Kuwait J. Sci. Eng. **27**(2) 2000

Characterization of best-weighted L_1-approximation

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ABSTRACT

A characterization theorem for best-weighted L_1-approximation is obtained. Under certain conditions the theorem leads to a method of obtaining a best approximation by means of interpolating at canonical points.

INTRODUCTION

Given $f(x) \in C[a, b]$ and a set of m real-valued continuous functions $\phi_j(x)$ defined on $[a, b]$, let

$$F(A, x) = \sum_{j=1}^{m} a_j \phi_j(x), \qquad \text{for any } A = (a_1, a_2, \ldots, a_m).$$

The general weighted L_1-approximation problem is to determine A^* such that

$$\int_a^b w(x) \mid F(A^*, x) - f(x) \mid dx \leq \int_a^b w(x) \mid F(A, x) - f(x) \mid dx$$

for all $A \in R^m$, $w(x)$ being a non-negative weight function defined on $[a, b]$.. In the unweighted case $w(x) \equiv 1$.

This problem has received great attention in the past few decades. See Carroll & Braess (1974), Cheney (1982), Kioustelidis & Spyropoulos (1978), Pinkus (1989), Rice (1964), Watson (1980) and the references therein. Results on fixed points of interpolation are well documented. Young & Kiountouzis (1979) proved a characterization theorem for best-weighted L_1-approximation over the space of polynomials, which was shown to include many previous results as special cases. They based their derivation on linear programming formulation. The main result of this paper is to present a generalization of the theorem of Young & Kiountouzis (1979). Our derivation is based on methods of analysis originating from approximation theory. Moreover, our results concern a finite dimensional linear space that is more general than the space of polynomials.

CHARACTERIZATION THEOREMS

Theorem 1. *Let M be a finite dimensional subspace of $C[a, b]$, $f \in C[a, b]$, and $w(x)$ a positive continuous weight function on $[a, b]$. If $p \in M$ coincides with f at no more than a finite number of points, then p is a best-weighted L_1-approximation to f if and only if the follow orthogonality condition is satisfied:*

$$\int_a^b w(x)h(x)\,\mathrm{sgn}(f(x) - p(x))\,dx = 0, h \in M$$

where

$$\mathrm{sgn}(z) = \begin{cases} 1 & z>0 \\ 0 & z=0 \\ -1 & z<0 \end{cases}.$$

To prove this theorem we need the following lemma:

Lemma 2. *Let f and h be elements of $C[a, b]$, and let $w(x)$ be a positive continuous weight function on $[a, b]$. If f has at most a finite number of roots and if*

$$\int_a^b wh\,\mathrm{sgn}(f)\,dx \neq 0,$$

then

$$\int_a^b w \mid f - \lambda h \mid dx < \int_a^b w \mid f \mid dx$$

for some λ.

Proof. Since f has only a finite number of roots, then $\mathrm{sgn}(f)$ has only a finite number of discontinuities and is therefore integrable. The functions wf and wh are elements of $C[a, b]$. Since $w(x) > 0$, the function wf has at most a finite number of roots and $\mathrm{sgn}(wf) = \mathrm{sgn}(f)$ on $[a, b]$. Now a lemma of Cheney (1982) is applied on the functions wf and wh and the result readily follows.

We now present the proof of the above theorem.

Proof. If the orthogonality condition fails, then for some $h \in M$

$$\int_a^b wh\,\mathrm{sgn}(f - p)dx \neq 0.$$

By the above lemma we may find a λ such that

$$\int_a^b w \mid f - p - \lambda h \mid dx < f \int_a^b w \mid f - p \mid dx$$

which means that p is not a best approximation of f.

