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An efficient three-node triangular element to study the response of coupled bending-twisting effects for symmetrically laminated plates

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ABSTRACT

A new efficient three-node triangular plate element with three degrees of freedom—one transverse displacement and two rotations—at each node suitable for a bending-twisting response of moderately-thick and thin symmetrically laminated composite plates is presented. The deformation formulation of the plate element is based on Reissner and Mindlin's theory that incorporates the effects of transverse shear deformations. The element normally shows a severe shear locking effect in thin plate situations using isoparametric interpolation functions with full integration scheme. A constant transverse shear strain criterion is now implemented in its formulation along with a recently developed transverse shear strain correction expression to circumvent such locking effects. The element shows a robustness for regular and cross-diagonal triangular mesh configurations in both moderately-thick and thin situations. The element shows fast convergence, and remains stable for the three-point integration scheme. For the purpose of illustration, the numerical results are presented for symmetrically angle-ply laminated simply-supported and fully-clamped edge plates that produce bending-twisting coupling. The results are compared with recently available double Fourier series and generalized Navier's – based analytical solutions, and a well-established finite element solution.

INTRODUCTION

Beneficial properties of advanced continuous fiber reinforced composite materials are well praised (Jones 1975; Bert & Chen 1978; Reddy 1984; Kapania & Raciti 1989; Vinson & Sierakowski, 1989). These materials are ranked as the most promising candidates for future growth of such highly technological industries as aerospace, aircraft, automobile, ocean-based, sporting goods, and other civil, mechanical, and bio-engineering applications. Analyses of structural responses of plates fabricated with such materials for arbitrary lamination stackings are complicated in nature due to various coupling effects, e.g., in-plane-bending, bending-twisting, etc., first studied by Ambartsumyan (1953, 1964) in the context of anisotropic plates and shells. Further complications are added when the effects of transverse shear deformations are introduced into their formulations. In industry, there is a general tendency to avoid such complications by introducing symmetric lamination patterns. However, even for such laminations, for example [45/-45]s, bending-twisting couplings may not be avoided. The analytical solutions for such laminations, for the case of simply-supported and fully-clamped plates, were only recently reported by Kabir (1994) and Kabir & Chaudhuri (1994), respectively. Regular plate structures (e.g., square or rectangular shapes) are not commonly used in industry, resulting in limited applications of these analytical solutions. However, they have been used to check the accuracy of popular finite element or finite difference methods. There are various finite plate elements that are available in industry to analyze laminated plate structures. But the most popular one is a four-node quadrilateral element, based on the formulation of Bathe & Dvorkin (1985), mainly due to the simplicity arising from a powerful auto mesh generation technique. The element considers the effects of transverse shear deformations using a separate set of interpolation functions. The element performs very well for both moderately-thick and thin situations, but industry would much prefer a three-node triangular element. An isoparametric-based three-node triangular element that takes account of transverse shear deformation produces an overstiff stiffness matrix in thin situations when a full integration scheme is used (Batoz *et al.* 1980). This overstiffness is attributed to a shear locking phenomenon. Batoz *et al.* (1980) have reported poor performance of the element even with selective reduced integration schemes. The three-node element of Belytschko *et al.* (1984) developed using the concept of decomposing the deformation into two distinctive modes, bending and transverse shear, has shown excellent performance for plates with simply supported boundary conditions at all edges with (i) all types of triangular mesh configurations (type A, type B, & type C) Belytschko *et al.* (1984), and for (ii) fully clamped boundary conditions at all edges with only cross-diagonal mesh configuration (type C). Numerical results for simply-supported circular and rectangular plates with isotropic materials have been presented. Zienkiewicz & Lefebvre (1987), and Papdopolus & Taylor (1990) among others, have formulated the element using the Hellinger/Reissner-based mixed variational principle and expressed transverse deformation and rotations terms in the form of cubic and quadratic polynomials, respectively.

