Kuwait J. Sci. Eng. **28**(1) 2001

On the error in surface spline interpolation of a compactly supported function

MICHAEL J. JOHNSON

Department of Mathematics and Computer Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait. johnson@mcc.sci.kuniv.edu.kw

ABSTRACT

We show that the $L_p(\Omega)$-norm of the error in surface spline interpolation of a compactly supported function in the Sobolev space W_2^{2m} decays like $O(\delta^{\gamma_p+m})$ where $\gamma_p := \min\{m, m+d/p-d/2\}$ and m is a parameter related to the smoothness of the surface spline. In case $1 \le p \le 2$, the achieved rate of $O(\delta^{2m})$ matches that of the error when the domain is all of $\mathbb{R}^d$ and the interpolation points form an infinite grid.

Keywords: Approximation order; interpolation; scattered data: surface spline.

1. INTRODUCTION

Let Ξ be a finite set of scattered points in $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{C}$ be a function which is known only on Ξ. A problem of practical importance is that of constructing a smooth function which interpolates the known data $f_{|\Xi}$ and provides a good approximation to f on any domain which is near Ξ. There are a number of methods which are currently being investigated in the literature for which the reader is referred to the surveys of Foley & Hagen (1994), Buhmann (1993), and Powell (1992). In this paper we restrict ourselves to the method known as surface spline interpolation which we now describe.

Let m be an integer greater than $d/2$, and let H be the set of continuous functions $s : \mathbb{R}^d \to \mathbb{C}$ all of whose derivatives of total order m are square integrable. Let $|||\cdot|||$ be the semi-norm defined on H by

$$|||s||| := \| |\cdot|^m \hat{s} \|_{L_2(\mathbb{R}^d\backslash 0)},$$

where $\hat{s}$ denotes the Fourier transform of s given formally by $\hat{s}(w) := \int_{\mathbb{R}^d} s(x)e^{-iw\cdot x}\,dx,\ w \in \mathbb{R}^d$. Duchon (1977) has shown that if $f \in H$ and Ξ is a bounded subset of $\mathbb{R}^d$ satisfying

$$\forall q \in \Pi_{m-1}(q_{|\Xi} = 0 \Rightarrow q = 0), \tag{1.1}$$

where $\Pi_k := \{\text{polynomials of total degree} \le k\}$, then there exists a unique $s \in H$ which minimizes $|||s|||$ subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. The function s is called the *surface spline interpolant to f at* Ξ and will be denoted by $T_\Xi f$. When

Ξ contains only finitely many points, Duchon further shows that $T_\Xi f$ is the unique function in $S(\phi; \Xi)$ which interpolates f at Ξ. Here $\phi : \mathbb{R}^d \to \mathbb{R}$ is the radially symmetric function given by

$$\phi := \begin{cases} |\cdot|^{2m-d} & \text{if } d \text{ is odd} \\ |\cdot|^{2m-d}\log|\cdot| & \text{if } d \text{ is even,} \end{cases}$$

and $S(\phi; \Xi)$ denotes the space of all functions of the form

$$q + \sum_{\xi\in\Xi} \lambda_\xi \phi(\cdot - \xi),$$

where $q \in \Pi_{m-1}$ and the λ_ξ's satisfy[1]

$$\sum_{\xi\in\Xi} \lambda_\xi r(\xi) = 0, \quad \forall r \in \Pi_{m-1}. \tag{1.2}$$

In order to discuss the extent to which $T_\Xi f$ approximates f, let us assume that $\Omega \subset \mathbb{R}^d$ is an open bounded domain over which the error between f and $T_\Xi f$ is measured. We assume that $\Xi \subset \bar{\Omega}$ and define the 'density' of Ξ in Ω to be the number

$$\delta := \delta(\Xi; \Omega) := \sup_{x\in\Omega} \inf_{\xi\in\Xi} |x - \xi|.$$

A common means of describing the asymptotic approximation attributes of an interpolation method is via the notion of L_p-approximation orders. Surface spline interpolation in Ω is said to provide *L_p-approximation of order γ* if

$$\|f - T_\Xi f\|_{L_p(\Omega)} = O(\delta^\gamma) \quad \text{as } \delta \to 0$$

for all sufficiently smooth functions f. Duchon (1978) has shown that if Ω is connected, has the cone property, and has a Lipschitz boundary, then surface spline interpolation in Ω provides L_p-approximation of order at least

$$\gamma_p := \min\{m, m + d/p - d/2\}$$

for $p \in [1 \,..\, \infty]$. More precisely, it was shown that for all $f \in H$ and $p \in [1 \,..\, \infty]$,

$$\|f - T_\Xi f\|_{L_p(\Omega)} \le \text{const}(m, \Omega)\delta^{\gamma_p} |||T_\Omega f - T_\Xi f|||, \tag{1.3}$$

for sufficiently small δ, and

$$|||T_\Omega f - T_\Xi f||| \to 0 \quad \text{as } \delta \to 0. \tag{1.4}$$

The lengthy assumptions on Ω were employed because Duchon only wanted to assume that $f \in W_2^m(\Omega)$. These assumptions assured the existence of a function in H whose restriction to Ω agreed with f. If one assumes straight off that $f \in H$, then (1.3) and (1.4) hold provided that Ω is a bounded open subset of $\mathbb{R}^d$ having the cone property. In the limiting case when the points Ξ are taken as the infinite grid $h\mathbb{Z}^d$ and Ω is taken as all of $\mathbb{R}^d$, it is known (Buhmann 1990, Jia & Lei 1993) that $\|f - T_\Xi f\|_{L_p(\mathbb{R}^d)} = 0(h^{2m})$ for all sufficiently smooth f.

[1] In case Ξ is infinite, we require additionally that only finitely many of the λ_ξ's are nonzero.

The gap between γ_p and $2m$ is rather substantial, and it has been my aim of late to narrow this gap. An upper bound on the possible L_p- approximation order of surface spline interpolation is obtained in Johnson (1998) for the special case when $\Omega = B := \{x \in \mathbb{R}^d : |x| < 1\}$. It is shown that there exists a C^∞ function f such that

$$\|f - T_\Xi f\|_{L_p(\Omega)} \neq o(\delta^{m+1/p}) \quad \text{as } \delta \to 0.$$

Interestingly, what is actually proved is that $\|f - T_\Xi f\|_{L_p(B\setminus(1-h)B)} \neq o(\delta^{m+1/p})$ where $B\setminus(1-h)B$ can be interpreted as the boundary layer within Ω of depth h. Thus it appears that our inability to achieve L_p-approximation of order $2m$ is due primarily to boundary effects. This corroborates experimental evidence reported in Powell (1994). It becomes interesting now to see if it is possible to approach L_p-approximation of order $2m$ if one changes the rules of the game so as to disable the boundary effects. One approach is to measure the error not on all of Ω, but rather on a compact subset of Ω. Bejanceu (1999) has considered the case when Ω is the open unit cube $(0\,..\,1)^d$ and the interpolation points are those points of the grid $h\mathbb{Z}^d$ which lie in the closed cube $[0\,..\,1]^d$. He shows that if K is a compact subset of $(0\,..\,1)^d$ and f is sufficiently smooth, then

$$\|f - T_\Xi f\|_{L_\infty(K)} = O(h^{2m}) \quad \text{as } h \to 0.$$

In the present work, we use an alternate means of disabling the boundary effects. We assume that f, the function being interpolated, is supported on $\bar{\Omega}$. Before stating our main result (see Corollary 5.3 for a more general statement), we define the Sobolev spaces W_2^γ.

Definition 1.5. The Sobolev space W_2^γ, $\gamma \geq 0$, is the set of all $f \in L_2 := L_2(\mathbb{R}^d)$ for which

$$\|f\|_{W_2^\gamma} := \|(1 + |\cdot|^2)^{\gamma/2}\hat{f}\|_{L_2} < \infty.$$

Theorem 1.6. *Let Ω be an open bounded subset of $\mathbb{R}^d$ having the cone property. If $\Xi \subset \bar{\Omega}$ satisfies (1.1) and $f \in W_2^{2m}$ is supported in $\bar{\Omega}$, then*

$$\|f - T_\Xi f\|_{L_p(\Omega)} \leq \text{const}(\Omega, m)\delta^{\gamma_p+m}\|f\|_{W_2^{2m}},$$

for sufficiently small $\delta := \delta(\Xi, \Omega)$.

