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New results on the approximations of the generalised elliptic-type integrals

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ABSTRACT

The generalised elliptic-type integral $R_\mu(k, \alpha, \gamma)$

$$R_\mu(k, \alpha, \gamma) = \int_0^\pi \frac{\cos_{(\theta/2)}^{2\alpha-1} \sin_{(\theta/2)}^{2\gamma-2\alpha-1}}{(1 - k^2 \cos \theta)^{\mu+1/2}} \, d\theta$$

where $0 \leqslant k < 1$, $\text{Re}(\gamma) > \text{Re}(\alpha) > 0$, $\text{Re}(\mu) > -0.5$ has been represented in terms of the Gauss hypergeometric function by Kalla *et al.* (1986). Furthermore, Kalla *et al.* (1987) derived a simple-structured single term approximation for this function in the neighbourhood of $k^2 = 1$ in some range of the parameters α, γ and μ. In this paper, a different technique is used to derive efficient two-term approximations in closed form in the neighbourhood of $k^2 = 1$ for $R_\mu(k, \alpha, \gamma)$, which may be considered as an extension of the concept of the single term approximation mentioned above. Evidently, a closed form reduces computations considerably, and the improvement in accuracy by having two terms instead of a single one is manifested by the reduction of the error from $O(h^2)$ to $O(h^4)$, where $h = (1 - k^2)/(2k^2) \ll 1$. The technique used in the approximation may be interpreted as a rational approximation to a function that matches the two rational terms with four terms of the Taylor expansion. Results show that the proposed technique is superior to existing approximations for the same number of terms. The formulation presented in this work has potential application for a wide class of special functions.

INTRODUCTION

Kalla *et al.* (1986) have treated a family of integrals of the form:

$$R_\mu(k, \alpha, \gamma) = \int_0^\pi \frac{\cos_{(\theta/2)}^{2\alpha-1} \sin_{(\theta/2)}^{2\gamma-2\alpha-1}}{(1 - k^2 \cos \theta)^{\mu+1/2}} \, d\theta \tag{1}$$

where $0 \leqslant k < 1$, $\text{Re}(\gamma) > \text{Re}(\alpha) > 0$, $\text{Re}(\mu) > -0.5$.

These integrals have applications in many physical and engineering problems, for example, in radiation field problems (Berger & Lamkin 1958, Hubbell *et al.* 1961).

Other forms of well known elliptic-type integrals are special cases of such representations. If, for example, $\gamma = 2\alpha = 1$, $\mu = j$, a non-negative integer, then:

$$R_j(k, \tfrac{1}{2}, 1) = \int_0^\pi (1 - k^2 \cos\theta)^{-j-1/2}\, d\theta = \Omega_j(k),$$

which is the Epstein-Hubbel function (1963). Complete elliptic integrals of the first and second kind are related to $\Omega_0(k)$, and $\Omega_1(k)$ respectively (Kalla *et al.* 1986).

Kalla & Al-saqabi (1986) have treated the family of integrals:

$$K_\mu(k, n) = \int_0^\pi \frac{\cos^{2n}\theta}{(1 - k^2\cos\theta)^{\mu+1/2}}\, d\theta$$

where $0 \leqslant k < 1$, $\text{Re}(\mu) > -\frac{1}{2}$, which is related to the generalised elliptic-type integral by

$$K_\mu(k, n) = \sum_{m=0}^{2n} (-1)^m 2^m \binom{2n}{m} R_\mu(k, \tfrac{1}{2}, m + 1)$$

If $\gamma = 2\alpha$, then (1) gives:

$$R_\mu(k, \alpha, 2\alpha) = 2^{1-2\alpha} S_\mu(k, \alpha - \tfrac{1}{2})$$

where:

$$S_\mu(k, \nu) = \int_0^\pi \frac{\sin^{2\nu}\theta}{(1 - k^2\cos\theta)^{\mu+1/2}}\, d\theta, \qquad \text{Re}(\nu) > -\frac{1}{2}.$$

This was studied by Kalla (1984).

In Epstein & Hubbell (1963), Kalla (1984), Kalla & Al-saqabi (1986), Kalla *et al.* (1986), Kalla *et al.* (1987) recurrence relations, series expansions, asymptotic approximations in the neighbourhood of $k^2 = 1$ and relations with various functions have been given. A simple-structured single term asymptotic approximation in the parameter subspace where $\text{Re}(\mu + \frac{1}{2} + \alpha - \gamma) > 0$ has been given by Kalla *et al.* (1987).

