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# On the optimal knots of first degree splines

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## ABSTRACT

Suppose that $f \in C[a, b]$ is a strictly convex function. Let $U = \{u_0, ..., u_n\}$ be a set of knots on $[a, b]$, and $S(U, .)$ the linear spline with knots set $U$. In this paper we consider best $L_1$-approximation with at most $n - 1$ varying partition points $a = u_0 < u_1 < ... < u_n = b$.

A novel algorithm is given to obtain optimal knots of $L_1$-approximation. starting with any of knot-set $U^{(0)}$, we construct a sequence of knot-sets $U^{(i)}$. The corresponding sequence of linear splines $S^{(i)}$ are constructed such that the corresponding sequence of $L_1$-errors will be a decreasing one.

## 1 . INTRODUCTION

Let $f$ be a given continuous function defined on the interval $[a, b]$. Given a set of continuous functions $\phi_j(x)$ defined on $[a, b]$, we can form a linear approximating function $F(A, x) = \sum_{j=1}^{n} a_j \phi_j(x)$ for any real set $A = \{a_1, ..., a_n\}$.

The approximation problem in the $L_1$-norm is to determine $A^*$ such that:

$$\int_a^b |f(x) - F(A^*, x)| dx \leq \int_a^b |F(x) - F(A, x)| dx \quad \text{for all } A.$$

Kripke and Rivlin (1965) showed that if both $\{\phi_j\}$ and $\{\phi_j, f\}$ form Chebyshev sets and if $F(A^*, x)$ is the unique best $L_1$-approximation to $f(x)$, then $f(x) - F(A^*, x)$ changes sing in $[a, b]$ at exactly $n$ distinct points which are independent of $f$ (see Pinkus 1989).

This paper focuses on the class of strictly convex functions $f \in C[a, b]$ and considers the problem of approximating $f$ by first degree spline functions.

## 2 . STATEMENT OF THE PROBLEM

Suppose that $f \in C[a, b]$ is a strictly convex function. For a given $n$, let $U = \{u_0, ..., u_n\}$ be a set of knots on $[a, b]$, i.e.: $a = u_0 < u_1 < ... < u_n = b$. Let $S(U, x)$ denote the linear spline with knot set $U$. that is

$$S(U,x) = a_1 + a_2 x + \sum_{k=1}^{n} b_k (x - u_k)_+$$

$$\text{where } (x-u_k)_+ = \begin{cases} 0, & x < u_k \\ x - u_k, & x \geq u_k \end{cases} .$$

Then the problem is to find $S(U^*, x)$ such that:

$$\int_a^b |f(x) - S(U^*, x)| dx \leq \int_a^b |f(x) - S(U, x)| dx. \qquad (2.1)$$

## 3. THE BASIC ALGORITHM AND THEORY

THEOREM 3.1. *For any function $f \in C[a,b]$ which is strictly convex in $[a,b]$, the optimal straight line approximation to $f$ in $[a,b]$ interpolates $f$ at points one-quarter and three-quarters of the way between $a$ and $b$.*

*Proof*. The optimal $L_1$ approximation straight line can be determined easily by applying a theorem mentioned in Rice (1964, page 106). According to that theorem the interpolating points $\xi_1$ and $\xi_2$ are the roots of the equations.

$$\int_a^{\xi_1} \phi_i(x)dx - \int_{\xi_1}^{\xi_2} \phi_i(x)dx + \int_{\xi_2}^{b} \phi_i(x)dx = 0 \qquad i = 1, 2$$

where $\phi_1(x) = 1,\ \phi_2(x) = x$. This gives

$$\xi_1 = \frac{3a+b}{4}, \qquad \xi_2 = \frac{a+3b}{4} .$$

We call the straight line, which interpolates a strictly convex function $f \in C[a,b]$ at points one-quarter and three-quarters of the way between $a$ and $b$, locally optimal.