Note that, for $p \in [1\,..\,2]$, the exponent of δ is $2m$. Although $\gamma_p + m < 2m$ when $2 < p \leq \infty$, we at least have $\gamma_p + m > m + 1/p$. Our proof of Theorem 1.6 is accomplished by showing that the factor $|||T_\Omega f - T_\Xi f|||$, on the right side of (1.3), decays like $O(\delta^m)$. For this, it suffices to show that there exists $s \in S(\phi; \Xi)$ such that $|||T_\Omega f - s||| = O(\delta^m)$. We do this by first showing, in Section 3, that there exists an $s_h \in S(\phi; h\mathbb{Z}^d)$ such that $|||T_\Omega f - s_h||| = O(\delta^m)$, where h is a multiple of δ. Then, in Section 4, we show that there exists $s \in S(\phi; \Xi)$ such that $|||s_h - s||| = O(\delta^m)$. The final result, Corollary 5.3, is then proved in Section 5.

We mention that Schaback (1999), using quite different methods, has obtained improved error estimates for a general class of radial basis function interpolants provided that the function f belongs to a certain subspace of the 'native space'. When his results are specialized to surface spline interpolation, they show that $\|f - T_\Xi f\|_{L_\infty(\Omega)} = O(\delta^{2m-d})$. Note that Theorem 1.6 provides a slightly higher order of approximation when $p = \infty$, namely $O(\delta^{\gamma_\infty + m}) = O(\delta^{2m-d/2})$.

Throughout this paper we use standard multi-index notation:

$$D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}.$$

The natural numbers are denoted $\mathbb{N} := \{1, 2, 3 \ldots\}$, and the non-negative integers are denoted $\mathbb{N}_0$. For multi-indices $\alpha \in \mathbb{N}_0^d$, we define $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$, while for $x \in \mathbb{R}^d$, we define

$$|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$

For multi-indices α, we employ the notation $(\ )^\alpha$ to represent the monomial $x \mapsto x^\alpha$, $x \in \mathbb{R}^d$. The space of polynomials of total degree $\le k$ can then be expressed as $\Pi_k := \operatorname{span}\{(\ )^\alpha : |\alpha| \le k\}$. For $x \in \mathbb{R}^d$, we define the complex exponential e_x by $e_x(t) := e^{ix \cdot t}$, $t \in \mathbb{R}^d$. The Fourier transform of a function f can then be expressed as $\hat{f}(w) := \int_{\mathbb{R}^d} e_{-w}(x) f(x) dx$. The space of compactly supported C^∞ functions is denoted $C_c^\infty(\mathbb{R}^d)$. If μ is a distribution and g is a test function, then the application of μ to g is denoted $\langle g, \mu \rangle$. We employ the notation const to denote a generic constant in the range $(0 \,..\, \infty)$ whose value may change with each occurrence. An important aspect of this notation is that const depends only on its arguments if any, and otherwise depends on nothing.

2. PRELIMINARIES

The Besov spaces, which we now define, play an essential role in our theory.

Definition 2.1. Let $A_0 := \bar{B}$, and for $k \in \mathbb{N}$, let $A_k := 2^k \bar{B} \backslash 2^{k-1} B$. The Besov space $B_{2,q}^\gamma$, $\gamma \in \mathbb{R}$, $1 \le q \le \infty$, is defined to be the set of all tempered distributions f for which

$$\|f\|_{B_{2,q}^\gamma} := \|k \mapsto 2^{k\gamma} \|\hat{f}\|_{L_2(A_k)}\|_{l_q(\mathbb{N}_0)} < \infty.$$

The spaces $B_{2,q}^\gamma$ are Banach spaces; the reader is referred to Peetre (1976) for a general reference.

Definition. For $\gamma \in (0 \,..\, m]$, let $\mathcal{M}_\gamma$ be the set of all compactly supported distributions μ which satisfy

$$\langle q, \mu \rangle = 0 \quad \forall q \in \Pi_{m-1} \tag{2.2}$$

and $\|\mu\|_{\mathcal{M}_\gamma} < \infty$ where

$$\|\mu\|_{\mathcal{M}_\gamma} := \begin{cases} \|\mu\|_{B_{2,\infty}^{\gamma-m}} & \text{if } 0 < \gamma < m \\ \|\mu\|_{L_2} & \text{if } \gamma = m. \end{cases}$$

The set of all $\mu \in \mathcal{M}_\gamma$ for which $\operatorname{supp} \mu \subset A$ is denoted $\mathcal{M}_\gamma(A)$.

For $\mu \in \mathcal{M}_\gamma$, we define the convolution $\phi * \mu$ by

$$(\phi * \mu)\hat{} := \hat{\phi}\hat{\mu}.$$

Proposition 2.3. *Let* $\gamma \in (0\,..\,m]$. *If* $\mu \in \mathcal{M}_\gamma$, $q \in \Pi_{m-1}$, *and* $0 < h \leq 1$, *then*

(*i*) $\phi * \mu + q \in H$,
(*ii*) $\||\cdot|^{-m}\hat{\mu}\|_{L_2(\mathbb{R}^d \setminus h^{-1}B)} \leq \mathrm{const}(m,\gamma) h^\gamma \|\mu\|_{\mathcal{M}_\gamma}$, *and*
(*iii*) $\|\hat{\mu}\|_{L_2(h^{-1}B)} \leq \mathrm{const}(m,\gamma) h^{\gamma - m} \|\mu\|_{\mathcal{M}_\gamma}$.

Proof. The proofs of Lemma 2.3 and Proposition 2.4 in Johnson (2000) can be adapted in a straightforward fashion to obtain (i). For (ii), (iii) we have

$$\||\cdot|^{-m}\hat{\mu}\|_{L_2(\mathbb{R}^d \setminus h^{-1}B)} \leq h^m \|\hat{\mu}\|_{L_2} = h^m \|\mu\|_{\mathcal{M}_m}, \quad \text{and}$$
$$\|\hat{\mu}\|_{L_2(h^{-1}B)} \leq \|\hat{\mu}\|_{L_2} = \|\hat{\mu}\|_{\mathcal{M}_m}$$

which proves (ii) and (iii) for the case $\gamma = m$. So assume $0 < \gamma < m$, and let l be the least integer for which $2^l > h^{-1}$. Then

$$\begin{aligned}\||\cdot|^{-m}\hat{\mu}\|_{L_2(\mathbb{R}^d \setminus h^{-1}B)} &\leq \sum_{k=l}^{\infty} \||\cdot|^{-m}\hat{\mu}\|_{L_2(A_k)} \leq 2^m \sum_{k=l}^{\infty} 2^{-km} \|\hat{\mu}\|_{L_2(A_k)} \\ &\leq 2^m \sum_{k=l}^{\infty} 2^{-km} 2^{k(m-\gamma)} \|\mu\|_{\mathcal{M}_\gamma} \leq \mathrm{const}(m,\gamma) 2^{-l\gamma} \|\mu\|_{\mathcal{M}_\gamma} \\ &\leq \mathrm{const}(m,\gamma) h^\gamma \|\mu\|_{\mathcal{M}_\gamma},\end{aligned}$$

and

$$\begin{aligned}\|\hat{\mu}\|_{L_2(h^{-1}B)} &\leq \sum_{k=0}^{l} \|\hat{\mu}\|_{L_2(A_k)} \leq \sum_{k=0}^{l} 2^{k(m-\gamma)} \|\mu\|_{\mathcal{M}_\gamma} \leq \mathrm{const}(m,\gamma) 2^{l(m-\gamma)} \|\mu\|_{\mathcal{M}_\gamma} \\ &\leq \mathrm{const}(m,\gamma) h^{\gamma - m} \|\mu\|_{\mathcal{M}_\gamma}\end{aligned}$$

which completes the proof of (ii) and (iii).

