

Extreme order statistics under power normalization and random sample size

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ABSTRACT

The present work deals with the weak convergence of the extreme order statistics under power normalization when the sample size itself is a random variable. The convergence conditions as well as the limit forms are obtained under nonrandom power normalization and when the interrelation between the basic variables and the random size is not restricted.

Key words: P -max stable laws; power normalization; sample of random size; weak convergence.

INTRODUCTION

Let $\{X_n, n \leq 1\}$ be a sequence of independent and identically distributed random variables (r.v.'s) with common distribution function (d.f.) $F(x) = P(X_n \leq x)$. Let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ denote the order statistics of X_1, X_2, \dots, X_n and let Φ_1 be a nondegenerate d.f. Suppose there exist real norming constants $a_n > 0, b_n$ such that, as $n \rightarrow \infty$,

$$F^n(a_n x + b_n) \xrightarrow{w} \Phi_1(x), \quad (1.1)$$

where (\xrightarrow{w}) denotes the weak convergence. We call $\Phi_1(x)$ a max stable d.f. under linear normalization or simply ℓ -max stable d.f. It will be known that every ℓ -max stable d.f. (Gnedenko 1943). $\Phi_1(x)$ can only have the form (in which x may be replaced by $ax + b$ for any $a > 0, b$) $\Phi_1(x) = \exp(-\mathcal{U}_{i;\alpha}(x))$, $i = 1, 2, 3, \alpha > 0$, where

$$\begin{aligned} \text{Types I) : } \mathcal{U}_{i;\alpha}(x) &= \begin{cases} \infty, & x \leq 0, \\ x^{-\alpha}, & x > 0; \end{cases} \\ \text{Types II) : } \mathcal{U}_{2;\alpha}(x) &= \begin{cases} (-x)^\alpha, & x \leq 0, \\ 0, & x > 0; \end{cases} \\ \text{Type III) : } \mathcal{U}_3(x) &= \mathcal{U}_{3;\alpha}(x) = e^{-x}, \quad \forall x. \end{aligned} \quad (1.2)$$

If (1.1) holds, we write $F \in D_\ell(\Phi_1)$ to indicate that F belongs to the max domain of attraction of Φ_1 under linear normalization. Moreover, Gnedenko (1943), showed that (1.1) holds with $\Phi_1(x) = \exp(-\mathcal{U}_{i,\alpha}(x))$, if and only if

$$n(1 - F(a_n x + b_n)) \rightarrow \mathcal{U}_{i,\alpha}(x). \tag{1.3}$$

Smirnov (1952) studied in depth the asymptotic behaviour of the r th upper extreme X_{n-r+1} when $n - r + 1$ is fixed. Namely, Smirnov proved that the d.f. of the normalized tern $\frac{1}{a_n}(X_{n-r+1:n} - b_n)$ converges weakly to a nondegenerate d.f. $\Phi_r(x)$ if and only if (1.3) holds. Moreover, the limit d.f. $\Phi_r(x)$ must have the form $\Phi_r(x) = 1 - \Gamma_r(\mathcal{U}_{i,\alpha}(x))$, where $\Gamma_r(\cdot)$ is an incomplete gamma function. All results obtained for $X_{n-r+1:n}$ imply analogous results for $X_{r:n}$. Following Pantcheva (1985) and Mohan and Ravi (1992) a d.f. F is said to belong to the max domain of attraction of a d.f. $\Psi_1(x)$ under power normalization if there exist norming constants $a_n > 0$ and $b_n > 0$, such that, as $n \rightarrow \infty$,

$$P\left(\left|\frac{X_{n:n}}{a_n}\right|^{b_n} \mathcal{S}(X_{n:n}) \leq x\right) = F^n(a_n|x|^{b_n} \mathcal{S}(x)) \xrightarrow{w} \Psi_1(x), \tag{1.4}$$

where $\mathcal{S}(x) = -1$ if $x < 0$, $= 0$ if $x = 0$ and $= 1$ if $x > 0$. In this case we write $F \in D_p(\Psi_1)$. We call $\Psi_1(x)$ a max stable d.f. under power normalization or simply p -max stable d.f. if (1.4) holds. We say that two d.f.'s F_1 and F_2 are of the same p -type if there exist $A > 0$ and $B > 0$ such that $F_1(x) = F_2(A|x|B\mathcal{S}(x)) \forall x$. Pantcheva (1985) showed that any p -max stable d.f. $\Psi_1(x)$ can be a p -type of one of the six d.f.'s $\exp(-\mathcal{V}_{i;\beta}(x))$, $i = 1, 2, \dots, 6, \beta > 0$, where

$$\begin{aligned} \text{Types I): } \mathcal{V}_{1;\beta}(x) &= \begin{cases} \infty, & x \leq 1, \\ (\log x)^{-\beta}, & x > 1; \end{cases} \\ \text{Types II): } \mathcal{V}_{2;\beta}(x) &= \begin{cases} \infty, & x \leq 0 \\ (-\log x)^\beta, & 0 < x \leq 1, \\ 0, & x > 1; \end{cases} \\ \text{Types III): } \mathcal{V}_{3;\beta}(x) &= \begin{cases} \infty, & x \leq -1, \\ (-\log(x))^{-\beta}, & -1 < x \leq 0 \\ 0, & x > 0; \end{cases} \end{aligned}$$

$$\text{Types IV): } \mathcal{V}_{4;\beta}(x) = \begin{cases} (\log(-x))^\beta, & x \leq -1, \\ 0, & x > -1; \end{cases}$$

$$\text{Type V): } \mathcal{V}_{5;\beta}(x) = \mathcal{V}_5(x) = \mathcal{U}_{1;1}(x);$$

$$\text{Type VI): } \mathcal{V}_{6;\beta}(x) = \mathcal{V}_6(x) = \mathcal{U}_{2;1}(x). \tag{1.5}$$