On the other hand, if the orthogonality condition is fulfilled, then for any $p_1 \in M$:

$$\begin{aligned}\int_a^b w \mid f - p_1 \mid dx &\geq \int_a^b w(f - p_1) \operatorname{sgn}(f - p_1) \operatorname{sgn}(f - p) dx \\ &= \int_a^b w(f - p + p - p_1) \operatorname{sgn}(f - p) dx \\ &= \int_a^b w(f - p) \operatorname{sgn}(f - p) dx + \int_a^b w(p - p_1) \operatorname{sgn}(f - p) dx \\ &= \int_a^b w(f - p) \operatorname{sgn}(f - p) dx \\ &= \int_a^b w \mid f - p \mid dx\end{aligned}$$

which means that p is a best approximation to f.

Corollary 3. *Let $w(x)$, $f(x)$ and $p(x)$ be as in Theorem 1. The integral*

$$\int_a^b w(x)(q(x) - p(x)) \operatorname{sgn}(f(x) - p(x)) dx$$

cannot be negative for all q in M, nor positive for all q in M, where $q \neq p$ on $[a, b]$.

Proof.

$$\begin{aligned}\int_a^b w \mid f - p \mid dx &= \int_a^b w(f - p) \operatorname{sgn}(f - p) dx \\ &= \int_a^b w(f - q + q - p) \operatorname{sgn}(f - p) dx \\ &= \int_a^b w(f - q) \operatorname{sgn}(f - p) dx + \int_a^b w(q - p) \operatorname{sgn}(f - p) dx\end{aligned}$$

This gives

$$\int_a^b w \mid f - p \mid dx \leq \int_a^b w \mid f - q \mid dx + \int_a^b w(q - p) \operatorname{sgn}(f - p) dx. \tag{1}$$

1. If

$$\int_a^b w(q - p) \operatorname{sgn}(f - p) dx < 0, \forall q \in M,$$

then inequality 1 implies that p is a best approximation to f which contradicts the orthogonality condition.

2. If

$$\int_a^b w(q-p)\,\mathrm{sgn}(f-p)dx > 0, \forall q \in M,$$

then, replacing $2p - q$ for q, we have

$$\int_a^b w(p-q)\,\mathrm{sgn}(f-p)dx > 0 \Rightarrow \int_a^b w(q-p)\,\mathrm{sgn}(f-p)dx < 0,$$

which contradicts the stated assumption.

We present a necessary condition for best-weighted L_1-approximation.

Theorem 4. *Let M be a finite dimensional subspace of $C[a,b]$, $f \in C[a,b]$, and $w(x)$ a positive continuous weight function on $[a,b]$. If $p \in M$ coincides with f in no more than a finite number of points, then p is a best approximation to f only if*

$$\int_a^b w(q-p)\,\mathrm{sgn}(f-q)dx \leq 0, \forall q \in M.$$

Moreover, if

$$\int_a^b w(q-p)\,\mathrm{sgn}(f-q)dx = 0, \quad \text{for some } q \in M$$

then q is another best approximation.

Proof. Let $p \in M$ be a best approximation to f. Now:

$$\begin{aligned}
\int_a^b w \mid f-p \mid dx &\geq \int_a^b w(f-p)\,\mathrm{sgn}(f-q)dx \\
&= \int_a^b w(f-q+q-p)\,\mathrm{sgn}(f-q)dx \\
&= \int_a^b w(f-q)\,\mathrm{sgn}(f-q)dx + \int_a^b w(q-p)\,\mathrm{sgn}(f-q)dx \\
&= \int_a^b w \mid f-q \mid dx + \int_a^b w(q-p)\,\mathrm{sgn}(f-q)dx \\
&\Rightarrow \int_a^b w \mid f-p \mid dx \geq \int_a^b w \mid f-q \mid dx \\
&\quad + \int_a^b w(q-p)\,\mathrm{sgn}(f-q)dx. \qquad (2)
\end{aligned}$$

If for some $q \in M$

$$\int_a^b w(q-p)\,\mathrm{sgn}(f-q)dx > 0,$$

then by the last inequality p cannot be a best approximation to f. Hence,

$$\int_a^b w(q-p)\,\mathrm{sgn}(f-q)dx \leq 0.$$

However, if p is a best approximation to f and the integral

$$\int_a^b w(q-p)\,\mathrm{sgn}(f-q)dx = 0$$

for some $q \in M$, then the inequality (2) shows that q is another best approximation to f.