3. THE GRIDDED SURFACE SPLINE $s_h(\mu)$

Let $\eta \in C_c(\mathbb{R}^d)$ and $\sigma \in C_c^\infty(\mathbb{R}^d)$ satisfy

$$\sup_{j \in \mathbb{Z}^d} |\delta_{0,j} - \hat{\eta}(w - 2\pi j)| \leq \mathrm{const}(d,m)|w|^m, \quad w \in \mathbb{R}^d \tag{3.1}$$

$$|1 - \hat{\sigma}(w)| \leq \mathrm{const}(d,m) \frac{|w|^m}{1 + |w|^{3m}}, \quad w \in \mathbb{R}^d, \tag{3.2}$$

and put

$$\psi := \eta * \sigma.$$

The existence of such functions η and σ is known. For example, η can be realized as a finite linear combination of the translates of a box spline (see de Boor *et al.* (1993)) and σ can be realized as a finite linear combination of the translates of any function in $C_c^\infty(\mathbb{R}^d)$ having nonzero mean.

For $\mu \in \mathcal{M}_\gamma$ and $h > 0$, we define

$$s_h(\mu) := \sum_{j\in\mathbb{Z}^d} [\psi(\cdot/h) * \mu](hj)\phi(\cdot - hj).$$

The proof of the following result is motivated by the techniques developed in de Boor *et al.* (1994).

Proposition 3.3. *Let* $\gamma \in (0\,..\,m]$, $h \in (0\,..\,1]$. *If* $\mu \in \mathcal{M}_\gamma$, $q \in \Pi_{m-1}$, *and* $f := \phi * \mu + q$, *then*

(*i*) $s_h(\mu) \in S(\phi; h\mathbb{Z}^d \cap (h \operatorname{supp} \psi + \operatorname{supp} \mu))$ *and*
(*ii*) $|||f - s_h(\mu)||| \le \operatorname{const}(m, \gamma) h^\gamma \|\mu\|_{\mathcal{M}_\gamma}$.

Proof. Put $\mu_h := \psi(\cdot/h) * \mu$. Since $\operatorname{supp} \mu_h \subset h \operatorname{supp} \psi + \operatorname{supp} \mu$, it is clear that $s_h(\mu) \in \operatorname{span}\{\phi(\cdot - \xi) : \xi \in h\mathbb{Z}^d \cap (h \operatorname{supp} \psi + \operatorname{supp} \mu)\}$. Hence, in order to prove (i), it remains only to show that $\sum_{j\in\mathbb{Z}^d} \mu_h(hj)r(j) = 0$ for all $r \in \Pi_{m-1}$. If we put $g := \mu_h(h\cdot)r$, then we obtain from Poisson's summation formula (Stein & Weiss 1971, Chapter 7) that $\sum_{j\in\mathbb{Z}^d} g(j) = \sum_{j\in\mathbb{Z}^d} \hat{g}(2\pi j)$. Now $\widehat{\mu_h} = h^d \hat{\psi}(j\cdot)\hat{\mu}$; hence, if $r = \sum_{|\alpha|<m} i^{-|\alpha|} a_\alpha ()^\alpha$, then

$$\hat{g} = \sum_{|\alpha|<m} a_\alpha D^\alpha (h^{-d} \widehat{\mu_h}(\cdot/h)) = \sum_{|\alpha|<m} a_\alpha D^\alpha [\hat{\eta}\hat{\sigma}\hat{\mu}(\cdot/h)].$$

Condition (3.1) ensures that $D^\alpha[\hat{\eta}\hat{\sigma}\hat{\mu}(\cdot/h)] = 0$ at $2\pi j$ whenever $j \in \mathbb{Z}^d\backslash 0$ and $|\alpha| < m$. On the other hand, (2.2) ensures that $D^\alpha[\hat{\eta}\hat{\sigma}\hat{\mu}(\cdot/h)] = 0$ at 0 for all $|\alpha| < m$. Hence, $\sum_{j\in\mathbb{Z}^d} \mu_h(hj)r(j) = \sum_{j\in\mathbb{Z}^d} \hat{g}(2\pi j) = 0$ which proves (i). We turn now to (ii). For brevity, let us write s_h in place of $s_h(\mu)$. According to Gelfand & Shilov (1964), $\hat{\phi}$ can be identified on $\mathbb{R}^d\backslash 0$ with $c_\phi|\cdot|^{-2m}$ where c_ϕ is a nonzero constant depending only on m and d. For $w \in \mathbb{R}^d\backslash 0$, we have $\widehat{s_h}(w) = \sum_{j\in\mathbb{Z}^d} \hat{\phi}(w)\mu_h(hj)e^{-ihj\cdot w}$. If we define $g := \mu_h(h\cdot)e_{-hw}$, then we obtain from Poisson's summation formula that $\sum_{j\in\mathbb{Z}^d} g(j) = \sum_{j\in\mathbb{Z}^d} \hat{g}(2\pi j)$. Hence,

$$\begin{aligned}
\widehat{s_h}(w) &= \hat{\phi}(w) \sum_{j\in\mathbb{Z}^d} g(j) \\
&= \hat{\phi}(w) \sum_{j\in\mathbb{Z}^d} \hat{g}(2\pi j) \\
&= \hat{\phi}(w) \sum_{j\in\mathbb{Z}^d} h^{-d} \widehat{\mu_h}(w + 2\pi j/h) \\
&= \hat{\phi}(w) \sum_{j\in\mathbb{Z}^d} \hat{\psi}(hw + 2\pi j)\hat{\mu}(w + 2\pi j/h).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{|c_\phi|}|||f-s_h||| &= \frac{1}{|c_\phi|}\||\cdot|^m(\hat{f}-\widehat{s_h})\|_{L_2(\mathbb{R}^d\backslash 0)} \\
&= \left\| |\cdot|^{-m}\left[\hat{\mu}-\sum_{j\in\mathbb{Z}^d}\hat{\psi}(h\cdot+2\pi j)\hat{\mu}(\cdot+2\pi j/h)\right]\right\|_{L_2} \\
&\le \||\cdot|^{-m}(1-\hat{\psi}(h\cdot))\hat{\mu}\|_{L_2} \\
&\quad + \left\| |\cdot|^{-m}\left[\sum_{j\in\mathbb{Z}^d\backslash 0}\hat{\psi}(h\cdot+2\pi j)\hat{\mu}(\cdot+2\pi j/h)\right]\right\|_{L_2} \\
&=: I + II.
\end{aligned}$$

We consider first I. It follows from (3.1) and (3.2) that

$$|1-\hat{\psi}(w)| \le \mathrm{const}(d,m)\frac{|w|^m}{1+|w|^m}, \quad w\in\mathbb{R}^d.$$

Consequently,

$$\begin{aligned}
I^2 &= \||\cdot|^{-m}(1-\hat{\psi}(h\cdot))\hat{\mu}\|^2_{L_2(h^{-1}B)} + \||\cdot|^{-m}(1-\hat{\psi}(h\cdot))\hat{\mu}\|^2_{L_2(\mathbb{R}^d\backslash h^{-1}B)} \\
&\le \mathrm{const}(d,m)\||\cdot|^{-m}|h\cdot|^m\hat{\mu}\|^2_{L_2(h^{-1}B)} + \mathrm{const}(d,m)\||\cdot|^{-m}\hat{\mu}\|^2_{L_2(\mathbb{R}^d\backslash h^{-1}B)} \\
&= \mathrm{const}(d,m)h^{2m}\|\hat{\mu}\|^2_{L_2(h^{-1}B)} + \mathrm{const}(d,m)\||\cdot|^{-m}\hat{\mu}\|^2_{L_2(\mathbb{R}^d\backslash h^{-1}B)} \\
&\le \mathrm{const}(d,m,\gamma)h^{2\gamma}\|\mu\|^2_{\mathcal{M}_\gamma}
\end{aligned}$$

by Proposition 2.3 (ii), (iii). Let $C := [-\frac{1}{2}\ .\ .\ \frac{1}{2})^d$. In order to estimate II, we employ the partition $\mathbb{R}^d = \bigcup_{k\in\mathbb{Z}^d} 2\pi h^{-1}(k+C)$ to write

$$II^2 = \sum_{k\in\mathbb{Z}^d}\left\| |\cdot|^{-m}\sum_{j\in\mathbb{Z}^d\backslash 0}\hat{\psi}(h\cdot+2\pi j)\hat{\mu}(\cdot+2\pi j/h)\right\|^2_{L_2(2\pi h^{-1}(k+C))}.$$