This leads to the following algorithm

**Algorithm 3.1:** Given a strictly convex function $f \in C[a,b]$ and $U = \{u_0, ..., u_n\}$, a set of arbitrary points such that

$$a = u_0 < u_1 < ...u_n = b,$$

the algorithm constructs the optimal $L_1$ approximation lines in the subintervals as follows:

for $i = 1$ to $n$ do

- Compute the best interpolating points of hte $i^{th}$ interval $[u_{i-1}, u_i]$ as follows

$$\xi_{i1} = \frac{3}{4}u_{i-1} + \frac{1}{4}u_i \ , \tag{3.1}$$

$$\xi_{i2} = \frac{1}{4}u_{i-1} + \frac{3}{4}u_i \ .$$

- Calculate the approximating straight lines

$$l_i(x) = \frac{f(\xi_{i2}) - f(\xi_{i1})}{\xi_{i2} - \xi_{i1}}(x - \xi_{i1}) + f(\xi_{i1})$$

## 4. THE OPTIMALLY SITED KNOTS

Let $f \in C[a, b]$ be a strictly convex function in $[a, b]$ and $\{u_i\}_{i=0}^{n} n+1$ varying knots. Kioustelidis and Spyropoulos (1978) have given a method for locating the optimally sited knots for first degree splines in order to obtain best approximations for strictly convex functions. They showed that $L_1 - error$ is a continuously differentiable function of the knots and therefore must have vanishing first order partial derivatives at any point where a minimum occurs. This leads to a tri-diagonal system of non-linear equations for the knots.

We present here a constructive method for locating these knots. Our algorithm is simple and does not require the use of first derivatives.

We prove:

THEOREM 4.1. *The resulting sequence of errors $E^{*(k)}$, generated by the following procedure, is convergent.*

*Proof*. Consider the subintervals $[u_{j-1}, u_j]$ and $[u_j, u_{j+1}]$. Applying algorithm 3.1 to these subintervals leads to the unique piecewise line approximations $l_j(x)$ and $l_{j+1}(x)$ respectively, which minimize the $L_1$ norm. We vary the knots so that the approximation becomes a spline by setting

$$\{u_j^* = (x_j : l_j(x) = l_{j+1}(x)\} \qquad j = n-1, n-2, ..., 1 \quad .$$

Since the approximation errors before and after intersecting the lines differ only on the subinterval $[\xi_{j,2}, \xi_{j+1,1}]$, and noting that the convex function $f$ lies above the lines $l_j$ and $l_{j+1}$ in the subintervals $[\xi_{j,2}, u_j]$ and $[u_j, \xi_{j+1,1}]$, the local error terms before and after intersecting the lines may be written as:

$$e_j = \int_{\xi_{j,2}}^{u_j} (f(x) - l_j(x))dx + \int_{u_j}^{\xi_{j+1,1}} (f(x) - l_{j+1}(x))dx \quad ,$$

$$e_j^* = \int_{\xi_{j,2}}^{u_j^*} (f(x) - l_j(x))dx + \int_{u_j^*}^{\xi_{j+1,1}} (f(x) - l_{j+1}(x))dx \quad .$$

Writing $e_j$ as:

$$e_j = \int_{\xi_{j,2}}^{u_j^*} (f(x) - l_j(x))dx + \int_{u_j^*}^{\xi_{j+1,1}} (f(x) - l_{j+1}(x))dx + \int_{u_j^*}^{u_j} (l_{j+1}(x) - l_j(x))dx,$$

we have:

$$e_j^* = e_j - \int_{u_j^*}^{u_j} (l_{j+1}(x) - l_j(x))dx \ .$$

Now the integral $\int_{u_j^*}^{u_j} (l_{j+1}(x) - l_j(x))dx$ (which represents the area of the triangle bounded by the lines $l_j$ and $l_{j+1}$) is a positive quantity. To show that, let $x$ lie between $u_j^*$ and $u_j$. Since $f$ is a strictly convex function, the slope of $l_{j+1}$ is greater than that of $l_j$, i.e.:

$$\frac{l_{j+1}(x) - l_{j+1}(u_j^*)}{x - u_j^*} > \frac{l_j(x) - l_{j+1}(u_j^*)}{x - u_j^*} \ .$$

If $x > u_j^*$ then $l_{j+1}(x) > l_j(x)$ and the integral $\int_{u_j^*}^{u_j} (l_{j+1}(x) - l_j(x))dx$ is positive. Similar arguments hold if $x < u_j^*$.

