D}}, \mathcal{D}, \bar{\mathcal{D}}) = 0 \tag{14}$$

where $\bar{\mathcal{D}}$ denotes the non-trivial orthogonal complementary subbundle of $\bar{\mathcal{D}}^{\perp}$ in $T^{\perp}M$ and $\mathcal{D}$ denotes the non trivial orthogonal complementary distribution of $J\mathcal{D}^{\perp}$ *in M*.

Proof. Let M be a non-anti-invariant non-anti-holomorphic CR-submanifold of $\bar{M}(\mu, a)$. Then form (1) for $X \in \mathcal{D}$, $U, V, \in \bar{\mathcal{D}}$ we get

$$2R(U, JV)X = (a - \mu)g(U, V)JX.$$

Since $JX \in TM$, for each $N \in \bar{\mathcal{D}}$ above equation implies (14).

Conversely, if the maximal anti-invariant subspaces $\bar{\mathcal{D}}_x^{\perp}$ of $T_x^{\perp}M$, such that $J\bar{\mathcal{D}}_x^{\perp} \subset T_xM$, define a subbundle $\bar{\mathcal{D}}^{\perp}$ in $T^{\perp}M$ such that (14) holds then for $X \in \mathcal{D}$, $N \in \bar{\mathcal{D}}$ we get

$$0 = 2R(N, JN, X, N) = (\mu - a)g(N, N)g(JN, X).$$

Since $\mu \neq a$, above equation implies that $J\bar{\mathcal{D}}$ is orthogonal to $\mathcal{D}$. Moreover, it can be seen that $J\bar{\mathcal{D}}$ is orthogonal to $\bar{\mathcal{D}}^{\perp}$ and $J\bar{\mathcal{D}}^{\perp}$ also. Thus $\bar{\mathcal{D}}$ is invariant by J and in view of Remark 2 and equivalence of statements (a) and (b) in Proposition 1 M becomes a non anti-invariant non-anti-holomorphic CR-submanifold of $\bar{M}(\mu, a)$.

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REFERENCES

Barros, M. & Urbano, F. 1979. CR-submanifolds of generalized complex space forms. Annals of Ştiint. University 'Al.I.Cuza' Iaşi Sect la Mat **25:** 355–363.

Bejancu, A. 1986. Geometry of CR-submanifolds. D. Reidel Publishing Company.

Blair, D.E. & Chen, B.Y. 1979. On CR-submanifolds of Hermitian manifolds. Israel Journal of Mathematics **34:** 353–363.

Hsu, C.J. 1984. Two theorems on CR-submanifolds of Kaehler manifolds. Bulletin of Institute of Mathematical Academy, Sinica **12:** 95–99.

Nomizu, K. 1973. Characteristic roots and vectors of a differentiable family of symmetric matrices. Linear and Multilinear algebra **1:** 159–162.

Stong, R.E. 1977. The rank of an f-structure. Kodai Mathematical Seminar Reports. **29:** 207–209.

Vanhecke, L. 1975–76. Almost Hermitian manifolds with J-invariant Riemann curvature tensor. Rend. Seminar in Mathematics Torino, **34:** 75–86.

Yano, K. & Kon, M. 1983. CR-submanifolds of Kaehler and Sasakian manifolds. Birkhäuser Boston.

Yano, K. & Kon, M. 1984 Structures on manifolds. World Scintific Co., Singapore, 1984.

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بعض تمييزات منطويات جزئية CR لأشكال فضائية عقدية معممة

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خلاصة

في هذا البحث، تم الحصول على شروط لازمة وكافية من أجل أن تكون منطويات جزئية لفضاء عقدي معمم منطويات جزئية CR.