The present paper derives simply structured and efficient two term approximations of (1) in the neighbourhood of $k^2 = 1$. The efficiency of the method is such that by adding just one more simple structured term, the error decreases from $O(h^2)$ to $O(h^4)$, where $h = (1 - k^2)/(2k^2) \ll 1$. It is sufficiently accurate to allow a semi-quantitative assessment of the dependence of (Eq. 1) on the parameters μ, $k \to 1$, α and γ, which is usually an important first step in physical applications.

ASYMPTOTIC EXPANSION

Expansion valid for the parameter subspace where

$$\text{Re}(\mu + \tfrac{1}{2} + \alpha - \gamma) > N,\ N \geqslant 1.$$

After some simple transformations, Eq. (1) can be written as (Kalla *et al.* 1987)

$$R_\mu(k, \alpha, \gamma) = \frac{1}{(1 - k^2)^{\mu+1/2}\kappa^{\gamma-\alpha}} \int_0^\kappa \frac{x^{\gamma-\alpha-1}(1 - x/\kappa)^{\alpha-1}}{(1 + x)^{\mu+1/2}}\, dx, \tag{2}$$

where

$$\kappa = \frac{2k^2}{1-k^2}.$$

By expanding binomially the factor $(1 - x/\kappa)^{\alpha-1}$ in powers of x/κ, for $|x| < 1$ and $|x/\kappa| < 1$, we obtain the following previous expression given by (Kalla 1987, Eq. (6), p. 273)

$$R_\mu(k, \alpha, \gamma) \approx \frac{\Gamma_{(\alpha)}}{(1-k^2)^{\mu+1/2}\kappa^{\gamma-\alpha}\Gamma_{(\mu+1/2)}} \sum_{n=0}^{N} \frac{(-1)^n \Gamma_{(\gamma-\alpha+n)}\Gamma_{(\mu+1/2+\alpha-\gamma-n)}}{n!\,\Gamma_{(\alpha-n)}\kappa^n} + R_N \quad (3)$$

where R_N is the remainder and $\mathrm{Re}(\mu + \frac{1}{2} + \alpha - \gamma - N) > 0$.

Single term approximation:
The single term approximation that has been given by Kalla *et al.* (1987, p. 247), can be written in the form of Beta function $\{B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y),\ \mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0\}$ as:

$$R_\mu(k, \alpha, \gamma) \approx \frac{\Gamma(\gamma-\alpha)\Gamma\left(\mu + 1/2 - \gamma + \alpha + \frac{\alpha-1}{\kappa}\right)}{(1-k^2)^{\mu+1/2}\kappa^{\gamma-\alpha}\Gamma\left(\mu + 1/2 + \frac{\alpha-1}{\kappa}\right)}$$
$$= \frac{B\left(\gamma-\alpha, \mu + 1/2 + \alpha - \gamma + \frac{\alpha-1}{\kappa}\right)}{(1-\kappa^2)^{\mu+1/2}\kappa^{\gamma-\alpha}}. \quad (4)$$

TWO TERM APPROXIMATIONS

Formulation valid for the parameter subspace where

$$\mathrm{Re}(\mu + \tfrac{1}{2} + \alpha - \gamma) > N,\ N \geqslant 1$$

Assume that there exist complex values C_1, C_2, z_1, z_2 such that

$$C_1(1+x)^{-z_1} + C_2(1+x)^{-z_2} \approx \left(1 - \frac{x}{\kappa}\right)^{\alpha-1}. \quad (5)$$

As in the remark following Eq. (3) this is also valid for $|x| < 1$ and $|x/\kappa| < 1$. In comparison with the technique used in deriving Eq. (3), if we consider the case $N = 1$, which is the case for two terms expansion, the error in using Eq. (3) would be of the order $O(h^2)$, while the error in using (Eq. 5) would be of the order $O(h^4)$ for the same number of terms. This simple comparison shows an advantage of our approach. In the derivation of Eq. (4), Kalla *et al.* (1987, p. 274), used the approximation:

$$\left(1 - \frac{x}{\kappa}\right)^{\alpha-1} \approx (1+x)^{-(\alpha-1)/\kappa}.$$