For $j\in\mathbb{Z}^d\backslash 0$ and $k\in\mathbb{Z}^d\backslash\{-j\}$ we have

$$\begin{aligned}
&\||\cdot|^{-m}\hat{\psi}(h\cdot+2\pi j)\hat{\mu}(\cdot+2\pi j/h)\|_{L_2(2\pi h^{-1}(k+C))} \\
&\quad = \||\cdot-2\pi j/h|^{-m}\hat{\sigma}(h\cdot)\hat{\eta}(h\cdot)\hat{\mu}\|_{L_2(2\pi h^{-1}(k+j+C))} \\
&\quad \le \mathrm{const}(d,m)\left\|\frac{|h\cdot-2\pi(k+j)|^m}{|\cdot-2\pi j/h|^m}\hat{\sigma}(h\cdot)\hat{\mu}\right\|_{L_2(2\pi h^{-1}(k+j+C))} \\
&\quad \le \mathrm{const}(d,m)\left\|\frac{|h\cdot-2\pi(k+j)|^m}{|\cdot-2\pi j/h|^m}\hat{\sigma}(h\cdot)|\cdot|^m\right\|_{L_\infty(2\pi h^{-1}(k+j+C))}\||\cdot|^{-m}\hat{\mu}\|_{L_2(2\pi h^{-1}(k+j+C))} \\
&\quad \le \mathrm{const}(d,m)\|\hat{\sigma}\|_{L_\infty(2\pi(k+j+C))}\left\||\cdot|^m\frac{|\cdot+2\pi(k+j)|^m}{|\cdot+2\pi k|^m}\right\|_{L_\infty(2\pi C)}\||\cdot|^{-m}\hat{\mu}\|_{L_2(2\pi h^{-1}(k+j+C))} \\
&\quad \le \mathrm{const}(d,m)\|\hat{\sigma}\|_{L_\infty(2\pi(k+j+C))}\frac{|k+j|^m}{1+|k|^m}\||\cdot|^{-m}\hat{\mu}\|_{L_2(2\pi h^{-1}(k+j+C))}.
\end{aligned}$$

Therefore,

$$\left\| \sum_{j\in\mathbb{Z}^d\setminus\{0,-k\}} |\cdot|^{-m}\hat{\psi}(h\cdot+2\pi j)\hat{\mu}(\cdot+2\pi j/h)\right\|_{L_2(2\pi h^{-1}(k+C))}$$

$$\leq \mathrm{const}(d,m)\sum_{j\in\mathbb{Z}^d\setminus\{0,-k\}}\|\hat{\sigma}\|_{L_\infty(2\pi(k+j+C))}\frac{|k+j|^m}{1+|k|^m}\||\cdot|^{-m}\hat{\mu}\|_{L_2(2\pi h^{-1}(k+j+C))}$$

$$\leq \frac{\mathrm{const}(d,m)}{1+|k|^m}\sqrt{\sum_{j\in\mathbb{Z}^d\setminus\{0,-k\}}\|\hat{\sigma}\|_{L_\infty(2\pi(k+j+C))}|k+j|^{2m}}$$

$$\times\sqrt{\sum_{j\in\mathbb{Z}^d\setminus\{0,-k\}}\||\cdot|^{-m}\hat{\mu}\|_{L_2(2\pi h^{-1}(k+j+C))}}$$

$$\leq \mathrm{const}(d,m)\frac{1}{1+|k|^m}\||\cdot|^{-m}\hat{\mu}\|_{L_2(\mathbb{R}^d\setminus 2\pi h^{-1}C)}\leq \mathrm{const}(d,m,\gamma)\frac{h^\gamma}{1+|k|^m}\|\hat{\mu}\|_{\mathcal{M}_\gamma}$$

by Proposition 2.3 (ii). Now if $k\neq 0$ and $j=-k$, then

$$\||\cdot|^{-m}\hat{\psi}(h\cdot+2\pi j)\hat{\mu}(\cdot+2\pi j/h)|_{L_2(2\pi h^{-1}(k+C))}$$

$$=\||\cdot+2\pi k/h|^{-m}\hat{\psi}(h\cdot)\hat{\mu}\|_{L_2(2\pi h^{-1}C)}$$

$$\leq \mathrm{const}(d,m)\||\cdot+2\pi k/h|^{-m}\|_{L_\infty(2\pi h^{-1}C)}\|\hat{\mu}\|_{L_2(2\pi h^{-1}C)}$$

$$\leq \mathrm{const}(d,m)\frac{h^m}{1+|k|^m}\|\hat{\mu}\|_{L_2(2\pi h^{-1}C)}\leq \mathrm{const}(d,m,\gamma)\frac{h^\gamma}{1+|k|^m}\|\hat{\mu}\|_{\mathcal{M}_\gamma}$$

by Proposition 2.3 (iii). Therefore,

$$II^2\leq \mathrm{const}(d,m,\gamma)(h^\gamma\|\hat{\mu}\|_{\mathcal{M}_\gamma})^2\sum_{k\in\mathbb{Z}^d}\frac{1}{(1+|k|^m)^2}\leq \mathrm{const}(d,m,\gamma)(h^\gamma\|\mu\|_{\mathcal{M}_\gamma})^2$$

since $m>d/2$; hence, $I+II\leq \mathrm{const}(d,m,\gamma)h^\gamma\|\mu\|_{\mathcal{M}_\gamma}$.

4. AN APPROXIMATION TO $s_h(\mu)$ FROM $S(\phi;\Xi)$

Let $\mathcal{N}$ be the set $\mathcal{N}:=\{\frac{1}{2m}j : j\in\mathbb{Z}^d,\ j_i>0, \text{ and } j_i+\cdots j_d\leq m\}$. It is known from de Boor & Ron (1992) that $\mathcal{N}$ is 'correct' for interpolation in Π_m; consequently, we have the following:

Lemma 4.1. *There exists* $\varepsilon_1\in(0\,..\,1/4)$ *(depending only on d, m) such that if* $x\in r\bar{B}$, *and* $\#\tilde{\mathcal{N}}=\#\mathcal{N}$ *and* $\delta(\tilde{\mathcal{N}};\mathcal{N})\leq\varepsilon_1$, *then there exists* $\{a_\xi\}_{\xi\in\tilde{\mathcal{N}}}$ *such that*

$$\max_{\xi\in\tilde{\mathcal{N}}}|a_\xi|\leq \mathrm{const}(d,m,r)\quad\text{and}\quad q(x)=\sum_{\xi\in\tilde{\mathcal{N}}}a_\xi q(\xi)\quad\forall q\in\Pi_m.$$

The following is equivalent to the standard definition of the cone property.

Definition 4.2. A set $\Omega \subset \mathbb{R}^d$ is said to have the *cone property* if there exists ε_Ω, $r_\Omega \in (0 \,..\, \infty)$ such that for all $x \in \Omega$ there exists $y \in \Omega$ such that $|x - y| = \varepsilon_\Omega$ and

$$(1 - t)x + ty + r_\Omega t B \subset \Omega \quad \forall t \in [0 \,..\, 1].$$

The purpose of this section is to prove the following:

Proposition 4.3. *Let Ω be a bounded, open subset of $\mathbb{R}^d$ having the cone property. If Ξ is a finite subset of $\bar{\Omega}$ satisfying $\delta := \delta(\Xi; \Omega) \le \varepsilon_1 r_\Omega$, then for all $\gamma \in (0 \,..\, m]$, $\mu \in \mathcal{M}_\gamma(\bar{\Omega})$, there exists $s \in S(\phi; \Xi)$ such that*

$$|||s_h(\mu) - s||| \le \text{const}(\Omega, m, \psi, \gamma)\delta^\gamma \|\mu\|_{\mathcal{M}_\gamma},$$

where $h := \delta/\varepsilon_1$.