The necessary and sufficient conditions for a d.f. to belong to $D_p(\cdot)$ for each of the six p -max stable laws is obtained by Mohan and Ravi (1992) and Subramanya (1994). In these papers the results of Gnedenko (1943) and de Haan (1971) concerning linear normalization were extended to p -stable laws. They showed that every d.f. attracted to ℓ -max stable law is necessarily attracted to some p -max stable, and that p -max stable laws, in fact, attract more. Therefore, the advantage of normalizing with power functions is given by the fact that p -stable laws attract more d.f.'s than linear-stable laws. A unified approach to the results of Mohan and Ravi (1992) and Subramanya (1994) has been obtained by Christoph and Falk (1996). It is not difficult now to obtain the following slight generalization of the above results for the extreme order statistics under power normalization.

Theorem 1.1. For suitable normalizing constants $a_n > 0$ and $b_n > 0$, the d.f. of the normalized term $\left| \frac{X_{n-r+1:n}}{a_n} \right|^{\frac{1}{b_n}} \mathcal{S}(X_{n-r+1:n})$ converges weakly to a nondegenerate d.f. $\Psi_r(x)$ if and only if

$$n \left(1 - F(a_n |x|^{b_n} \mathcal{S}(x)) \right) \rightarrow \mathcal{V}_{i;\beta}(x), \quad i \in \{1, 2, \dots, 6\}. \tag{1.6}$$

Moreover, $\Psi_r(x) = 1 - \Gamma_r(\mathcal{V}_{i;\beta}(x))$.

Proof. The proof follows immediately from the results of Pantcheva (1985) and Theorem 2.2.1 of Leadbetter *et al.* (1983). This paper deals with the weak convergence of the extremes when the sample size itself is a random variable ν_n . Many authors considered this problem for extremes under linear normalization in two different cases. The first case is: when the basic variables X_1, X_2, \dots, X_n and the random sample size ν_n are independent and the d.f. of $\frac{\nu_n}{n}$ weakly converges to a nondegenerate d.f. (see, for example, Barndorff-Nielsen 1964, Dziubdziela 1972, Gnedenko & Gnedenko 1982, Barakat 1990, 1998a, Barakat & Nigm 1991, 1999). The second case is: when the interrelation of the basic variables X_1, X_2, \dots, X_n and the random sample size ν_n is not restricted and $\frac{\nu_n}{n}$ converges in probability (\xrightarrow{P}) to a positive r.v. τ (see, for example, Galambos 1978, 1987, Barakat & El-Shandidy

1990, Barakat 1998a, Barakat & Nigm 1991, 1999). Under the conditions of the second case, Theorem 2.1 of this paper gives the sufficient conditions for the convergence of the extremes under nonlinear normalization as well as the limit forms of the d.f.'s, where the normalizing constants do not contain the random size ν_n (stability of the normalizing constants).

MAIN RESULTS

Theorem 2.1. Let $\frac{\nu_n}{n} \xrightarrow{p} \tau$, where τ is a positive r.v. Assume that there are sequences $a_n > 0$ and $b_n > 0$ such that $\left| \frac{X_{n-r+1:n}}{a_n} \right|^{\frac{1}{b_n}} \mathcal{S}(X_{n-r+1:n})$ converges weakly to a nondegenerate d.f. $\Psi_r(x) = 1 - \Gamma_r(\mathcal{V}_{i;\beta}(x))$, $i \in \{1, 2, \dots, 6\}$. Then, as $n \rightarrow \infty$,

$$P\left(\left| \frac{X_{\nu_n-r+1:\nu_n}}{a_n} \right|^{1/b_n} \mathcal{S}(X_{\nu_n-r+1:\nu_n}) \leq x\right) \xrightarrow{w} \mathcal{L}_r^{(i)}(x) = 1 - \int_{-\infty}^{\infty} \Gamma_r(z \mathcal{V}_{i;\beta}(x)) dP(\tau \leq z). \quad (2.1)$$

Remark 2.1. It is natural to ask what limitations on ν_n the limit distribution of $X_{n-r+1:n}$ (suitably power normalized) preserves with the substitution n by ν_n . From Theorem 1.1 it immediately follows that for the validity of this substitution, it is sufficient to have the equation $P(\tau = 1) = 1$, which means the almost randomness of ν_n . This fact is of considerable interest for physical applications, since the r.v. ν_n often has a Poisson d.f. with mean n and this situation leads to the equation $P(\tau = 1) = 1$ (see Barakat & El-Shandidy 1990).

The proof of Theorem 2.1 will follow from a sequence of Lemmas, the ideas of which are given in Galambos (1978, 1987) and Barakat & El-Shandidy (1990).