Apart from the isoparametric approach, Lynn & Dhillion (1971) have developed a shear deformation-based three-node triangular plate bending element. The displacement and rotation terms were assumed to vary quadratically and linearly, respectively. This element has shown a very good response in moderately-thick situations. Hughes & Taylor (1981) also have developed three-node triangular elements with the polynomial of transverse displacement terms one order higher than the rotation terms. The element has shown good performance in cross-diagonal mesh configurations. Lardeur & Batoz (1989) have reported a three-node triangular finite element with nine degrees of freedom for symmetrically laminated composite plates based on discrete shear triangle (DST) theory, where displacement and rotation terms are assumed in the form of cubic and quadratic polynomials, respectively. Recently, Onate *et al.* (1992) have given a general solution to a group of shear flexible elements. Linear and quadratic variations of transverse displacement and rotations, respectively, have been considered for the three-node triangular element.

It is apparent from an extensive literature survey that a shear locking free isoparametric formulation-based three-node triangular bending element for general symmetrically laminated thick/thin plates is yet to be reported that uses a full integration scheme. Therefore, the first objective is to develop such an element for symmetric laminated plates to study the bending-coupling effects. The second objective is to use a full integration scheme (three point) to obtain a stable element (Onate *et al.* 1992). The third objective is to obtain shear locking free situations for mesh types A and B

(Belytschko *et al.* 1984) for fully-clamped rectangular plates. The final objective is to compare the numerical results with the recently-available analytical (Kabir 1994, Kabir & Chaudhuri 1994) and finite element solutions. The finite element solutions are obtained using the four-node quadrilateral plate element based on the work of Bathe & Dvorkin (1985) that has been implemented in NISA-II (1992), a commercially available general purpose finite element package.

THEORETICAL BACKGROUND FOR THE FINITE ELEMENT FORMULATION

A three-node triangular element of total thickness h is shown in Fig. 1. The natural coordinate axes ξ and η are placed at middle surface (reference surface) of the plate thickness at node 1. Axis ξ is along nodes 1 and 2, while axis η is along nodes 1 and 3. ζ is perpendicular to the ξ—η plane. x', y', and z' are local Cartesian axes. Axes x' and y' are in the same plane as axes ξ and η. Axes x, y and z represent a global Cartesian coordinate system. Based on the assumptions of transverse shear deformation (Reissner 1944 and Mindlin 1951), the displacement components at any point (x, y, z) of the plate thickness may be written as:

$$u(x, y, z) = z\varphi_y(x, y) \tag{1 a}$$

$$v(x, y, z) = -z\,\varphi_x(x, y) \tag{1 b}$$

$$\overline{w}(x, y, z) = w(x, y) \tag{1 c}$$

where u and v are in-plane displacements along the x and y axes, respectively; $\overline{w}$ and w are mid-plane transverse displacements. φ_y and φ_x are rotations of reference surface about y and x axes, respectively. Strain and displacement relations are

$$\varepsilon_x(x, y, z) = z\kappa_x(x, y) \tag{2 a}$$

$$\varepsilon_y(x, y, z) = z\kappa_y(x, y) \tag{2 b}$$

$$\varepsilon_{xz}(x, y, z) = \varepsilon^o_{xz}(x, y) \tag{2 c}$$

$$\varepsilon_{yz}(x, y, z) = \varepsilon^o_{yz}(x, y) \tag{2 d}$$

$$\varepsilon_{xy}(x, y, z) = z\kappa_{xy}(x, y) \tag{2 e}$$

where ε_x and ε_y are normal strains, ε_{xy} denotes in-plane shear strain, and ε_{xz} and ε_{yz} are transverse shear strains at any point of the plate thickness. ε^o_{xz} and ε^o_{yz}, and κ_x, κ_y and κ_{xy} are given as:

$$\varepsilon^o_{yz} = w_{,x} + \varphi_y \tag{3 a}$$

$$\varepsilon^o_{xz} = w_{,y} + \varphi_x \tag{3 b}$$

$$\kappa_x = \varphi_{y,\,x} \tag{3 c}$$

$$\kappa_y = -\,\varphi_{x,\,y} \tag{3 d}$$

$$\kappa_{xy} = -\,\varphi_{x,\,x} + \varphi_{y,y} \tag{3 e}$$

in which the comma denotes partial differentiation.