Let r_0 be the smallest positive real number for which

$$\text{supp}\ \psi \subset r_0 \bar{B}.$$

Let Ω, μ, and Ξ satisfy the hypothesis of Proposition 4.3. Let $\mu_h \in C_c^\infty(\mathbb{R}^d)$ be given by $\mu_h := \psi(\cdot/h) * \mu$, and note that $\text{supp}\ \mu_h \subset \text{supp}\ \mu + h\ \text{supp}\ \psi \subset \bar{\Omega} + h r_0 \bar{B}$. For $j \in \mathbb{Z}^d$ satisfying $\mu_h(hj) \ne 0$, there exists $x_j \in \Omega$ such that $|x_j - hj| \le h r_0$. By Definition 4.2, there exists $y_j \in \Omega$ such that $|x_j - y_j| = \varepsilon_\Omega$ and

$$(1 - t)x_j + t y_j + r_\Omega t B \subset \Omega, \quad \forall t \in [0 \,..\, 1].$$

Substituting $t = h/r_\Omega$ (necessarily ≤ 1) we obtain $z_j + hB \subset \Omega$, where $z_j := (1 - h/r_\Omega)x_j + (h/r_\Omega)y_j$. Note that

$$|j - h^{-1}z_j| \le |hj - x_j|/h + |x_j - z_j|/h \le r_0 + \varepsilon_\Omega/r_\Omega =: r_1, \tag{4.4}$$

and $\delta(h^{-1}\Xi; h^{-1}z_j + B) \le h^{-1}\delta(\Xi; \Omega) = \varepsilon_1$. Since $\mathcal{N} \subset B$, there exists $\mathcal{N}_j \subset \Xi$ such that $\#\mathcal{N}_j = \#\mathcal{N}$ and $\delta(h^{-1}\mathcal{N}_j - h^{-1}z_j; \mathcal{N}) \le \varepsilon_1$. By Lemma 4.1, there exists $\{a_{j,\xi}\}_{\xi \in \mathcal{N}_j}$ such that

$$\max_{\xi \in \mathcal{N}_j} |a_{j,\xi}| \le \text{const}(d, m, r_1) \tag{4.5}$$

and

$$q(j - h^{-1}z_j) = \sum_{\xi \in \mathcal{N}_j} a_{j,\xi}\, q(h^{-1}(\xi - z_j)), \quad \forall q \in \Pi_m. \tag{4.6}$$

Two easily proved consequences of (4.6) are that for all $q \in \Pi_m$,

$$q(0) = \sum_{\xi \in \mathcal{N}_j} a_{j,\xi}\, q(\xi/h - j) \quad \text{and} \quad q = \sum_{\xi \in \mathcal{N}_j} a_{j,\xi}\, q(\cdot - (\xi/h - j)). \tag{4.7}$$

Noting that $s_h(\mu)$ can be written as $s_h(\mu) = \sum_{j \in \mathbb{R}^d} \mu_h(hj)\phi(\cdot - hj)$, Dyn & Ron (1995) have suggested that in order to approximate $s_h(\mu)$ from $S(\phi; \Xi)$, one should first find 'pseudo-shifts' $\phi_j \in \text{span}\{\phi(\cdot - \xi) : \xi \in \Xi\}$ which approximate $\phi(\cdot - hj)$ and then put $s := \sum_{j \in \mathbb{Z}^d} \mu_h(hj)\phi_j$.

Definition. For $j \in \mathbb{Z}^d$ satisfying $\mu_h(hj) \neq 0$, define

$$\phi_j := \sum_{\xi \in \mathcal{N}_j} a_{j,\xi}\phi(\cdot - \xi),$$
$$\zeta_j := \sum_{\xi \in \mathcal{N}_j} a_{j,\xi}\zeta(\cdot - \xi),$$

where

$$\zeta := \begin{cases} |\cdot|^{m-d} & \text{if } m-d \notin 2\mathbb{N}_0 \\ |\cdot|^{m-d}\log|\cdot| & \text{if } m-d \in 2\mathbb{N}_0 \end{cases}.$$

Lemma 4.8. *If*

$$s := \sum_{j \in \mathbb{Z}^d} \mu_h(hj)\phi_j,$$

then $s \in S(\phi; \Xi)$ *and*

$$|||s_h(\mu) - s||| \leq \text{const}(d, m)\left\| \sum_{j \in \mathbb{Z}^d} \mu_h(hj)(\zeta(\cdot - hj) - \zeta_j)\right\|_{L_2}. \tag{4.9}$$

Proof. It is clear that $s \in \text{span}\{\phi(\cdot - \xi) : \xi \in \Xi\}$, so in order to show that $s \in S(\phi; \Xi)$, it suffices to show that $\sum_{j\in\mathbb{Z}^d} \mu_h(hj) \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}q(\xi) = 0$, for all $q \in \Pi_{m-1}$. It was shown in the proof of Proposition 3.3 that $s_h(\mu) \in S(\phi; h\mathbb{Z}^d \cap \text{supp}\,\mu_h)$; hence $\sum_{j\in\mathbb{Z}^d} q(hj)\mu_h(hj) = 0$ for all $q \in \Pi_{m-1}$. Therefore, if $q \in \Pi_{m-1}$, then $\sum_{j\in\mathbb{Z}^d} \mu_h(hj) \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}q(\xi) = \sum_{j\in\mathbb{Z}^d} \mu_h(hj)q(hj) = 0$ which proves that $s \in S(\phi; \Xi)$. Now, if $\| \sum_{j\in\mathbb{Z}^d} \mu_h(h)(\zeta(\cdot - hj) - \zeta_j)\|_{L_2} = \infty$ then the inequality is clear; so assume $\sum_{j\in\mathbb{Z}^d} \mu_h(hj)(\zeta(\cdot - hj) - \zeta_j) \in L_2$. Then

$$\begin{aligned} |||s_h(\mu) - s||| &= \left|\left|\left| \sum_{j\in\mathbb{Z}^d} \mu_h(hj)\left[\phi(\cdot - hj) - \sum_{j\in\mathcal{N}_j} a_{j,\xi}\phi(\cdot - \xi)\right]\right|\right|\right| \\ &= |c_\phi| \left\| |\cdot|^m \sum_{j\in\mathbb{Z}^d} \mu_h(hj)\left[|\cdot|^{-2m}e_{-hj} - \sum_{j\in\mathcal{N}_j} a_{j,\xi}|\cdot|^{-2m}e_{-\xi}\right]\right\|_{L_2} \\ &= |c_\phi| \left\| \sum_{j\in\mathbb{Z}^d} \mu_h(hj)\left[|\cdot|^{-m}e_{-hj} - \sum_{j\in\mathcal{N}_j} a_{j,\xi}|\cdot|^{-m}e_{-\xi}\right]\right\|_{L_2} \\ &= \text{const}(d, m)\left\| \sum_{j\in\mathbb{Z}^d} \mu_h(hj)(\zeta(\cdot - hj) - \zeta_j)\right\|_{L_2}, \end{aligned}$$

since $|\hat{\zeta}| = \text{const}(d, m)|\cdot|^{-m}$ on $\mathbb{R}^d\backslash 0$ (Gelfand & Shilov 1964).

The problem of estimating the right side of (4.9) would be much simpler if the function $\zeta - \zeta_j(\cdot + hj)$ was independent of j. The following lemma, proposition, and lemma will allow us to carry forth our desired estimate despite the dependence of $\zeta - \zeta_j(\cdot + hj)$ on j.