Lemma 5. *Let $f(x) \in C[a,b]$, $\Xi = \{x_1, x_2, \ldots, x_k\}$ be the roots of f in (a,b), and let $h(x)$ be a bounded function on $[a,b]$ whose set of discontinuity points is a subset of Ξ. If*

$$\int_a^b w(x)h(x)\,\mathrm{sgn}(f(x))dx \neq 0,$$

then there exists λ such that

$$\int_a^b w\mid f-\lambda h\mid dx < \int_a^b w\mid f\mid dx.$$

Proof. Since $w(x) > 0$, the function wf has at most a finite number of roots and $\mathrm{sgn}(wf) = \mathrm{sgn}(f)$ in $[a,b]$. Let

$$A = [a+\varepsilon, x_1-\varepsilon] \cup [x_1+\varepsilon, x_2-\varepsilon] \cup \cdots \cup [x_k+\varepsilon, b-\varepsilon],$$

where $\varepsilon > 0$. Denote by B the complement of A in $[a,b]$. Suppose that

$$\int_a^b wh\,\mathrm{sgn}(f)dx > 0,$$

(in the negative case we would take λ with different sign). Now, since $\int_B \mid wh\mid dx$ is a continuous function of ε and $\lim_{\varepsilon\to 0}\int_B \mid wh\mid dx = 0$, we may select ε small enough such that

$$\int_A wh\,\mathrm{sgn}(f)dx > \int_B \mid wh\mid dx.$$

Since A is closed and contains no roots of f, the number $\delta = \min\{|f(x)| : x \in A\}$ is positive. As $h(x)$ is bounded on A, we may choose λ such that:

$$0 \leq \lambda \mid h(x) \mid < \delta \leq |f(x)|$$

on A, and consequently $\mathrm{sgn}(f - \lambda h) = \mathrm{sgn}(f)$ on A. Now

$$\begin{aligned}
\int_a^b w \mid f - \lambda h \mid dx &= \int_B w \mid f - \lambda h \mid dx + \int_A w \mid f - \lambda h \mid dx \\
&= \int_B w \mid f - \lambda h \mid dx + \int_A w(f - \lambda h)\,\mathrm{sgn}(f - \lambda h)dx \\
&= \int_B w \mid f - \lambda h \mid dx + \int_A w(f - \lambda h)\,\mathrm{sgn}(f)dx \\
&= \int_B w \mid f - \lambda h \mid dx + \int_A w \mid f \mid dx - \lambda \int_A wh\,\mathrm{sgn}(f)dx \\
&= \int_B w \mid f - \lambda h \mid dx - \int_B w \mid f \mid dx \\
&\quad + \int_a^b w \mid f \mid dx - \lambda \int_A wh\,\mathrm{sgn}(f)dx \\
&\leq \lambda \int_B w \mid h \mid dx - \lambda \int_A wh\,\mathrm{sgn}(f)dx \\
&\quad + \int_a^b w \mid f \mid dx < \int_a^b w \mid f(x) \mid dx.
\end{aligned}$$

Theorem 6. *Let* $f(x) \in C[-1, 1]$

$$F(A, x) = \sum_{j=1}^{m} a_j \phi_j(x)$$

be an approximation to $f(x)$*; each* $\phi_j(x) \in C[-1, 1]$, *and* $\Lambda = \{\xi_1, \xi_2, \ldots, \xi_m\}$ *be a set of m points in* $[-1, 1]$ *such that*