In comparison with our approximation, this case corresponds to the special case, $C_1 = 1$, $C_2 = 0$ and $z_1 = (\alpha - 1)/\kappa$ in Eq. (5). By performing binomial expansions

Eq. (5) gives:

$$\sum_{n=0}^{\infty}\left[C_1\binom{-z_1}{n}+C_2\binom{-z_2}{n}\right]x^n=\sum_{n=0}^{\infty}\binom{\alpha-1}{n}(-1)^n\frac{1}{\kappa^n}x^n$$

or

$$C_1(z_1)_n+C_2(z_2)_n=(-1)^n\frac{(1-\alpha)_n}{\kappa^n},\qquad n=0,1,2,3. \tag{6}$$

where:

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=a(a+1)(a+2)\ldots(a+n-1)$$

and

$$\binom{-a}{n}=(-1)^n\frac{(a)_n}{n!}.$$

This leads to a system of four non-linear equations, which can be written in the form:

$$C_1z_1^n+C_2z_2^n=p_n,\ n=0,1,2,3. \tag{7}$$

with

$$\left.\begin{aligned}
p_0&=1\\
p_1&=\frac{(\alpha-1)}{\kappa}\\
p_2&=\frac{(\alpha-1)}{\kappa}\left(\frac{(\alpha-2)}{\kappa}-1\right)\\
p_3&=\frac{(\alpha-1)}{\kappa}\left(\frac{(\alpha-2)(\alpha-3)}{\kappa^2}-\frac{3(\alpha-2)}{\kappa}+1\right)
\end{aligned}\right\}. \tag{8}$$

Simplified solution, suitable for α (real) > 1:
The first three equations of (Eq. 7) are solved by setting $C_1=C_2=\frac{1}{2}$; this certainly assumes z_1 and z_2 are complex conjugates. Setting $z_1=re^{i\theta}$, $z_2=re^{-i\theta}$ and solving these equations we arrive at

$$\begin{cases}
r=\dfrac{\sqrt{(\alpha-1)(\alpha+\kappa)}}{\kappa}\\[2ex]
\theta=\cos^{-1}\left(\sqrt{\dfrac{\alpha-1}{\alpha+\kappa}}\right)
\end{cases} \tag{9}$$

from which

$$\begin{cases}
z_1=\dfrac{(\alpha-1)}{\kappa}+\dfrac{i\sqrt{(\alpha-1)(1+\kappa)}}{\kappa}\\[2ex]
z_2=\dfrac{(\alpha-1)}{\kappa}-\dfrac{i\sqrt{(\alpha-1)(1+\kappa)}}{\kappa}
\end{cases} \tag{10}$$

Using $C_1 = C_2 = \frac{1}{2}$ and (Eq. 10) in (Eq. 5) and performing the integration, the two term approximation for α(real) > 1 is given by:

$$R_\mu(k, \alpha, \gamma) \approx \frac{B\{\gamma - \alpha, \mu + 1/2 + \alpha - \gamma + z_1\} + B\{\gamma - \alpha, \mu + 1/2 + \alpha - \gamma + z_2\}}{2(1 - k^2)^{\mu + 1/2} \kappa^{\gamma - \alpha}}. \quad (11)$$

Simplified solution, suitable for complex α:

The first three equations of (Eq. 7) are now solved by setting $C_1 = C_2 = \frac{1}{2}$, with $\alpha = \alpha_x + i\alpha_y$, where α_x and α_y are the real and imaginary components of α.

Setting

$$r_\alpha = \sqrt{(\alpha_x - 1)^2 + \alpha_y^2},\ \theta_\alpha = \tan^{-1}\left(\frac{\alpha_y}{\alpha_x - 1}\right),$$

results in

$$\begin{cases} z_1^* = \dfrac{1}{\kappa}\left\{\alpha_x - 1 - \sqrt{(1+\kappa) r_\alpha}\, \sin\left(\dfrac{\theta_\alpha}{2}\right)\right\} + \dfrac{i}{\kappa}\left\{\alpha_y + \sqrt{(1+\kappa) r_\alpha}\, \cos\left(\dfrac{\theta_\alpha}{2}\right)\right\} \\ z_2^* = \dfrac{1}{\kappa}\left\{\alpha_x - 1 + \sqrt{(1+\kappa) r_\alpha}\, \sin\left(\dfrac{\theta_\alpha}{2}\right)\right\} + \dfrac{i}{\kappa}\left\{\alpha_y - \sqrt{(1+\kappa) r_\alpha}\, \cos\left(\dfrac{\theta_\alpha}{2}\right)\right\} \end{cases}. \quad (12)$$