Let $\rho : \mathbb{R}^d \to [0\,..\,\infty)$ be given by $\rho(x) := 0$ if $x \in (1 + r_1)B$ and

$$\rho(x) := \max\left\{\left|\zeta(x) - \sum_{\xi\in\tilde{\mathcal{N}}} a_\xi \zeta(x - \xi)\right|\right\}, \quad \text{if } x \notin (1 + r_1)B,$$

where the maximum is taken over all z, $\tilde{\mathcal{N}}$ satisfying $z \in r_1\bar{B}$, $\#\tilde{\mathcal{N}} = \#\mathcal{N}$, $\delta(\tilde{\mathcal{N}} - z, \mathcal{N}) \le \varepsilon_1$, and the coefficients $\{a_\xi\}_{\xi\in\tilde{\mathcal{N}}}$ are determined by the requirement $q(0) = \sum_{\xi\in\tilde{\mathcal{N}}} a_\xi q(\xi)$, $\forall q \in \Pi_m$. We will show that ρ belongs to the space $\mathcal{L}_2$ which was first introduced by Jia & Micchelli (1991) as the set of all $g \in L_2$ for which

$$\|g\|_{\mathcal{L}_2} := \left\|\sum_{j\in\mathbb{Z}^d} |g(\cdot - j)|\right\|_{L_2(C)} < \infty,$$

where $C := [-1/2\,..\,1/2)^d$.

Lemma 4.10. $\|\rho\|_{\mathcal{L}_2} \le \text{const}(d, m, r_1)$.

Proof. Let $x \in \mathbb{R}^d \backslash (1 + r_1)B$, and let $z, \tilde{\mathcal{N}}$ be such that $\rho(x) = |\zeta(x) - \sum_{\xi\in\tilde{\mathcal{N}}} a_\xi \zeta(x - \xi)|$, where the coefficients $\{a_\xi\}$ are as described in the definition of ρ. Since $\delta(\tilde{\mathcal{N}} - z, \mathcal{N}) \le \varepsilon_1$, $z \in r_1\bar{B}$, and $q(-z) = \sum_{\xi\in\tilde{\mathcal{N}}} a_\xi q(\xi - z)$ for all $q \in \Pi_m$, it follows by Lemma 4.1 that $\max_{\xi\in\tilde{\mathcal{N}}} |a_\xi| \le \text{const}(d, m, r_1)$. Note that since $\mathcal{N} \subset \frac{1}{2}\bar{B}$ and $\varepsilon_1 \in (0\,..\,1/4)$, it follows that $\tilde{\mathcal{N}} \subset (r_1 + \frac{3}{4})B$. Define the difference operator T by $Tg := g - \sum_{\xi\in\tilde{N}} a_\xi g(\cdot - \xi)$. It follows from the requirement $q(0) = \sum_{\xi\in\tilde{\mathcal{N}}} a_\xi q(\xi)$ $\forall q \in \Pi_m$ that $Tq = 0$ $\forall q \in \Pi_m$. Let $q \in \Pi_m$ be the mth-degree Taylor polynomial of ζ at x. Then

$$\begin{aligned}
\rho(x) &= |T\zeta(x)| \\
&= |T(\zeta - q)(x)| \\
&= \left|\zeta(x) - q(x) + \sum_{\xi\in\tilde{\mathcal{N}}} a_\xi(\zeta(x - \xi) - q(x - \xi))\right| \\
&\le \left(\max_{\xi\in\tilde{\mathcal{N}}} |a_\xi|\right) \sum_{\xi\in\tilde{\mathcal{N}}} |\zeta(x - \xi) - q(x - \xi)| \\
&\le \text{const}(d, m, r_1) \max\{|D^\alpha\zeta(w)| : |\alpha| = m + 1 \text{ and } w \in x + (r_1 + 3/4)\bar{B}\} \\
&\le \text{const}(d, b, r_1)|x|^{-d-1}(1 + \log|x|).
\end{aligned}$$

It follows from this that $\|\rho\|_{\mathcal{L}_2} \le \text{const}(d, m, r_1)$.

The following proposition, which demonstrates the utility of the space $\mathcal{L}_2$, was proved in Jia & Micchelli (1991).

Proposition 4.11. *If $c \in l_2(\mathbb{Z}^d)$ and $g \in \mathcal{L}_2$, then*

$$\left\| \sum_{j \in \mathbb{Z}^d} c_j g(\cdot - j) \right\|_{L_2} \leq \|c\|_{l_2(\mathbb{Z}^d)} \|g\|_{\mathcal{L}_2}.$$

Lemma 4.12. *If $j \in \mathbb{Z}^d$ is such that $\mu_h(hj) \neq 0$ and $\rho_j := \zeta - \zeta_j(\cdot + hj)$, then*

(*i*) $|\rho_j(x)| \leq h^{m-d} \rho(x/h) \quad \forall x \in \mathbb{R}^d \backslash h(1+r_1)B$ *and*
(*ii*) $\|\rho_j\|_{L_2(h(1+r_1)B)} \leq \text{const}(d, m, r_1) h^{m-d/2}$.

Proof. We first establish the identity

$$\rho_j(x) = h^{m-d} \left[\zeta(x/h) - \sum_{\xi \in \mathcal{N}_j} a_{j,\xi} \zeta(x/h - (\xi/h - j)) \right], \quad x \notin \{0\} \cup (\mathcal{N}_j - hj). \quad (4.13)$$

If $m - d \notin 2\mathbb{N}_0$, then (4.13) is simply a consequence of the fact that $\zeta(y) = h^{m-d}\zeta(y/h)$. If $m - d \in 2\mathbb{N}_0$, then

$$\zeta(x) = \zeta(hx/h) = h^{m-d}\zeta(x/h) + h^{m-d}|x/h|^{m-d} \log h$$

and hence

$$\begin{aligned} \rho_j(x) = h^{m-d} &\left[\zeta(x/h) - \sum_{\xi \in \mathcal{N}_j} a_{j,\xi} \zeta(x/h - (\xi/h - j)) \right] \\ &+ h^{m-d} \log h \left[|x/h|^{m-d} - \sum_{\xi \in \mathcal{N}_j} a_{j,\xi} |x/h - (\xi/h - j)|^{m-d} \right]. \quad (4.14) \end{aligned}$$

Let T be the difference operator defined by $Tg := g - \sum_{\xi \in \mathcal{N}_j} a_{j,\xi} g(\cdot - (\xi/h - j))$, and note that $Tq = 0 \ \ \forall q \in \Pi_m$ by (4.7). In particular, since $|\cdot|^{m-d} \in \Pi_m$, it follows that $|x/h|^{m-d} - \sum_{\xi \in \mathcal{N}_j} a_{j,\xi}|x/h - (\xi/h - j)|^{m-d} = [T(|\cdot|^{m-d})](x/h) = 0$ which, in view of (4.14), completes the proof of (4.13). In order to establish (i), let $z = h^{-1}z_j - j$. Then $(h^{-1}\mathcal{N}_j - j) - z = h^{-1}\mathcal{N}_j - h^{-1}z_j$ and hence $\delta((h^{-1}\mathcal{N}_j - j) - z, \mathcal{N}) \leq \varepsilon_1$. Now (i) follows from (4.13) since by the definition of ρ and with (4.4), (4.7) in view,

$|\zeta(x/h) - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\xi(x/h - (\xi/h - j))| \le \rho(x/h)$. For (ii), we note that

$$\begin{aligned}\|\rho_j\|_{L_2(h(1+r_1)B)} &= h^{m-d}\left\|\zeta(\cdot/h) - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\zeta(\cdot/h - (\xi/h - j))\right\|_{L_2(h(1+r_1)B)}, \qquad \text{by (4.13)},\\ &= h^{m-d/2}\left\|\zeta - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\zeta(\cdot - (\xi/h - j))\right\|_{L_2((1+r_1)B)}\\ &\le \mathrm{const}(d, m, r_1)h^{m-d/2}\|\zeta\|_{L_2(2(1+r_1)B)} = \mathrm{const}(d, m, r_1)h^{m-d/2}, \text{ by (4.5).}\end{aligned}$$