Lemma 2.1. Let F_n be a sequence of d.f.'s and T_1 a nondegenerate d.f. Let $a_n > 0$ and $b_n > 0$ be constants such that

$$F_n(a_n |x|^{b_n} \mathcal{S}(x)) \xrightarrow{w} T_1(x). \quad (2.2)$$

Then, for some nondegenerate d.f. T_2 and constants $\alpha_n > 0$ and $\beta_n > 0$,

$$F_n(\alpha_n |x|^{\beta_n} \mathcal{S}(x)) \xrightarrow{w} T_2(x) \quad (2.3)$$

if and only if

$$\left(\frac{\alpha_n}{a_n}\right)^{\frac{1}{b_n}} \rightarrow A > 0 \text{ and } \frac{\beta_n}{b_n} \rightarrow B > 0 \quad (2.4)$$

for some $A > 0$ and $B > 0$, and then

$$T_2(x) = T_1\left(A|x|^B \mathcal{S}(x)\right). \tag{2.5}$$

Proof. By writing $A_n = \left(\frac{\alpha_n}{a_n}\right)^{\frac{1}{b_n}}$; $B_n = \frac{\beta_n}{b_n}$ and $F_n^*(x) = F_n\left(a_n|x|^{b_n} \mathcal{S}(x)\right)$, we may rewrite (2.2), (2.3) and (2.4) respectively as

$$F_n^*(x) \xrightarrow{w} T_1(x), \tag{2.2}'$$

$$F_n^*(A_n|x|^{B_n} \mathcal{S}(x)) = F_n(\alpha_n|x|^{\beta_n} \mathcal{S}(x)) \xrightarrow{w} T_2(x), \tag{2.3}'$$

$$A_n \rightarrow A > 0 \text{ and } B_n \rightarrow B > 0, \text{ for some } A, B > 0. \tag{2.4}'$$

If (2.2)' and (2.4)' hold, then obviously so does (2.3)', with $T_2(x) = T_1\left(A|x|^B \mathcal{S}(x)\right)$ (since, $0 < A, B < \infty$). Thus (2.2) and (2.4) imply (2.3) and (2.5). The proof of the lemma will be complete if we show that (2.2)' and (2.3)' imply (2.4)', for then (2.5) will also hold, as above. Since T_2 is assumed nondegenerate, there are two distinct points x' and x'' (which may be taken to be continuity points of T_2) such that $0 < T_2(x'), T_2(x'') < 1$. The sequence $\left\{A_n|x'|^{B_n} \mathcal{S}(x')\right\}$ must be bounded. For if not, a sequence $\{n_k\}$ could be chosen so that $A_{n_k}|x'|^{B_{n_k}} \mathcal{S}(x') \rightarrow \pm\infty$, which by (2.2)' (since T_1 is a d.f.) would clearly imply that the limit of $F_{n_k}^*\left(A_{n_k}|x'|^{B_{n_k}} \mathcal{S}(x')\right)$ is zero or one-contradicting (2.3)' for $x = x'$. Hence $\left\{A_n|x'|^{B_n} \mathcal{S}(x')\right\}$ is bounded, and similarly so is $\left\{A_n|x''|^{B_n} \mathcal{S}(x'')\right\}$, which together show that the sequence $\{A_n\}$ and $\{B_n\}$ are each bounded. Thus there are constants A and B and a sequence $\{n_k\}$ of integers such that $A_{n_k} \rightarrow A$ and $B_{n_k} \rightarrow B$, and it follows as above that

$$F_{n_k}^*\left(A_{n_k}|x|^{B_{n_k}} \mathcal{S}(x)\right) \xrightarrow{w} T_1\left(A|x|^B \mathcal{S}(x)\right),$$

Whence since by (2.3)', $T_1\left(A|x|^B \mathcal{S}(x)\right) = T_2(x)$ a d.f., we must have $A > 0$ ($A \neq 0$) and $B > 0$ ($B \neq 0$). On the other hand, if another sequence $\{m_k\}$ of integers gave $A_{m_k} \rightarrow A' > 0$ and $B_{m_k} \rightarrow B' > 0$, we would have

$T_1(A'|x|^{B'} \mathcal{S}(x)) = T_2(x) = T_1(A|x|^B \mathcal{S}(x))$. To prove that $A = A'$ and $B = B'$, we can choose $y_1 < y_2$ and $-\infty < x_1 < x_2 < \infty$ by (iii) of Lemma 1.2.1 of Leadbeter *et al.* (1983) so that $x_1 = T_1^{-1}(y_1)$ and $x_2 = T_1^{-1}(y_2)$ (since T_1 is nondegenerate d.f.), where $\mathcal{G}^{-1}(y) = \inf \{x : \mathcal{G}(x) \geq y\}$, for any nondegenerate d.f. $\mathcal{G}(x)$. On the other hand if $\mathcal{H}(x) = \mathcal{G}(A|x|^B \mathcal{S}(x))$, then it is not difficult to show that $\mathcal{H}^{-1}(y) = A^{-\frac{1}{B}} |\mathcal{G}^{-1}(y)|^{\frac{1}{B}} \mathcal{S}(\mathcal{G}^{-1}(y))$. Therefore, by taking inverses of $T_1(A|x|^B \mathcal{S}(x))$ and $T_1(A'|x|^{B'} \mathcal{S}(x))$ we have $A^{-\frac{1}{B}} |T_1^{-1}(y)|^{\frac{1}{B}} \mathcal{S}(T_1^{-1}(y)) = A'^{-\frac{1}{B'}} |T_1^{-1}(y)|^{\frac{1}{B'}} \mathcal{S}(T_1^{-1}(y))$, for all y . Applying this to y_1 and y_2 in turn, we obtain $A^{-\frac{1}{B}} |x_1|^{\frac{1}{B}} = A'^{-\frac{1}{B'}} |x_2|^{\frac{1}{B'}}$, from which it follows that $A = A'$ and $B = B'$. Thus $A_n \rightarrow A$ and $B_n \rightarrow B$, as required to complete the proof.