The strain energy in the plate is defined by:

$$U=\frac{1}{2}\int_x\int_y\int_z \{\sigma\}^T\{\varepsilon\}\,dx\;dy\;dz \tag{4}$$

where

$$\{\sigma\}^T=\{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}\} \tag{5}$$

in which σ_x and σ_y denote in-plane stresses, σ_{xy} is an in-plane shear stress, σ_{xz} and σ_{yz} represent transverse shear stresses, and

$$\{\varepsilon\}=\{\varepsilon_1, \varepsilon_2, \varepsilon_6, \varepsilon_4, \varepsilon_5\}^T. \tag{6}$$

For an orthotropic lamina, stresses are related to strains by

$$\{\sigma_x, \sigma_y, \sigma_{xy}\}=[D_b]\,\{\varepsilon_x, \varepsilon_y, \varepsilon_{xy}\}^T \tag{7 a}$$

and

$$\{\sigma_{xz}, \sigma_{yz}\}=[D_s]\,\{\varepsilon_{xz}, \varepsilon_{yz}\}^T. \tag{7 b}$$

Where the constitutive matrices $[D_b]$ and $[D_s]$ are as defined in Appendix A.

THE FINITE ELEMENT FORMULATION

For the three-node isoparametric triangular elements, coordinates, displacements, and rotations are related to their corresponding nodal values using interpolations functions in the following form:

$$(x,y)=\sum_{j=1}^{3} N_j(x^{(j)},y^{(j)}) \tag{8 a}$$

$$w=\sum_{j=1}^{3} N_j w^{(j)} \tag{8 b}$$

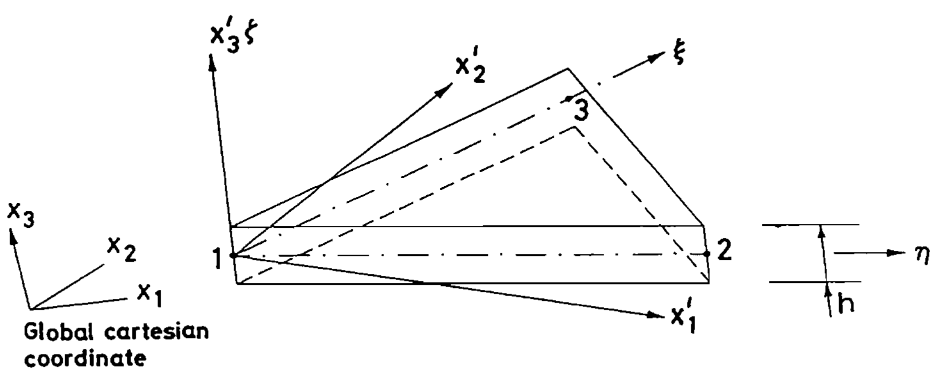

Fig. 1. A triangular element.

$$(\varphi_x, \varphi_y) = \sum_{j=1}^{3} N_j(\varphi_x^{(j)}, \varphi_y^{(j)}) \tag{8 c}$$

where j (= *1,2,3*) denote nodes. The interpolation functions, N_j, are (Bathe 1982)

$$N_1 = 1 - \xi - \eta \tag{8 d}$$

$$N_2 = \xi \tag{8 e}$$

$$N_1 = \eta. \tag{8 f}$$

With those interpolation functions, transverse shear strains (ε_{xz} and ε_{yz}) contain the rotation terms (first order polynomial) and the derivative of transverse displacement terms (zero order polynomial).