Substitution of these values in Eq. (5) and performing the integration, the corresponding two term approximation is given by:

$$R_\mu(k, \alpha, \gamma) \approx \frac{B_{\{\gamma - \alpha, \mu + 1/2 + \alpha - \gamma + z_1*\}} + B_{\{\gamma - \alpha, \mu + 1/2 + \alpha - \gamma + z_2*\}}}{2(1 - k^2)^{\mu + 1/2} \kappa^{\gamma - \alpha}}. \quad (13)$$

Note: Equation 11 is a special case of Eq. (13) when α is real, $\alpha > 1$.

Efficient two term approximation:

$$\mathrm{Re}(\mu + \tfrac{1}{2} + \alpha - \gamma) > N,\ N \geqslant 1$$

The solution of the four equations of (Eq. 7) will be performed by considering:

$$\begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}, \quad (14)$$

$$\begin{bmatrix} z_1^2 & z_2^2 \\ z_1^3 & z_2^3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} p_2 \\ p_3 \end{bmatrix}. \quad (15)$$

Solution of Eq. (14) leads to

$$C_1 = \frac{z_2 - p_1}{z_2 - z_1}, \qquad C_2 = \frac{p_1 - z_1}{z_2 - z_1}. \quad (16)$$

The substitution of (Eq. 16) in (Eq. 15) gives

$$\left.\begin{aligned} p_1(z_1 + z_2) - z_1 z_2 &= p_2 \\ p_1(z_1 + z_2)^2 - z_1 z_2[\mathrm{p}_1 + (z_1 + z_2)] &= p_3 \end{aligned}\right\}. \quad (17)$$

Setting

$$z_1 + z_2 = 2p_s, \qquad z_1 z_2 = p_p \quad (18)$$

will resolve the non linearity in Eq. 17; after simplifications we arrive at

$$p_s = \frac{p_3 - p_1 p_2}{2(p_2 - p_1^2)}, \qquad p_p = \frac{p_1 p_3 - p_2^2}{p_2 - p_1^2}. \tag{19}$$

The substitution for p_1, p_2 and p_3 in Eq. 19 results in

$$\left.\begin{aligned} p_s &= \frac{\alpha - 2}{\kappa} - \frac{1}{2} \\ p_p &= \frac{(\alpha - 1)(\alpha - 2)}{\kappa^2} \end{aligned}\right\}. \tag{20}$$

Finally z_1 and z_2 are given by:

$$z_1 = p_s + \sqrt{p_s^2 + p_p}, \qquad z_2 = p_s - \sqrt{p_s^2 + p_p}. \tag{21}$$

Thus, the substitution of (Eq. 20) in (Eq. (21) gives

$$\left.\begin{aligned} z_1 &= \frac{\alpha - 2}{\kappa} - \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4(\alpha - 2)}{\kappa}\left(1 + \frac{1}{\kappa}\right)} \\ z_2 &= \frac{\alpha - 2}{\kappa} - \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4(\alpha - 2)}{\kappa}\left(1 + \frac{1}{\kappa}\right)} \end{aligned}\right\}. \tag{22}$$

Substitution of (Eq. 22) and (Eq. 8) in (Eq. 16) results in

$$\left.\begin{aligned} C_1 &= \frac{1}{2} + \frac{\left(\frac{1}{2} + \frac{1}{\kappa}\right)}{\sqrt{1 - \frac{4(\alpha - 2)}{\kappa}\left(1 + \frac{1}{\kappa}\right)}} \\ C_2 &= \frac{1}{2} - \frac{\left(\frac{1}{2} + \frac{1}{\kappa}\right)}{\sqrt{1 - \frac{4(\alpha - 2)}{\kappa}\left(1 + \frac{1}{\kappa}\right)}} \end{aligned}\right\}. \tag{23}$$

Then, the double term approximation in the parameter subspace $\mathrm{Re}(\mu + \frac{1}{2} + \alpha - \gamma) > N$, $N \geqslant 1$, is given by:

$$R_\mu(k, \alpha, \gamma) \approx \frac{C_1 B\{\gamma - \alpha, \mu + 1/2 + \alpha - \gamma + z_1\} + C_2 B\{\gamma - \alpha, \mu + 1/2 + \alpha - \gamma + z_2\}}{(1 - k^2)^{\mu + 1/2} \kappa^{\gamma - \alpha}} \tag{24}$$

where $\mathrm{Re}(\mu + 1/2 + \alpha - \gamma + z_2) > 0$ and C_1, C_2 are given by (Eq. 23).