Lemma 2.2. Let a_n and b_n be sequences of positive numbers, for which

$$P\left(\left|\frac{X_{n-3+1:n}}{a_n}\right|^{\frac{1}{b_n}} \mathcal{S}(X_{n-r+1:n}) \leq x\right) \xrightarrow{w} \Psi_r(x), \quad (2.6)$$

where $\Psi_r(x)$ is any nondegenerate d.f. Then, for arbitrary $m \geq 1$, the limits

$$\lim_{n \rightarrow \infty} \left(\frac{a_n}{a_{[nm]}}\right)^{\frac{1}{b_{[nm]}}} = A_m > 0$$

and

$$\lim_{n \rightarrow \infty} \frac{b_n}{b_{[nm]}} = B_m > 0$$

are finite, where $[\theta]$ denotes the largest integer not exceeding θ . Furthermore,

$$\Gamma_r(\mathcal{V}_{i;\beta}(x)) = \Gamma_r\left(m\mathcal{V}_{i;\beta}\left(A_m|x|^{B_m} \mathcal{S}(x)\right)\right), \quad i \in \{1, 2, \dots, 6\}.$$

Proof. By Theorem 1.1 and Lemma 2.1, the proof of this lemma follows closely that of Theorem 2.2.1 of Galambos (1978, 1987) (see also Lemma 2.3 of Barakat & El-Shandidy 1990).

Lemma 2.3. Let $V_{r,n}^m = \left\lfloor \frac{X_{m-r+1:n}}{a_n} \right\rfloor \frac{1}{b_n} \mathcal{S}(X_{m-r+1:n})$. Assume that (2.6) holds with $\Psi_r(x) = 1 - \Gamma_r(\mathcal{V}_{i;\beta}(x))$, $i \in \{1, 2, \dots, 6\}$. Then, for every event \mathcal{E} , as $n \rightarrow \infty$,

$$P\left(V_{r,n}^m \leq x \cap \mathcal{E}\right) \xrightarrow{w} \Psi_r(x)P(\mathcal{E}). \quad (2.7)$$

Proof. In view of Lemma 6.2.1 of Galambos (1978, 1987) and Remark 1.1 of Barakat & Nigm (1996), Lemma 2.3 follows from the relation

$$P\left(V_{r,n}^{(n)} \succ x/V_{r,k}^{(k)} \succ x\right) \xrightarrow{w} 1 - \Psi_r(x) = \Gamma_r(\mathcal{V}_{i;\beta}(x)) \quad (2.8)$$

for all $k = r, r + 1, \dots$. Now we prove (2.8). Note that

$$\begin{aligned} P\left(V_{r,n}^{(n)} \succ x/V_{r,k}^{(k)} \succ x\right) &= P\left(V_{r,n}^{(n)} \succ x, V_{r,n}^{(k)} \leq x/V_{r,k}^{(k)} \succ x\right) \\ &+ P\left(V_{r,n}^{(n)} \succ x, V_{r,n}^{(k)} \succ x/V_{r,k}^{(k)} \succ x\right). \end{aligned} \quad (2.9)$$

Bearing in mind that all X_i , $i = 1, 2, \dots$ are independent and identical, the first term in (2.9) can be written in the form

$$\begin{aligned} P\left(V_{r,n}^{(k)} \succ x, V_{r,n}^{(k)} \leq x/V_{r,k}^{(k)} \succ x\right) &= P\left(V_{r,n}^{*(n-k)} \succ x, V_{r,n}^{(k)} \leq x/V_{r,k}^{(k)} \succ x\right) \\ &= P\left(V_{r,n}^{*(n-k)} \succ x\right) - P\left(V_{r,n}^{*(n-k)} \succ x, V_{r,n}^{(k)} \succ x/V_{r,k}^{(k)} \succ x\right), \end{aligned}$$

where

$$V_{r,n}^{*(n-k)} = \left\lfloor \frac{\text{(rth largest of } (X_{r+1}, X_{r+2}, \dots, X_n))}{a_n} \right\rfloor \frac{1}{b_n} \mathcal{S}(\text{rth largest of } (X_{r+1}, X_{r+1}, \dots, X_n)).$$

Therefore, in view of (2.9), we have

$$P\left(V_{r,n}^{(n)} \succ x/V_{r,k}^{(k)} \succ x\right) = P\left(V_{r,n}^{*(n-k)} \succ x\right) - \theta_n, \quad (2.10)$$

where

$$\theta_n = P\left(V_{r,n}^{(n)} \succ x, V_{r,n}^{(k)} \succ x/V_{r,k}^{(k)} \succ x\right) - P\left(V_{r,n}^{*(n-k)} \succ x, V_{r,n}^{(k)} \succ x/V_{r,k}^{(k)} \succ x\right).$$