The mismatch of the order of polynomials in rotation and displacement terms in the transverse shear strain formulation is supposed to be responsible for the shear locking phenomenon. Verwood & Kok (1990) have reported such a mismatch for the case of six-node isoparametric triangular elements formulated for isotropic materials. A shear correction term was introduced to circumvent such a mismatch for six-node elements using reduced integration schemes. These six-node triangular elements perform reasonably well with reduced integration schemes. Kabir (1992) has extended such a correction term to three-node isoparametric triangular elements for isotropic materials, and provided a general expression for the correction term. Following such a procedure, the following correction term is assumed (Kabir 1992):

$$\Delta w = a\, x'^2 + b\, y'^2 + c\, x'y' + d\, x' + e\, y' + f. \tag{9}$$

This term is then added to transverse strains. For example, for the case of ε_{xz}, the modified transverse shear strain may be written as follows:

$$\begin{aligned} \varepsilon^*_{xz} &= \varepsilon_{xz} + \Delta\varepsilon_{xz} \\ &= \varphi_x + w_{,y} + \Delta w_{,y} \end{aligned} \tag{10}$$

where ε^*_{xz} denotes the modified transverse shear strain. The coefficients of equation (9) are obtained by: (i) imposing a constant transverse shear strain condition in equation (10) as follows

$$\varepsilon^*_{n,s} = 0 \tag{11 a}$$

where n and s represent normal and tangential components at an edge of an element, and (ii) reducing Δw to zero at the three nodes (Kabir 1992).

The strain energy of an element may be written as

$$U^e = U^e_b + U^e_s \tag{11 b}$$

where

$$U^e_b = \frac{1}{2}\int_A \{\kappa\}^T [D_b] \{\kappa\}\, dA \tag{12 a}$$

$$U_s^e = \frac{1}{2}\int_A \left\{\varepsilon^*\right\}^T [D_s] \left\{\varepsilon^*\right\} dA \tag{12 b}$$

in which {{κ}, and {ε*} are as given in Appendix A.

The total potential energy can be written as

$$\Pi^e = \frac{1}{2}\int_A \{\kappa\}^T [D_b] \{\kappa\} \, dA + \frac{1}{2}\int_A \left\{\varepsilon^*\right\}^T [D_s] \left\{\varepsilon^*\right\} dA - W \tag{13}$$

where W is the potential energy due to applied loads. Minimization of Π^e with respect to nodal variables, $\{u\} = (w, \varphi_x, \varphi_y)$ supplies the following equation:

$$[K]^e \{u\}^e = \{f\}^e \tag{14}$$

where $[K]^e$ is a stiffness matrix and $\{f\}^e$ is a load vector.

The stiffness matrix can be written as:

$$[K]^e = [K_b]^e + [K_s]^e \tag{15}$$

in which

$$[K_B]^e = \frac{1}{2}\int_A [B_b]^T [D_{b]} [B_b] \, dA \tag{16}$$

$$[K_s]^e = \frac{1}{2}\int_A [B_s]^T [D_{s]} [B_b] \, dA \tag{17}$$

where $[B_b]$ and $[B_s]$ represent strain-displacement matrices.

NUMERICAL RESULTS AND DISCUSSIONS

The numerical results are presented for the following material properties characterizing each orthotropic lamina (Whitney & Pagano 1970):