Note: as $\kappa \to \infty$, $z_1 \to 0$, $z_2 \to -1$, $C_1 \to 1$, $C_2 \to 0$ and $(1 - x/\kappa)^{\alpha - 1} \to 1$, a requirement for convergence.

RESULTS AND CONCLUSIONS

The proposed algorithm has been implemented using Turbo Pascal programming language with extended precision (better than double precision) for more accurate results. Computer runs have been made for some real as well as for some complex

Table 1. Comparison of the numerical values of $R_\mu(k, \alpha, \gamma)$ near $k^2 = 1$ for some parameter values computed using Kalla (1986) Eq. (21), Kalla (1986) Eq. (24), and Eqs (4), (11) and (24) respectively.

For real parameter values in the range: $\alpha + \mu - \gamma > 0.0$

(1) k	$\alpha = (0.900, 0.000)$ $\gamma = (1.000, 0.000)$ $\mu = (0.850, 0.000)$ Kalla 1986 Eq. (21)	Kalla 1986 Eq. (24)	Eq. (4)	Eq. (11)	$\mathrm{Re}(\alpha + \mu + 1/2 - \gamma = 1.25)$ Eq. (24)
0.99	1.210 422E + 03	1.209 931E + 03	1.210 498E + 03	1.210 581E + 03	1.210 886E + 03
0.999	2.136 422E + 04	2.136 337E + 04	2.136 403E + 04	2.136 417E + 04	2.136 465E + 04
0.9999	3.796 268E + 05	3.796 253E + 05	3.796 262E + 05	3.796 264E + 05	3.796 272E + 05

(2) k	$\alpha = (5.900, 0.000)$ $\gamma = (6.200, 0.000)$ $\mu = (2.000, 0.000)$ Kalla 1986 Eq. (21)	Kalla 1986 Eq. (24)	Eq. (4)	Eq. (11)	$\mathrm{Re}(\alpha + \mu + 1/2 - \gamma = 2.20)$ Eq. (24)
0.99	1.106 853E + 04	1.118 958E + 04	1.111 249E + 04	1.108 410E + 04	1.106 495E + 04
0.999	1.745 831E + 06	1.747 946E + 06	1.746 595E + 06	1.746 135E + 06	1.745 819E + 06
0.9999	2.765 751E + 08	2.766 086E + 08	2.765 871E + 08	2.765 798E + 08	2.765 747E + 08

(3) k	$\alpha = (1.100, 0.000)$ $\gamma = (2{,}800, 0.000)$ $\mu = (3.500, 0.000)$ Kalla 1986 Eq. (21)	Kalla 1986 Eq. (24)	Eq. (4)	Eq. (11)	$\mathrm{Re}(\alpha + \mu + 1/2 - \gamma = 2.30)$ Eq. (24)
0.9	3.444 503E + 00	3.506 083E + 00	3.520 160E + 00	3.504 459E + 00	3.483 416E + 00
0.99	4.595 303E + 02	4.601 659E + 02	4.598 788E + 02	4.597 165E + 02	4.595 665E + 02
0.999	8.810 573E + 04	8.811 750E + 04	8.811 173E + 04	8.810 868E + 04	8.810 596E + 04

(4) k	$\alpha = (1.800, 0.000)$ $\gamma = (3.800, 0.000)$ $\mu = (4.900, 0.000)$ Kalla 1986 Eq. (21)	Kalla 1986 Eq. (24)	Eq. (4)	Eq. (11)	$\mathrm{Re}(\alpha + \mu + 1/2 - \gamma = 2.40)$ Eq. (24)
0.9	6.640 246E + 00	7.225 418E + 00	6.875 321E + 00	6.736 296E + 00	6.641 078E + 00
0.99	1.050 541E + 04	1.057 719E + 04	1.053 254E + 04	1.051 492E + 04	1.050 549E + 04
0.999	2.521 706E + 07	2.523 399E + 07	2.522 345E + 07	2.521 931E + 07	2.521 714E + 07