By using the well-known inequality

$$V_{r,n}^{*(n-k)} \leq V_{r,n}^{(n)}$$

and

$$P(E_2 \cap E_3) - P(E_1 \cap E_3) \leq P(E_2) - P(E_1),$$

for any three events E_1 , E_2 and E_3 , for which $E_1 \subseteq E_2$, we get

$$0 \leq \theta_n P\left(V_{r,k}^{(k)} \succ x\right) \leq P\left(V_{r,n}^{(n)} \succ x\right) - P\left(V_{r,k}^{*(n-k)} \succ x\right). \quad (2.11)$$

On the other hand, since $\forall s$'s for which $\mathcal{V}_{i,\beta}(x) < \infty$, $n\left(1 - F\left(a_n|x|^{b_n} \mathcal{S}(x)\right)\right) \rightarrow \mathcal{V}_{i,\beta}(x)$, $i \in \{1, 2, \dots, 6\}$ implies $(n-k)\left(1 - F\left(a_n|x|^{b_n} \mathcal{S}(x)\right)\right) \rightarrow \mathcal{V}_{i,\beta}(x)$, we have, by virtue of Theorem 1.1,

$$\lim_{n \rightarrow \infty} P\left(V_{r,k}^{*(n-k)} \succ x\right) = P\left(V_{r,k}^{(n-k)} \succ x\right) = 1 - \Psi_r(x) = \Gamma_r(\mathcal{V}_{i,\beta}(x)). \quad (2.12)$$

By combining (2.10), (2.11) and (2.12) the proof of (2.8) follows immediately.

Lemma 2.4. Assume that (2.6) holds with $\Psi_r(x) = 1 - \Gamma_r(\mathcal{V}_{i,\beta}(x))$, $i \in \{1, 2, \dots, 6\}$. Let $\frac{\nu_n}{n} \xrightarrow{P} \tau$, where τ is a positive r.v. Then for every event \mathcal{E} , as $n \rightarrow \infty$

$$P\left(V_{r,\nu_n}^{(\nu_n)} \leq x \cap \mathcal{E}\right) \xrightarrow{W} \Psi_r(x)P(\mathcal{E}).$$

Proof. Let $\varepsilon > 0$ be arbitrary. Choose $z_1 < z_2$ such that $P(z_1 < \tau \leq z_2) \geq 1 - \varepsilon$. From the assumptions it follows that there is an integer N such that for $n \geq N$, $P(z_1 < \frac{\nu_n}{n} \leq z_2) \geq 1 - 2\varepsilon$. Fix z_1 , z_2 and N and divide the interval $[z_1, z_2]$ by the points $z_1 = s_0 < s_1 < \dots < s_m = z_2$. Let $n(j) = [ns_j]$. We now have

$$\begin{aligned} & -2\varepsilon + \sum_{j=1}^m P\left(\left\{V_{r,\nu_n}^{(n(j))} \leq x\right\} \cap \mathcal{E} \cap \left\{s_{j-1} < \frac{\nu_n}{n} \leq s_j\right\}\right) \\ & \leq P\left(\left\{V_{r,\nu_n}^{(\nu_n)} \leq x\right\} \cap \mathcal{E}\right) \\ & \leq \sum_{j=1}^m P\left(\left\{V_{r,\nu_n}^{(n(j-1))} \leq x\right\} \cap \mathcal{E} \cap \left\{s_{j-1} < \frac{\nu_n}{n} \leq s_j\right\}\right) + 2\varepsilon. \end{aligned}$$

We make two modifications in the above inequalities. First, we replace $\frac{\nu_n}{n}$ by its limit τ . Because m does not depend on n the effect of replacing $\frac{\nu_n}{n}$ by its limit τ in the above inequalities is arbitrarily small if N is suitably chosen. Thus, the above inequalities hold if 2ε is replaced by 3ε , say and $\frac{\nu_n}{n}$ by τ . Let us write

$$\begin{aligned} \left\{ V_{r,\nu_n}^{(n(j-i))} \leq x \right\} &= \left\{ \left| \frac{V_{r,n(j-i)}^{(n(j-i))} |b_{in}^*|^{\frac{1}{b_{in}^*}}}{a_{in}^*} \mathcal{S}(V_{r,n(j-i)}^{(n(j-i))}) \leq x \right. \right\} \\ &= \left\{ V_{r,n(j-i)}^{(n(j-i))} \leq a_{in}^* |x|^{b_{in}^*} \mathcal{S}(x) \right\}, \end{aligned}$$

where $a_{in}^* = \left(\frac{a_{\nu_n}}{a_{n(j-i)}}\right)^{\frac{1}{b_{n(j-i)}}}$ and $b_{in}^* = \frac{b_{n(j-i)}}{b_{\nu_n}}$ and $i = 0, 1$. If we choose the points s_j , $0 \leq j \leq m$, sufficiently close, then by Lemma 2.2, for large n , $|a_{in}^* - 1| < \delta_i$ and $|b_{in}^* - 1| < \delta_i$, $i = 0, 1$, where $\delta_0 > 0$ and $\delta_1 > 0$ are again arbitrary. Thus we can conclude that if the s_j 's are sufficiently close and if n is large

$$\begin{aligned} -3\varepsilon &+ \sum_{j=1}^m P\left(\left\{ V_{r,n(j)}^{(n(j))} \leq (1 - \delta_0) |x|^{1-\delta_0} \mathcal{S}(x) \right\} \cap \mathcal{E} \cap \left\{ s_{j-1} < \frac{\nu_n}{n} \leq s_j \right\}\right) \\ &\leq P\left(\left\{ V_{r,\nu_n}^{(\nu_n)} \leq x \right\} \cap \mathcal{E}\right) \\ &\leq \sum_{j=1}^m P\left(\left\{ V_{r,n(j-1)}^{(n(j-1))} \leq (1 + \delta_1) |x|^{1+\delta_1} \mathcal{S}(x) \right\} \cap \mathcal{E} \cap \left\{ s_{j-1} < \frac{\nu_n}{n} \leq s_j \right\}\right) + 3\varepsilon. \end{aligned}$$