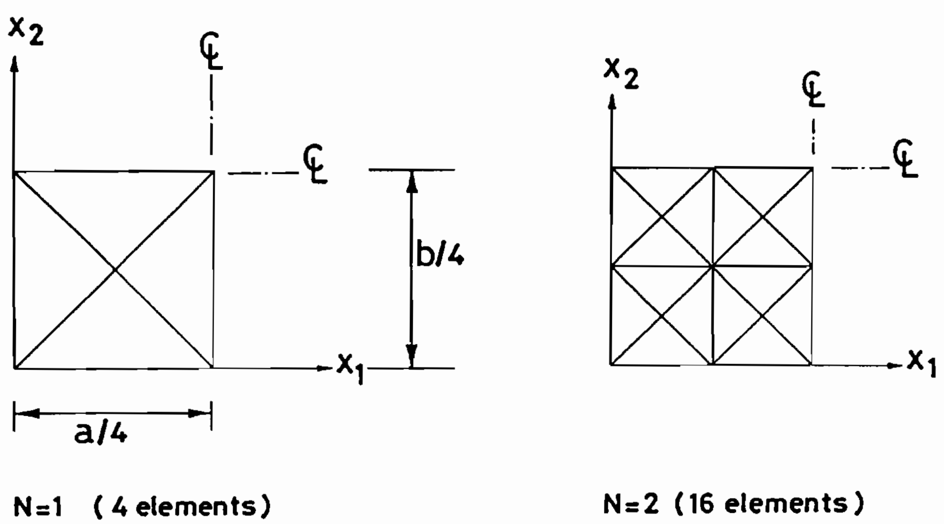

Fig. 2. A rectangular plate modeled with cross-diagonal triangular elements.

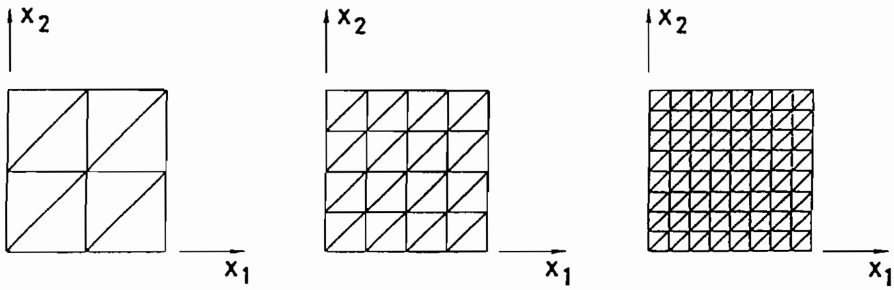

Fig. 3a. Regular triangular mesh type A.

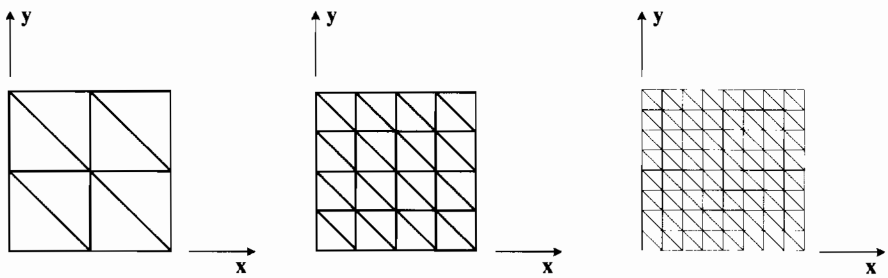

Fig. 3b. Regular triangular mesh type B.

$$\frac{E_1}{E_2} = 25; \frac{G_{12}}{E_2} = \frac{G_{13}}{E_2} = 0.5; \tag{18 a, b}$$

$$\frac{G_{23}}{E_2} = 0.2 \, ; \, \nu_{12} = 0.25 \, ; \, k_1^2 = k_2^2 = 1.0 \tag{18 c–e}$$

where E_1 and E_2 are Young moduli along major and minor axes, respectively, of the material properties of a lamina. G_{12} is an in-plane shear modulus, while G_{13} and G_{23} are transverse shear moduli.

ν_{12} is a Poisson's ratio. ν_{12} is related to ν_{21} by

Table 1. Convergence of normalized central transverse displacements (w*) for a square simply-supported plate subjected to a uniform distributed load with *a/h = 10* and *1000*.

N	w* Span-to-thickness ratio = 10		Span-to-thickness ratio = 1000	
	With shear correction	Without shear correction	With shear correction	Without shear correction
1	44.2480	24.8583	0.01956	0.00475
2	5.1569	5.3481	2.3982	2.0534
4	6.5421	5.2259	3.9715	0.00175
8	6.9418	6.9733	4.4348	0.37128
16	7.0497	7.0234	4.6225	0.08403
Analytical (Kabir, 1994)	7.0359		4.7644 (for a/h = 100)	
Finite element (NISA-II, 1992)	7.0599		4.7520 (for a/h = 100)	

Table 2. Convergence of normalized central moment (M^*_{xz}) for a square simply-supported plate subjected to a uniform distributed load with a/h = 10 and 1000.