(5) k	$\alpha = (0.450, 0.000)$ $\gamma = (2.300, 0.000)$ $\mu = (5.500, 0.000)$ Kalla 1986 Eq. (21)	Kalla 1986 Eq. (24)	Eq. (4)	Eq. (11)	$\mathrm{Re}(\alpha + \mu + 1/2 - \gamma = 4.15)$ Eq. (24)
0.9	2.411 049E + 01	2.299 221E + 01	2.371 229E + 01	2.394 027E + 01	2.406 342E + 01
0.99	1.898 106E + 05	1.891 848E + 05	1.896 169E + 05	1.897 551E + 05	1.898 105E + 05
0.999	2.537 167E + 09	2.536 363E + 09	2.536 933E + 09	2.537 113E + 09	2.537 184E + 09

(6) k	$\alpha = (0.700, 0.000)$ $\gamma = (1{,}200, 0.000)$ $\mu = (5.300, 0.000)$ Kalla 1986 Eq. (21)	Kalla 1986 Eq. (24)	Eq. (4)	Eq. (11)	$\mathrm{Re}(\alpha + \mu + 1/2 - \gamma = 4.3)$ Eq. (24)
0.9	4.133 850E + 03	4.116 341E + 03	4.130 749E + 03	4.133 158E + 03	4.133 785E + 03
0.99	5.844 504E + 08	5.842 442E + 08	5.844 200E + 08	5.844 463E + 08	5.844 517E + 08
0.999	1.127 955E + 14	1.127 918E + 14	1.127 952E + 14	1.127 957E + 14	1.127 958E + 14

parameter values μ, $k \to 1$, α and γ. The approximation formulas for $R_\mu(k, \alpha, \gamma)$ developed by the author are given in terms of the well known Beta function. Since the arguments of this function should have a positive real part (from its integral definition), different approximation formulas with different error measures ranging

Table 2. Comparison of the numerical values of $R_\mu(k, \alpha, \gamma)$ near $k^2 = 1$ for some parameter values computed using Kalla (1986) Eq. (21), Kalla (1986) Eq. (24), and Eqs (4), (13) and (24) respectively.

For complex parameter values in the range: $\mathrm{Re}(\alpha + \mu - \gamma) > 0.0$

(1)	$\alpha = (0.900, 0.700)$ $\gamma = (1.000, 0.800)$ $\mu = (0.850, 0.600)$ $\mathrm{Re}(\alpha + \mu + 1/2 - \gamma = 1.25)$				
k	Kalla 1986 Eq. (21)	Kalla 1986 Eq. (24)	Eq. (4)	Eq. 913)	Eq. (24)
0.900	(53.233 625, −6.280 356)	(53.534 379, −5.278 908)	(53.630 786, −6.008 611)	(53.532 530, −6.336 027)	(53.038 156, −6.203 533)
0.990	(473.282 878, 747.731 133)	(472.109 255, 748.812 009)	(472.931 028, 748.560 205)	(473.324 386, 748.297 231)	(473.195 063, 747.613 208)
0.999	(−8620.248 592, 13 028.779 82)	(−8622.831 955, 13 027.685 88)	(−8622.015 485, 13 028.875 15)	(−8621.325 519, 13 029.319 69)	(−8620.313 258, 13 028.686 62)
(2)	$\alpha = (0.700, 0.800)$ $\gamma = (1.000, 0.700)$ $\mu = (3.000, 1.700)$ $\mathrm{Re}(\alpha + \mu + 1/2 - \gamma = 3.2)$				
k	Kalla 1986 Eq. (21)	Kalla 1986 Eq. (24)	Eq. (4)	Eq. (13)	Eq. (24)
0.800	(−24.000 830, 37.553 806)	(−25.159 671, 37.763 667)	(−24.158 842, 37.801 935)	(−23.974 962, 37.645 515)	(−23.987 450, 37.533 648)
0.900	(−319.160 376, −58.900 526)	(−320.929 155, −62.063 334)	(−319.922 950, −59.161 555)	(−319.370 876, −58.810 854)	(−319.144 264, −58.874 237)
0.990	(180 265.7725, 381 195.851)	(180 081.096, 381 562.560)	(180 281.422, 381 282.123)	(180 281.297, 381 214.797)	(180 267.936, 381 200.619)
(3)	$\alpha = (4.000, 1.800)$ $\gamma = (6.500, 2.000)$ $\mu = (8.500, 5.700)$ $\mathrm{Re}(\alpha + \mu + 1/2 - \gamma = 6.5)$				
k	Kalla 1986 Eq. (21)	Kalla 1986 Eq. (24)	Eq. (4)	Eq. (13)	Eq. (24)
0.900	(51.695 508, 55.877 749)	(60.173 725, 62.276 061)	(53.659 487, 56.020 544)	(51.983 772, 55.723 464)	(51.659 125, 55.923 562)
0.990	(102 034 016.4, 71 234 905.9)	(103 325 460.9, 71 869 786.4)	(102 297 179.4, 71 216 634.1)	(102 063 909.5, 71 225 830.9)	(102 031 838.8, 71 256 335.9)
0.999	(0.287 549 + 15, 0.231 198 + 15)	(0.287 902 + 15, 0.231 463 + 15)	(0.287 617 + 15, 0.231 252 79 + 15)	(0.287 548 + 15, 0.231 251 + 15)	(287 539 + 15, 0.231 259 + 15)