Proof of Proposition 4.3. Let $s \in S(\phi; \Xi)$ be as in Lemma 4.8, and for brevity, put $\tilde{B} := (1 + r_1)B$. Then

$$\begin{aligned}\mathrm{const}(d, m)|||s_h(\mu) - s||| &\le \left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)(\zeta(\cdot - hj) - \zeta_j)\right\|_{L_2}\\ &= \left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)\rho_j(\cdot - hj)\right\|_{L_2}\\ &\le \left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)\chi_{h(j+\tilde{B})}\rho_j(\cdot - hj)\right\|_{L_2}\\ &\quad + \left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)\chi_{\mathbb{R}^d\setminus h(j+\tilde{B})}\rho_j(\cdot - hj)\right\|_{L_2}\\ &\le \mathrm{const}(d, r_1)\sqrt{\sum_{j\in\mathbb{Z}^d} |\mu_h(hj)|^2\|\rho_j(\cdot - hj)\|^2_{L_2(h(j+\tilde{B}))}}\\ &\quad + h^{m-d}\left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)\rho(\cdot/h - j)\right\|_{L_2}\\ &= \mathrm{const}(d, r_1)\sqrt{\sum_{j\in\mathbb{Z}^d} |\mu_h(hj)|^2\|\rho_j\|^2_{L_2(h\tilde{B})}}\\ &\quad + h^{m-d/2}\left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)\rho(\cdot - j)\right\|_{L_2}\\ &\le \mathrm{const}(d, m, r_1)h^{m-d/2}\|\mu_h\|_{l_2(h\mathbb{Z}^d)},\end{aligned}$$

by Lemma 4.12 (ii), Lemma 4.10, and Proposition 4.11. Therefore,

$$|||s_h(\mu) - s||| \le \mathrm{const}(d, m, r_1)h^{m-d/2}\|\mu_h\|_{l_2(h\mathbb{Z}^d)}. \tag{4.15}$$

Claim 4.16.

$$\|\mu_h\|_{l_2(h\mathbb{Z}^d)} = (h/2\pi)^{d/2}\left\|\sum_{j\in\mathbb{Z}^d}\hat{\psi}(h\cdot+2\pi j)\hat{\mu}(\cdot+2\pi j/h)\right\|_{L_2(2\pi h^{-1}C)}$$

Proof. Define $G: 2\pi h^{-1}C \to \mathbb{C}$ by $G := \sum_{j\in\mathbb{Z}^d}\mu_h(hj)e_{-hj}$ and note that $\|G\|_{L_2(2\pi h^{-1}C)} = (2\pi/h)^{d/2}\|\mu_h\|_{l_2(h\mathbb{Z}^d)}$. Hence

$$\|\mu_h\|_{l_2(h\mathbb{Z}^d)} = (h/2\pi)^{d/2}\|G\|_{L_2(2\pi h^{-1}C)}. \tag{4.17}$$

Fix $x \in 2\pi h^{-1}C$ and put $g := \mu_h(h\cdot)e_{-hx}$ and note that $G(x) = \sum_{j\in\mathbb{Z}^d} g(j) = \sum_{j\in\mathbb{Z}^d}\hat{g}(2\pi j)$ by Poisson's summation formula (Stein & Weiss 1971, Chapter 7). Now,

$$\hat{g} = (\mu_h(h\cdot))\hat{}\,(\cdot+hx) = h^{-d}\widehat{\mu_h}(\cdot/h+x) = \hat{\psi}(\cdot+hx)\hat{\mu}(\cdot/h+x).$$

Therefore, $G(x) = \sum_{j\in\mathbb{Z}^d}\hat{g}(2\pi j) = \sum_{j\in\mathbb{Z}^d}\hat{\psi}(2\pi j+hx)\hat{\mu}(2\pi j/h+x)$ which, in view of (4.17), proves the claim.

Now,

$$\begin{aligned}
&\left\|\sum_{j\in\mathbb{Z}^d}\hat{\psi}(h\cdot+2\pi j)\hat{\mu}(\cdot+2\pi j/h)\right\|_{L_2(2\pi h^{-1}C)} \\
&\quad\le \sum_{j\in\mathbb{Z}^d}\|\hat{\psi}(h\cdot+2\pi j)\hat{\mu}(\cdot+2\pi j/h)\|_{L_2(2\pi h^{-1}C)} \\
&\quad\le \|\hat{\psi}\|_{L_\infty(2\pi C)}\|\hat{\mu}\|_{L_2(2\pi h^{-1}C)} + \sum_{j\in\mathbb{Z}^d\backslash 0}\|\hat{\psi}(h\cdot)|\cdot|^m\|_{L_\infty(2\pi h^{-1}(j+C))}\||\cdot|^{-m}\hat{\mu}\|_{L_2(2\pi h^{-1}(j+C))} \\
&\quad\le \mathrm{const}(d,m,\psi)\|\hat{\mu}\|_{L_2(2\pi h^{-1}C)} \\
&\qquad + h^{-m}\sqrt{\sum_{j\in\mathbb{Z}^d\backslash 0}\|\hat{\psi}|\cdot|^m\|^2_{L_\infty(2\pi(j+C))}}\||\cdot|^{-m}\hat{\mu}\|_{L_2(\mathbb{R}^d\backslash 2\pi h^{-1}C)} \\
&\quad\le \mathrm{const}(d,m,\psi,\gamma)h^{\gamma-m}\|\mu\|_{\mathcal{M}_\gamma},
\end{aligned}$$

by Proposition 2.3, (3.1), (3.2), which, in view of (4.15) and Claim 4.16 completes the proof.

5. THE MAIN RESULTS

Combining Proposition 3.3 and Proposition 4.3 yields the following:

Theorem 5.1. *Let Ω be a bounded, open subset $\mathbb{R}^d$ having the cone property, and let $\Xi \subset \bar{\Omega}$ satisfy $\delta := \delta(\Xi;\Omega) \le \min\{\varepsilon_1 r_\Omega, \varepsilon_1\}$. If $f \in C(\mathbb{R}^d)$ is such that there exists $\gamma \in (0\,..\,m]$, $\mu \in \mathcal{M}_\gamma(\Omega)$, $q \in \Pi_{m-1}$ such that $f = \phi * \mu + q$ on Ω, then*

(*i*) $\phi * \mu + q = T_\Omega f$,
(*ii*) $|||\phi * \mu + q - T_\Xi f||| \le \mathrm{const}(\Omega, m, \gamma)\delta^\gamma\|\mu\|_{\mathcal{M}_\gamma}$, *and*
(*iii*) $\|f - T_\Xi f\|_{L_p(\Omega)} \le \mathrm{const}(\Omega, m, \gamma)\delta^{\gamma_p+\gamma}\|\mu\|_{\mathcal{M}_\gamma}$

for all $1 \le p \le \infty$.

Proof. Since (i) follows from (ii) via (1.4) and (iii) follows from (i) and (ii) via (1.3), it suffices to prove (ii). Duchon (1977) has shown that

$$|||\phi * \mu + q - T_\Xi f||| \leq |||\phi * \mu - s||| \quad \forall s \in S(\phi; \Xi). \tag{5.2}$$

Put $h = \delta/\varepsilon_1$ and recall from Proposition 3.3 (ii) that $|||\phi * \mu - s_h(\mu)||| \leq \text{const}(m, \gamma, \psi)h^\gamma \|\mu\|_{\mathcal{M}_\gamma}$. By Proposition 4.3, there exists $s \in S(\phi; \Xi)$ such that $|||s_h(\mu) - s||| \leq \text{const}(\Omega, m, \gamma, \psi)\delta^\gamma \|\mu\|_{\mathcal{M}_\gamma}$. Hence, by (5.2),

$$|||\phi * \mu + q - T_\Xi f||| \leq |||\phi * \mu - s_h(\mu)||| + |||s_h(\mu) - s||| \leq \text{const}(\Omega, m, \gamma, \psi)\delta^\gamma \|\mu\|_{\mathcal{M}_\gamma}$$

which, after a suitable choice of ψ, proves (ii).

Given a smooth f, the problem of finding $\mu \in \mathcal{M}_\gamma(\bar{\Omega})$, $q \in \Pi_{m-1}$ such that $\phi * \mu + q = f$ on Ω is quite difficult. In the special case $m = d = 2$, $\Omega = B$, Johnson (2000) has shown that if $f \in C^\infty(\mathbb{R}^2)$, then there exists $\mu \in \mathcal{M}_{1/2}(\bar{\Omega})$, $q \in \Pi_1$ such that $\phi * \mu + q = f$ on Ω. There is one special case in when μ is easily found. That is the case when f is a smooth function supported in $\bar{\Omega}$. The following corollary deals with this special case.