N	M^*_x			
	Span-to-thickness ratio = 10		Span-to-thickness ratio = 1000	
	With shear correction	Without shear correction	With shear correction	Without shear correction
1	57.853	13.936	0.089	0.026
2	15.827	13.068	21.232	5.407
4	32.937	19.344	37.982	0.012
8	35.854	35.332	39.085	2.459
16	37.255	37.069	39.888	0.834
Analytical (Kabir, 1994)	39.749		41.878 (for a/h = 100)	
Finite element (NISA-II, 1992)	38.775		41.878 (for a/h = 100)	

$$\nu_{12} E_2 = E_1 \nu_{21} \tag{19}$$

For the sake of convenience, the following dimensionless quantities are defined.

$$w^* = \frac{10^3 E_2 h^3 w}{qa^4}; \quad M_x^* = \frac{10^3 M_x}{qa^2}. \tag{20 a, b}$$

The numerical results for patch tests, and convergence of transverse displacement and moments for various isotropic plates, etc., are discussed in detail in Kabir (1992); therefore, they are excluded in this investigation. Instead, symmetrically angle-ply ($[45/-45]_s$) laminated square simply-supported and clamped plates subjected to uniform distributed loadings are considered for illustrative purposes, as they provide coupled bending-twisting. The plate is modeled with cross-diagonal (Fig. 2) and regular, type A (Fig. 3a), and type B (Fig. 3b) triangular mesh configurations. N = 1, 2, 4, 8, 16 are designated for a total of 4, 16, 64, 256, and 1024 cross-diagonal triangular elements, respectively, in the plate model (Fig. 2), while N = 2, 4, 8, 16, and 32 denote for a total of 8, 32, 128, 512, and 1024 regular type A or B elements. The results for the present elements are obtained using three-point Gaussian quadrature with and without the shear correction term. Results, when the shear correction term is not included, are shown to indicate the acerbity of shear locking effects. Tables 1 & 2 present convergence studies of normalized central transverse displacement (w^*) and central moment (M^*_{xz}), respectively, for span-to-thickness ratios 10 and 1000 with and without the shear correction term. The plate is modeled with cross-diagonal meshes. The severity of shear locking for the element without the shear locking term is quite evident. The converged results of the present element are compared with a recently available analytical (Kabir

Table 3. Normalized central displacement (w*) and central moment (M^*_{xz}) for a square simply-supported plate subjected to a uniform distributed load with various span-to-thickness ratios.

Span-to-thickness ratio	w*	M^*_{xz}
50	4.7941	39.912
100	4.6948	39.898
10000	4.6208	39.094

Table 4. Convergence of normalized central displacement (w^*) and central moment (M^*_{xz}) for a square simply-supported plate subjected to a uniform distributed load modeled with Type A and B mesh configurations (a/h = 10).

N	w*		M^*_{xz}	
	Type A	Type B	Type A	Type B
2	3.5050	2.9501	18.269	11.050
4	5.6856	5.6855	33.026	33.026
8	6.6260	6.6260	38.145	38.145
16	6.9237	7.0076	37.924	37.305
32	7.0253	7.0687	38.001	37.729
32 (Without shear correction)	6.9690	7.0443	37.649	37.512

1994) and finite element (NISA-II 1992) solutions. The finite element solutions (Kabir 1994; NISA-II 1992), used for comparisons, were obtained using four-node quadrilateral plate elements, the basic formulation of which was due to Bathe & Dvorkin (1985). This plate was modeled with 16 × 16 quadrilateral elements. The results that are shown in Tables 1&2 are very encouraging.

Variations of w^* and (M^*_{xz}), computed at the center of the plate for various span-to-thickness ratios are presented in Table 3. Even at a very thin situation (a/h = 10,000), the element performance is as good as in moderately-thick cases. The plate for this example (Table 3) was also modeled with cross-diagonal mesh configurations.