An application of Lemma 2.2, thus yields

$$\begin{aligned} -3\varepsilon &+ \Psi_r\left((1 - \delta_0) |x|^{1-\delta_0} \mathcal{S}(x)\right) P(\mathcal{E} \cap \{z_1 < \tau \leq z_2\}) \\ &\leq \liminf_{n \rightarrow \infty} P\left(\left\{ V_{r,\nu_n}^{(\nu_n)} \leq x \right\} \cap \mathcal{E}\right) \\ &\leq \limsup_{n \rightarrow \infty} P\left(\left\{ V_{r,\nu_n}^{(\nu_n)} \leq x \right\} \cap \mathcal{E}\right) \\ &\leq \Psi_r\left((1 + \delta_1) |x|^{1+\delta_1} \mathcal{S}(x)\right) P(\mathcal{E} \cap \{z_1 < \tau \leq z_2\}) + 3\varepsilon \end{aligned}$$

In view of the choice of z_1 and z_2 ,

$$|P(\mathcal{E} \cap \{z_1 < \tau \leq z_2\}) - P(\mathcal{E})| \leq P(\tau \leq y_1 \text{ or } \tau > z_2) < \varepsilon.$$

Thus, because $\varepsilon > 0$ and $\delta_i > 0$, $i = 0, 1$ were arbitrary and $\Psi_r(x)$ is continuous, the proof is completed.

Lemma 2.5. Under the assumptions of Lemma 2.4, as $n \rightarrow \infty$,

$$\left(\frac{a_n}{a_{\nu_n}}\right)^{\frac{1}{b_{\nu_n}}} \xrightarrow{P} A_\tau > 0, \quad \frac{b_n}{b_{\nu_n}} \xrightarrow{P} B_\tau > 0 \quad (2.13)$$

where A_τ and B_τ are defined by the relation

$$\Gamma_r(t \mathcal{V}_{i;\beta}(x)) = \Gamma_r\left(\mathcal{V}_{i;\beta}\left(A_t |x|^{B_t} \mathcal{S}(x)\right)\right), \quad i \in \{1, 2, \dots, 6\}.$$

Proof. Because there are only six possibilities in (2.6), a quick check shows that A_t and B_t are continuous and monotonic functions of t (in fact $B_t = t^\theta$, where $\theta = -\frac{1}{\beta}$, for the first and third types; $\theta = \frac{1}{\beta}$, for the second and fourth types; while $\theta = 0$, for the fifth and the sixth types. On the other hand $A_t = 1$, for the first-fourth types, while $A_t = t^\phi$, where $\phi = -1$, for the fifth type and $\phi = 1$, for the sixth type). Therefore, by using Theorem 1.1 and applying Lemma 2.1, the proof of the lemma follows closely that of Lemma 6.2.4 of Galambos (1978, 1987) and Lemma 2.6 of Barakat & El-Shandidy (1990).

Lemma 2.6. Under the assumptions of Lemmas 2.4 and 2.5

$$\lim_{n \rightarrow \infty} P\left(V_{r,\nu_n}^{(\nu_n)} \leq x\right) = \lim_{n \rightarrow \infty} P\left(A_\tau |V_{r,n}^{(\nu_n)}|^{B_\tau} \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right) \leq x\right). \quad (2.14)$$

Proof. Let us write

$$\begin{aligned} V_{r,\nu_n}^{(\nu_n)} &= \left(\frac{a_n}{a_{\nu_n}}\right)^{\frac{1}{b_{\nu_n}}} |V_{r,n}^{(\nu_n)}|^{\frac{b_n}{b_{\nu_n}}} \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right) \\ &= A_{\nu_n} |V_{r,n}^{(\nu_n)}|^{B_{\nu_n}} \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right), \end{aligned}$$

where $A_{\nu_n} = \left(\frac{a_n}{a_{\nu_n}}\right)^{\frac{1}{b_{\nu_n}}}$ and $B_{\nu_n} = \frac{b_n}{b_{\nu_n}}$. Therefore,

$$\begin{aligned} P\left(V_{r,\nu_n}^{(\nu_n)} \leq x\right) &= P\left(A_{\nu_n} |V_{r,n}^{(\nu_n)}|^{B_{\nu_n}} \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right) \leq x\right) \\ &= P\left(V_{r,n}^{(\nu_n)} \leq \left|\frac{x}{A_{\nu_n}}\right|^{\frac{1}{B_{\nu_n}}} \mathcal{S}(x)\right) \\ &= P\left(V_{r,n}^{(\nu_n)} \leq A_{\nu_n}^* \left(\left|\frac{x}{A_\tau}\right|^{\frac{1}{B_\tau}}\right)^{B_{\nu_n}^*} \mathcal{S}(x)\right), \end{aligned}$$