1. $F(A, \xi_j) = f(\xi_j), \forall j = 1, 2, \ldots, m,$
2. $F(A, x) \neq f(x), \forall x \in [-1, 1] \backslash \Lambda,$

and let w equal to $w_1 > 0$ *when* $F \geq f$ *and to* $w_2 > 0$ *when* $F < f$ *with* $w_1 + w_2 = 1$, *(*w_1 *and* w_2 *are constants).*

Then $F(A, x)$ *is a best approximation to* $f(x)$ *in the sense that*

$$\int_{-1}^{1} w(x) \mid F(A, x) - f(x) \mid dx,$$

is minimum if and only if

$$\alpha_k = \sum_{j=1}^{m} (-1)^{j+1} \int_{-1}^{\xi_j} \phi_k(x)dx + \theta \int_{-1}^{1} \phi_k(x)dx = 0, k = 1, \ldots, m, \tag{3}$$

where

$$\theta = \begin{cases} w & \text{if } m \text{ is even} \\ w-1 & \text{if } m \text{ is odd}. \end{cases}$$

Proof. Condition 3 is equivalent to $\int_{-1}^{1} w\phi_k(x)\,\mathrm{sgn}(f-F)dx = 0$ (see Young & Kiountouzis 1979). Moreover, if M is the subspace spanned by $\{\phi_1, \phi_2, \ldots, \phi_m\}$, then it is clear that the orthogonality condition

$$\int_{-1}^{1} wh\,\mathrm{sgn}(f-F)dx = 0, \forall h \in M, \tag{4}$$

is equivalent to $\int_{-1}^{1} w\phi_k(x)\,\mathrm{sgn}(f-F)dx = 0$, since every $h \in M$ is a linear combination of the basis $\{\phi_k\}, k = 1, \ldots, m$.

Assume $F(A, x) = \sum_{j=1}^{m} a_j\phi_j(x)$ is a best approximation to $f(x)$. If condition 4 fails for some $h \in M$, then

$$\int_{-1}^{1} wh\,\mathrm{sgn}(f-F)dx \neq 0.$$

By Lemma 5 we may find a λ such that

$$\int_{-1}^{1} w\,|\,f - F - \lambda h\,|\,dx < \int_{-1}^{1} w\,|\,f - F\,|\,dx.$$

This implies that F is not a best approximation to f.

On the other hand, if condition 4 is fulfilled, then for any $F_1 \in M$ we have

$$\begin{aligned}\int_{-1}^{1} w|f - F_1|dx &\geq \int_{-1}^{1} w(f - F_1)\,\mathrm{sgn}(f-F)dx \\ &= \int_{-1}^{1} w(f - F + F - F_1)\,\mathrm{sgn}(f-F)dx \\ &= \int_{-1}^{1} w(f-F)\,\mathrm{sgn}(f-F)dx + \int_{-1}^{1} w(F - F_1)\,\mathrm{sgn}(f-F)dx.\end{aligned}$$

Now using 4 we have $\int_{-1}^{1} w(F-F_1)\,\mathrm{sgn}(f-F)dx = 0$, thus

$$\int_{-1}^{1} w\,|\,f - F_1\,|\,dx \geq \int_{-1}^{1} w(f-F)\,\mathrm{sgn}(f-F)dx \int_{-1}^{1} w\,|\,f - F\,|\,dx$$

which means that F is a best approximation to f. This completes the proof.

In the special case of polynomial approximation, the interpolation points can be found as zeros of certain polynomials (Young & Hamideh 1982).

ACKNOWLEDGEMENTS

The authors would like to thank the referees for their helpful suggestions that resulted in considerable improvements to the paper.

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(Accepted 2 May 2000)

سمات التقريب L_1 الأمثل الموزون

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خلاصة

لقد كانت النقطة الرئيسية في هذا البحث هي الحصول على شرط لازم وكافي للتقريب الأمثل لدالة متصلة بالنظيم L_1 الموزون.

واعتماداً على هذه النظرية فإنه بالإمكان الحصول على التقريب الأمثل ضمن شروط معينة من خلال عملية الاستكمال.