from $O(h)$ to $O(h^4)$, $h = 1/\kappa$ have been developed to meet this requirement for arbitrary selections of parameter values. Implementation has been performed for the single and two term approximations. Error estimates for the different formulas implemented for Kalla *et al.* (1986, Eq. 24), single, and two term approximations are as follows: Kalla *et al.* (1986, Eq. 24) is a first order approximation with error of the order of $O(h)$,

For single term approximation:

(i) Since the single term matches two terms of the Taylor expansion, then error in approximating (1) using Eq. (4) is of the order of $O(h^2)$.

For double-term approximation:

(1) Since the two terms in Eqs (11 & 13) match three terms in the expansion, the error in approximating (1) using these equations is of the order of $O(h^3)$.

(2) Since the two terms of Eq. (24) match four terms in the expansion, the error in approximating (1) using Eq. (24) is of the order of $O(h^4)$.

Comparison of the numerical values of $R_\mu(k, \alpha, \gamma)$ near $k^2 = 1$ for some parameter values computed using different formulas are shown in Table 1 for some real parameter values, and in Table 2 for some complex parameter values. These results show the efficiency of the derived algorithms for arbitrary selections of parameter values, and the results agree fully with the error estimates described above, of which Eq. (24) is the most accurate. However, in the parameter subspace where $\mathrm{Re}(\mu + \frac{1}{2} + \alpha - \gamma)$ is near unity, the results are still acceptable for the two term-approximations, Eqs (19 & 21).

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نتائج جديدة في تقريبات تكاملات الدوال البيضية المعممة

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خلاصة

عبر كالا وهابيل (1986) عن التكامل البيضاوي المعمم ذات المتغيرات (ميو، ك، أ ، ج) بدلالة الدالة الهندسية الفوقية لجاوس وبالاضافة الى ذلك فقد أستنبط كالا وليوبنر وهابيل (1987) تقريب لهذه الدالة بحد واحد بسيط البنيان في مجال معين للمتغيرات أ و ج و ميو وذلك في المنطقة المحيطة بمربع المتغير : عندما يساوي واحد.

ويقدم الباحث في هذ الورقة طريقة جديدة لأستنباط حدين ذات كفاءة وفي صورة مضغوطة لعملية التقريب سالفة الذكر. ويمكن اعتبار هذه الطريقة بمثابة امتداد لمفهوم الحد الواحد المذكور أعلاه. ومن البديهي أن الحلول المضغوطة تقلل العمليات الحسابية بصورة ملحوظة والتحسن الذي طرأ على دقة التقريب نتيجة لاستخدام حدين بدلا من حد واحد ظهر في نقصان الخطأ من المرتبة الثانية الى المرتبة الرابعة. ويمكن تفسير الطريقة التي أستخدمت في عملية التقريب على أنها تقريب كسري لدالة يتطابق فيها الحدين الكسريين بأربعة حدود من مفكوك تايلور. وتوضح الدراسة النظرية والنتائج العددية أن الطريقة المقترحة أفضل من الطرق المنشورة لنفس عدد الحدود. هذا ويمكن تطبيق هذا البحث في مجالات واسعة من الدوال الخاصة.