The same plate was then modeled with regular type A (Fig. 3a) and B (Fig. 3b) mesh configurations. The numerical results on bending responses for w^* and M^*_{xz} (computed at the center) with span-to-thickness ratios 10 and 1000, respectively, are shown in Tables 4 & 5. Though the performance of type B appears to be better than that of type A, overall responses are in good agreement with their cross-diagonal counterpart, suggesting a robust characteristic in triangular meshing. The severity of shear locking for mesh configurations without the shear correction term is also evident in Tables 4 & 5. Table 6 shows the numerical results of fully-clamped plates for a/h = 1000, 100 and 10 modeled with type A mesh configuration. The results agree reasonably well with the analytical (Kabir & Chaudhuri 1994) and finite element (NISA-II, 1992) solutions for moments and displacements, and demonstrate no locking effects of such boundary conditions. It may be noted here that type A and type B mesh configurations produce almost the same results for fully-clamped conditions.

Table 5. Convergence of normalized central displacement (w^*) and central moment (M^*_{xz}) for a square simply-supported plate subjected to a uniformly distributed load modeled with Type A and B mesh configurations (a/h = 1000).

N	w*		M^*_{xz}	
	Type A	Type B	Type A	Type B
2	0.0010	0.0010	0.004	0.008
4	0.4919	0.4919	10.974	10.974
8	1.1889	1.1890	16.677	16.673
16	3.6688	3.6688	29.105	29.105
32	4.4525	4.5797	38.605	39.673
32 (Without shear correction)	0.1109	0.1139	12.658	9.811

Table 6. Converged normalized central displacement (w*) and central and edge moments (M^*_{xz}) for a square fully-clamped plate subjected to a uniformly distributed load modeled with Type A mesh configurations for various a/h.

Element		w*			M^*_1 central moment (maximum edge moment)	
	a/h = 1000	a/h = 100	a/h = 10	a/h = 1000	a/h = 100	a/h = 10
Present				22.68	23.156	25.908
Element	1.750	1.830	4.560			
N = 32				(39.093)	(41.654)	(34.417)
				23.310	23.446	25.808
NISA-II (1992)	1.800	1.840	5.044			
				(36.157)	(36.005)	(34.700)
Analytical					23.520	
(Kabir and		1.919				25.530
Chaudhuri			4.891		(36.678)	
		(a/h = 50)				(34.056)
1994					(a/h = 50)	

Fully-clamped plates with cross-diagonal mesh configurations also give excellent results, but are not reported here for the sake of brevity.

CONCLUSION

An isoparametric three-node triangular element free of shear locking, suitable for moderately-thick and thin symmetrically laminated plates that produce coupled bending-twisting, is presented. For the three degrees of freedom per node, Reissner and Mindlin theory takes account of transverse deformation in the formulation. The shear locking effects are taken care of by introducing a shear correction term and imposing a constant transverse shear strain criterion. The numerical results were found to be in good agreement with the available analytical and finite element solutions for fully-clamped and simply-supported boundary conditions.

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APPENDIX A—DEFINITIONS OF CERTAIN ARRAYS

$\{\kappa\}$, $\{\varepsilon^*\}$, $[D_b]$, and $[D_s]$ are defined as:

$$\{\kappa\} = \{\varepsilon_1^o, \varepsilon_2^o, \varepsilon_6^o, \kappa_1, \kappa_2, \kappa_6\}^T \tag{A1}$$

$$\{\varepsilon^*\} = \{\varepsilon_5^*, \varepsilon_4^*\}^T \tag{A2}$$

$$[D_b] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \tag{A3}$$

$$[D_s] = \begin{bmatrix} k_1^2 A_{55} & k_1^2 A_{45} \\ k_2^2 A_{54} & k_2^2 A_{44} \end{bmatrix} \tag{A4}$$

where D_{ij} ($i, j = 1, 2, 6$) are flexural rigidities, while A_{ij} ($i, j = 4, 5$) are the transverse shear rigidities (Jones 1975). k_1^2 and k_2^2 are transverse shear correction factors.