where $A_{\nu_n}^* = \left(\frac{A_\tau}{A_{\nu_n}}\right)^{\frac{1}{B_{\nu_n}^*}}$ and $B_{\nu_n}^* = \frac{B_\tau}{B_{\nu_n}}$ (clearly in view of Lemma 2.5, $A_{\nu_n}^* \xrightarrow{p} 1$ and $B_{\nu_n}^* \xrightarrow{p} 1$). Now, clearly (2.14) is trivially proved if $\mathcal{S}(x) = 0$ or $\mathcal{S}(x) \neq \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right)$ ($\lim P\left(V_{r,n}^{(\nu_n)} \leq x\right) = \lim P\left(A_\tau |V_{r,n}^{(\nu_n)}|^{B_\tau} \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right) \leq x\right) = 0$ or 1, if $\mathcal{S}(x) < \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right)$ or $\mathcal{S}(x) > \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right)$, respectively. On the other hand, if $\mathcal{S}(x) = \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right) \neq 0$, it is easy to show that

$$P\left(V_{r,n}^{(\nu_n)} \leq A_{\nu_n}^* \left(|\frac{x}{A_\tau}|^{B_\tau}\right)^{B_{\nu_n}^*} \mathcal{S}(x)\right) = P\left(|\frac{V_{r,n}^{(\nu_n)}}{A_{\nu_n}^*}|^{\frac{1}{B_{\nu_n}^*}} \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right) \leq |\frac{x}{A_\tau}|^{\frac{1}{B_\tau}}\right).$$

Therefore, (2.14) follows immediately from the following lemma.

Lemma 2.7. Let $\{U_n\}$, $\{\delta_{1n}\}$ and $\{\delta_{2n}\}$ be three sequences of r.v.'s. Assume that there is a nondegenerate d.f. $H(x)$ such that $P(U_n \leq x) \xrightarrow{w} H(x)$, as $n \rightarrow \infty$. Furthermore, assume that $\delta_{in} \xrightarrow{p} 1$, $i = 1, 2$, as $n \rightarrow \infty$. Then

$$P\left(|\frac{U_n}{\delta_{1n}}|^{\frac{1}{\delta_{2n}}} \mathcal{S}(U_n) \leq x\right) \xrightarrow{w} H(x), \text{ as } n \rightarrow \infty.$$

Proof. Clearly, we have only to prove the lemma when $\mathcal{S}(x) = \mathcal{S}(U_n)$ (the lemma is trivially proved when $\mathcal{S}(x) = 0$ or $\mathcal{S}(x) \neq \mathcal{S}(U_n)$). On the other hand, we have

$$P\left(|\frac{U_n}{\delta_{1n}}|^{\frac{1}{\delta_{2n}}} \mathcal{S}(U_n) \leq x\right) = \begin{cases} P(\delta_{1n}^* + \delta_{2n}^* \ln U_n \leq \ln x), & \text{if } \mathcal{S}(U_n) = \mathcal{S}(x) = 1, \\ P(\delta_{1n}^* + \delta_{2n}^* \ln |U_n| \geq \ln |x|), & \text{if } \mathcal{S}(U_n) = \mathcal{S}(x) = -1, \end{cases}$$

where $\delta_{1n}^* = -\frac{1}{\delta_{2n}} \ln \delta_{1n} \xrightarrow{p} 0$ and $\delta_{2n}^* = \frac{1}{\delta_{2n}} \xrightarrow{p} 1$, $n \rightarrow \infty$. On account of Lemmas 2.2.1 of Galambos (1978, 1987) and 3.2 of Barakat (1998b), we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} P\left(|\frac{U_n}{\delta_{1n}}|^{\frac{1}{\delta_{2n}}} \mathcal{S}(U_n) \leq x\right) = \\ & = \begin{cases} \lim_{n \rightarrow \infty} P(\delta_{1n}^* + \delta_{2n}^* \ln U_n \leq \ln x), & \text{if } \mathcal{S}(U_n) = \mathcal{S}(x) = 1, \\ \lim_{n \rightarrow \infty} P(\delta_{1n}^* + \delta_{2n}^* \ln |U_n| \geq \ln |x|), & \text{if } \mathcal{S}(U_n) = \mathcal{S}(x) = -1, \end{cases} \\ & = \begin{cases} \lim_{n \rightarrow \infty} P(\ln U_n \leq \ln x), & \text{if } \mathcal{S}(U_n) = \mathcal{S}(x) = 1, \\ \lim_{n \rightarrow \infty} P(\ln |U_n| \geq \ln |x|), & \text{if } \mathcal{S}(U_n) = \mathcal{S}(x) = -1, \end{cases} \\ & = \lim_{n \rightarrow \infty} P(U_n \leq x) = H(x). \end{aligned}$$

this completes the proof of lemma 2.7 as well as Lemma 2.6. We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $s_0 < s_1 < \dots < s_m$ be given real numbers. Define the events $\mathcal{E}_k = \{s_{k-1} < \tau \leq s_k\}$, $1 \leq k \leq m$, and let $\mathcal{E}_0 = \{\tau \leq s_0\}$ and $\mathcal{E}_{m+1} = \{\tau > s_m\}$. Then, starting with the result of Lemma 2.4. we have, for $1 \leq k \leq m+1$,

$$P\left(\left\{V_{r,n}^{(\nu_n)} \leq x\right\} \cap \mathcal{E}_k\right) \xrightarrow{w} \Psi_r(x)P(\mathcal{E}_k). \quad (2.15)$$