For $\gamma \in (0\,..\,m]$, let $\mathcal{F}_\gamma$ be the space given by

$$\mathcal{F}_\gamma := \begin{cases} B^{\gamma+m}_{2,\infty} & \text{if } \gamma \in (0\,..\,m) \\ W^{2m}_2 & \text{if } \gamma = m. \end{cases}$$

Corollary 5.3. *Let Ω and Ξ be as in Theorem 5.1, and let $\gamma \in (0\,..\,m]$. If $f \in \mathcal{F}_\gamma$ is supported in $\bar{\Omega}$, then*

(*i*) $f = T_\Omega f$ *and*
(*ii*) $|||f - T_\Xi f||| \leq \text{const}(\Omega, m, \gamma)\delta^\gamma \|f\|_{\mathcal{F}_\gamma}$,

where $\delta := \delta(\Xi, \Omega)$. Additionally, if δ is sufficiently small, then

(*iii*) $\|f - T_\Xi f\|_{L_p(\Omega)} \leq \text{const}(\Omega, m, \gamma)\delta^{\gamma_p+\gamma}\|f\|_{\mathcal{F}_\gamma}$,

for all $1 \leq p \leq \infty$.

Proof. As mentioned in the proof of Theorem 5.1, it suffices to prove (ii). Assume $f \in \mathcal{F}_\gamma$ is supported in $\bar{\Omega}$. Put $\mu := [(-1)^m/c_\phi]\Delta^m f$, where

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$$

denotes the Laplacian operator, and note that $\mu \in \mathcal{M}_\gamma(\bar{\Omega})$ and $\|\mu\|_{\mathcal{M}_\gamma} \leq \text{const}(d, m, \gamma)\|f\|_{\mathcal{F}_\gamma}$. We show that $f = \phi * \mu$. Since $\hat{f} = (\phi * \mu)\hat{}$ on $\mathbb{R}^d\backslash 0$, it follows that the difference $f - \phi * \mu$ is a polynomial. For $x \notin \text{supp}\, f$, it follows from Green's second identity (Gilbarg & Trudinger 1983, p. 5) that

$$\phi * \mu(x) = \frac{(-1)^m}{c_\phi}\int_{\text{supp}\, f} \phi(x - t)\Delta^m f(t)dt = \frac{(-1)^m}{c_\phi}\int_{\text{supp}\, f} \Delta^m \phi(x - t) f(t)dt = 0$$

since $\Delta^m \phi = 0$ on $\mathbb{R}^d\backslash 0$. Thus the polynomial $f - \phi * \mu = 0$ on $\mathbb{R}^d\backslash\text{supp}\, f$; hence

$f = \phi * \mu$. If $\delta > \min\{\varepsilon_1 r_\Omega, \varepsilon_1\}$, then choosing $s = 0$ in (5.2) yields

$$|||f - T_\Xi f||| \leq |||f||| \leq \text{const}(m, \gamma)\|f\|_{\mathcal{F}_\gamma} \leq \text{const}(\Omega, m, \gamma)\delta^\gamma \|f\|_{\mathcal{F}_\gamma}.$$

On the other hand, if $\delta \leq \min\{\varepsilon_1 r_\Omega, \varepsilon_1\}$, then by Theorem 5.1 (ii),

$$|||f - T_\Xi f||| \leq \text{const}(\Omega, m, \gamma)\delta^\gamma \|\mu\|_{\mathcal{M}_\gamma} \leq \text{const}(\Omega, m, \gamma)\delta^\gamma \|f\|_{\mathcal{F}_\gamma}.$$

ACKNOWLEDGEMENTS

This work was supported by Kuwait University Grant SM-159.

REFERENCES

Bejancu, A. 1999. Local accuracy for radial basis function interpolation on finite uniform grids. Journal of Approximation Theory **99**: 242–257.

De Boor, C., DeVore, R.A. & Ron, A. 1994. Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$. Transactions of the American Mathematical Society **341**: 787–806.

de Boor, C., Höllig, K. & Riemenschneider, S. 1993. Box Splines. Springer-Verlag, New York, NY, USA.

de Boor, C. & Ron, A. 1992. The least solution for the polynomial interpolation problem. Mathematische Zeitschrift **210**: 347–378.

Buhmann, M.D. 1990. Multivariate cardinal interpolation with radial basis functions. Constructive Approximation **8**: 225–255.

Buhmann, M.D. 1993. New developments in the theory of radial basis function interpolation. In: **Jetter, K. & Utreras, F.I. (Eds.)** Multivariate Approximation: From CAGD to Wavelets. Pp. 35–75. World Scientific, Singapore.

Duchon, J. 1977. Splines minimizing rotation-invariant seminorms in Sobolev spaces. In: **Schempp, W. & Zeller, K. (Eds.)** Constructive Theory of Functions of Several Variables, Lecture Notes in Mathematics 571. Pp. 85–100. Springer-Verlag, Berlin, Ger.

Duchon, J. 1978. Sur l'erreur d'interpolation des fonctions de plusieur variables par les D^m-splines. RAIRO Analyse Numerique **12**: 325–334.

Dyn, N. & Ron, A. 1995. Radial basis function approximation: from gridded centres to scattered centres. Proceedings of the London Mathematical Society **71**: 76–108.

Foley, T.A. & Hagen, H. 1994. Advances in scattered data interpolation. Surveys on Mathematics for Industry **4**: 71–84.

Gelfand, I.M. & Shilov, G.E. 1964. Generalized Functions, Vol. 1. Academic Press, New York, NY, USA.

Gilbarg, D. & Trudinger, N.S. 1983. Elliptic Partial Differential Equations of Second Order (2nd ed). Springer-Verlag, New York, NY, USA.

Jia, R.-Q. & Lei, J. 1993. Approximation by multiinteger translates of functions having global support. Journal of Approximation Theory **72**: 2–23.

Jia, R.-Q. & Micchelli, C.A. 1991. Using the refinement equation for the construction of pre-wavelets II: Powers of two. In: **Laurent, P.J., Le Méhauté, A. & Schumaker, L.L. (Eds.)** Curves and Surfaces. Pp. 209–246. Academic Press, New York, NY, USA.

Johnson, M.J. 1998. A bound on the approximation order of surface splines. Constructive Approximation **14**: 429–438.

Johnson, M.J. 2000. An improved order of approximation for thin-plate spline interpolation in the unit disk. Numerische Mathematik **84**: 451–474.

Peetre, J. 1976. New Thoughts on Besov Spaces. Math. Dept. Duke Univ., Durham, NC, USA.

Powell, M.J.D. 1992. The theory of radial basis function approximation in 1990. In: **Light, W.A. (Ed.)** Advances in Numerical Analysis II: Wavelets, Subdivision, and Radial Functions. Pp. 105–210. Oxford University Press, Oxford, UK.

Powell, M.J.D. 1994. The uniform convergence of thin plate spline interpolation in two dimensions. Numerische Mathematik **68**: 107–128.

Schaback, R. 1999. Improved error bounds for radial basis function interpolation. Mathematics of Computation **68**: 201–216.

Stein, E.M. & Weiss, G. 1971. Introduction to Fourier Analysis on Euclidean Spaces. Princeton Univ. Press, Princeton, NJ, USA.

(Accepted 13 November 2000)

حول الخطأ في استكمال شريحة سطحية لدالة متراصة التدعيم

مايكل ج. جونسون
قسم الرياضيات وعلم الحاسوب
كلية العلوم – جامعة الكويت
ص.ب. 5969 – صفاة 13060 – الكويت

خلاصة

في هذا البحث سنبرهن على أن المعيار $L_p(\Omega)$ للخطأ في استكمال شريحة سطحية لدالة متراصة التدعيم في فضاء سوبوليف W_2^{2m} يتضاءل مثل $O(\delta^{\gamma_p+m})$ حيث $\gamma_p := \min\{m, m+d/p-d/2\}$ و m بارامیتر منسوب إلى درجة نعومة الشريحة السطحية . وفي حالة $1 \le p \le 2$ يتفق المعدل $O(\delta^{2m})$ مع الخطأ عندما يغطي المجال كل $\mathbb{R}^d$ ونقاط الاستكمال من مشبك لا نهائي.