APPENDIX B—NOTATIONS

The following symbols are used in this paper:

A_{44}, A_{45}, A_{55}	= transverse shear rigidities;
A	= total area;
a	= a constant coefficient;
$[B_b]$	= bending part of strain-displacement matrices;
$[B_s]$	= transverse shear part of strain-displacement matrices;
b	= a constant coefficient;
c	= a constant coeffieient;
d	= a constant coefficient;
dA	= differential area;
e	= a constant coefficient;
f	= a constant coefficient;
$[D_b]$	= matrix for bending rigidities;
$[D_s]$	= matrix for shear rigidities;
E_1, E_2	= Young's moduli along major and minor material axes, respectively;
$\{f\}^e$	= a load vector for an element;
G_{12}, G_{13}, G_{23}	= shear moduli;
h	= total thickness of the plate;
j	= number of nodes;
$[K]^e$	= stiffness matrix for an element;
$[K_b]^e$	= bending part of an element stiffness matrix;
$[K_s]^e$	= transverse shear part of an element stiffness matrix;
k_1^2, k_2^2	= transverse shear correction factors;
M_x	= moment resultant about y axis;
N	= total number of elements in a plate model;
N_i ($i = 1 - 3$)	= shape functions;
U_e	= strain energy of an element;
U_b^e, U_s^e	= strain energy (bending, transverse shear);
$\{u\}$	= a displacement vector;
$\{u\}^e$	= displacement vector of an element;
u	= displacement along x-axis;
v	= displacement along y-axis;
W	= potential energy due to loads;
w	= displacement along z-axis;
(x, y, z)	= a Cartesian coordinate system (global);
x′, y′, z′	= a Cartesian coordinate system (local);
Δw	= a transverse shear correction term;
$\Delta\varepsilon_{xz}$	= $\Delta w_{,y}$;
ε_x, ε_y, ε_{xy}	= in-plane normal strains;
ε^*_{xz}	= modified transverse shear strain;
φ_x, φ_y	= rotations;

ν_{12}, ν_{21}	= in-plane Poisson's ratios;
Π^e	= total potential energy;
σ_x, σ_y, σ_{xy}	= in-plane normal stresses;
σ_{xz}, σ_{yz}	= transverse shear stresses;
(ξ, η, ζ)	= a natural coordinate system.

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العنصر المثلث الفعال ذي الثلاث عقد لدراسة الانثناء ـ الالتواء المزدوج للصفائح المتناظرة ذات الطبقات

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خلاصة

نقدم في هذا البحث عنصر جديد فعال، هو عنصر صفيحة مثلث ذو ثلاث عقد مع ثلاث درجات من الحرية هي: إزاحة مستعرضة ودورانين لكل عقدة، وهذا مناسب للانشاء والالتواء للصفائح معتدلة السمك والصفائح الرقيقة والتي تكون متناظرة وذات طبقات مركبة. إن صيغة التشوه لعنصر الصفيحة مبني على نظرية رايسنر ومندلين والتي تأخذ في الاعتبار تأثير تشوهات القص المستعرضة. إن العنصر الذي يظهر عليه تأثير القفل الحاد للقص في الصفائح الرقيقة فأنه باستخدام دالة استيفاء متجانسة مع التكامل التام واجهاد ثابت للقص المستعرض في صيغته، إضافة إلى تعبير حديث لتعديل القص المستعرض يمكن من التغلب على تأثيرات القفل. ويظهر العنصر قوة لشبكات العنصر المثلث العادية والمتقاطعة القطر في الصفائح معتدلة السمك والرقيقة. ويظهر العنصر تقاربا سريعا مع اثباته بالنسبة للتكامل ذي الثلاث نقاط. ونقدم في البحث الأمثلة للزوايا المثنية ذات الطبقات لصفائح مدعومة ببساطة أو مثبتة والتي ينتج عنها انثناء ـ التواء مزدوج ونقارن النتائج مع نتائج حديثة لمتتالية فوريير المضاعفة، أو حلول نافيير المعممة التحليلية وكذلك حلول العنصر المتناهي.