Thus, we can conclude from (2.15) and Lemma 2.6 that, $n \rightarrow \infty$,

$$P\left(\left\{A_\tau | V_{r,n}^{(\nu_n)} |^{B_\tau} \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right) \leq x\right\} \cap \mathcal{E}_k\right) \xrightarrow{w} \Psi_r(x)P(\mathcal{E}_k), \quad (2.16)$$

where $i \in \{1, 2, \dots, 6\}$. For $1 \leq k \leq m$, let $s_{k-1} \leq s(k) \leq s_k$ be fixed points. Then by the basic equations for A_τ and B_τ in Lemma 2.5 and by (2.16), as $n \rightarrow \infty$,

$$P\left(\left\{A_\tau | V_{r,n}^{(\nu_n)} |^{B_\tau} \mathcal{S}\left(V_{r,n}^{(\nu_n)}\right) \leq A_{s(k)} |x|^{B_{s(k)}} \mathcal{S}(x)\right\} \cap \mathcal{E}_k\right) \xrightarrow{w} (1 - \Gamma_r(s(k)\mathcal{V}_i(x)))P(\mathcal{E}_k).$$

Consequently, the continuity of the function $\Psi_r(x) = 1 - \Gamma_r(s(k)\mathcal{V}_i(x))$, $i \in \{1, 2, \dots, 6\}$, $A_\tau > 0$ and $B_\tau > 0$, implies that, if s_{k-1} and s_k are sufficiently close, then, for $1 \leq k \leq m$,

$$\left|P\left(\left\{V_{r,n}^{(\nu_n)} \leq x\right\} \cap \mathcal{E}_k\right) - (1 - \Gamma_r(s(k)\mathcal{V}_i(x)))P(\mathcal{E}_k)\right| < \frac{\varepsilon}{m}, \quad i \in \{1, 2, \dots, 6\},$$

for all large n . If we choose s_0 and s_m so that $P(\mathcal{E}_0) + P(\mathcal{E}_{m+1}) < \varepsilon$, then with the choice of s_k required above,

$$P\left(V_{r,n}^{(\nu_n)} \leq x\right) = \sum_{k=0}^{k=m+1} P\left(\left\{V_{r,n}^{(\nu_n)} < x\right\} \cap \mathcal{E}_k\right)$$

would deviate from

$$\sum_{k=1}^{k=m} (1 - \Gamma_r(s(k)\mathcal{V}_i(x)))P(\mathcal{E}_k)$$

by less than 2ε for all large n . But this latter sum is a Riemann sum of the integral

$$\int_{s_0}^{s_m} (1 - \Gamma_r(z\mathcal{V}_i(x)))dP(\tau \leq z).$$

Therefore, for all large n ,

$$|P(V_{r,n}^{(\nu_n)} \leq x) - \int_{-\infty}^{+\infty} (1 - \Gamma_r(zV_i(x)))dP(\tau \leq z)| < 3\varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, passing to infinity with n , we obtain (2.1) and this completes the proof of theorem.

Example 2.1. Let us consider an important practical situation, when the r.v. τ has an exponential distribution. This case may happen in practical situations when ν_n has a geometric distribution with mean n . It is easy to show that in this case

$$\mathcal{L}_1^{(1)}(x) = \begin{cases} 0, & x \leq 1, \\ \frac{(\ln x)^\beta}{1+(\ln x)^\beta}, & x > 1; \end{cases} \quad \mathcal{L}_1^{(2)}(x) = \begin{cases} 0, & x \leq 0, \\ \frac{1}{1+(-\ln x)^\beta}, & 0 < x \leq 1, \\ 1, & x > 0; \end{cases}$$

$$\mathcal{L}_1^{(3)}(x) = \begin{cases} 0, & x \leq -1, \\ \frac{(-\ln(-x))^\beta}{1+(-\ln(-x))^\beta}, & -1 < x \leq 0, \\ 1, & x > 0; \end{cases} \quad \mathcal{L}_1^{(4)}(x) = \begin{cases} \frac{1}{1+(\ln(-x))^\beta}, & x \leq -1, \\ 1, & x > -1; \end{cases}$$

$$\mathcal{L}_1^{(5)}(x) = \begin{cases} 0, & x \leq 0, \\ \frac{x}{1+x}, & x > 0; \end{cases} \quad \mathcal{L}_1^{(6)}(x) = \begin{cases} \frac{1}{1-x}, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

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الإحصاءات المرتبة المتطرفة تحت ثوابت اتزان القوى وحجم عينة عشوائي

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خلاصة

يتناول البحث دراسة التقارب الضعيف للإحصاءات المرتبة المتطرفة عندما يكون حجم العينة متغير عشوائي متقطع موجب ومستقل عن مفردات العينة المسحوبة. وقد تم إيجاد شروط التقارب وشكل النهايات تحت شروط عامة لم تدرس من قبل؛ مثل استخدام ثوابت اتزان غير خطية وعدم تحديد العلاقة بين المتغيرات العشوائية الأصلية (مفردات العينة) والحجم العشوائي للعينة.