By this movement we obtain a new set of partition points $U^{*(1)}$ with a corresponding set of optimal lines $L^{*(1)}$ and a corresponding total $L_1$ error $E^{*(1)}$. With these new sets we begin as before and repeat the process. The above discussion shows that the resulting sequence of errors $E^{*(k)}$ is decreasing and as it is bounded, it is convergent.

It is to be noted that the continuity condition is equivalent to:

$$f(\xi_{j,1}) - 3f(\xi_{j,2}) + 3f(\xi_{j+1,1}) - f(\xi_{j+1,2}) = 0 \ . \quad (4.1)$$

In particular if $f(x) = x^m$, where $m$ is an integer, and $[a, b]$ is chosen to correspond to a convex part of $f$, then the optimal knot $u \in [a, b]$ is a root of the degree $m - 1$ polynomial

$$p(z) = \frac{1}{4^m}[-(3a + z)^m + 3(a + 3z)^m - 3(3z + b)^m + (z + 3b)^m]$$

## 5. GEOMETRIC INTERPRETATION OF THE CONTINUITY CONDITION

THEOREM 5.1. *The continuity condition (4.1) holds if and only if the line which contains the two points* $(\xi_{i,1}, f(\xi_{i,1}))$ *and* $(\xi_{i+1,2}, f(\xi_{x+1,2}))$ *and the line which contains the two points* $(\xi_{i,2}, f(\xi_{i,2}))$ *and* $(\xi_{i+1,1}, f(\xi_{i+1,1}))$ *are parallel.*

*Proof*. The condition (4.1) can be written as:

$$\frac{f(\xi_{i,1}) - f(\xi_{i+1,2})}{3} = f(\xi_{i,2}) - f(\xi_{i+1,1}) \ .$$

Using Eq (3.1) and the above results, one gets

$$\frac{f(\xi_{i,1}) - f(\xi_{i+1,2})}{\xi_{i,1} - \xi_{i+1,2}} = \frac{f(\xi_{i,1}) - f(\xi_{i+1,2})}{\frac{3}{4}(u_{i-1} - u_{i+1})} = \frac{f(\xi_{i,2}) - f(\xi_{i+1,1})}{\frac{1}{4}(u_{i-1} - u_{i+1})}$$

$$= \frac{f(\xi_{i,2}) - f(\xi_{i+1,1})}{\xi_{i,2} - \xi_{i+1,1}} \ .$$

This implies

$$\frac{f(\xi_{i,1}) - f(\xi_{i+1,2})}{\xi_{i,1} - \xi_{i+1,2}} = \frac{f(\xi_{i,2}) - f(\xi_{i+1,1})}{\xi_{i,2} - \xi_{i+1,1}} \ .$$

The converse is also true.

## 6. THE OPTIMAL KNOTS FOR QUADRATIC FUNCTIONS

THEOREM 6.1. *The optimal knots by fitting n lines over some interval* $[a, b]$ *to a quadratic function are equally spaced.*

*Proof*. We can take a quadratic function $f(x)$ as $f(x) = ax^2 + bx + c$ where $a > 0$. The continuity condition implies

$$0 = 3\left\{a\left(\frac{3u_i + u_{i-1}}{4}\right)^2 + b\left(\frac{3u_i + u_{i-1}}{4}\right) + c\right\} - \left\{a\left(\frac{3u_{i-1} + u_i}{4}\right)^2 + b\left(\frac{3u_{i-1} + u_i}{4}\right) + c\right\}$$

$$- 3\left\{a\left(\frac{3u_i + u_{i+1}}{4}\right)^2 + b\left(\frac{3u_i + u_{i+1}}{4}\right) + c\right\} + a\left(\frac{3u_{i+1} + u_i}{4}\right)^2 + b\left(\frac{3u_{i+1} + u_i}{4}\right) + c \ .$$

After simplifying we obtain

$$(u_{i-1} - u_{i+1})(2u_i - u_{i-1} - u_{i+1}) = 0 \ .$$

Since $u_{i-1} - u_{i+1} \neq 0$, we have

$$u_i - u_{i-1} = u_{i+1} - u_i \ .$$

This implies:

$$u_i = u_0 + \frac{i}{n}(u_n - u_0) \ .$$

The uniformity of the spacing of the optimal knots for quadratic functions is a unique case within the convex functions, as the following theorem states:

THEOREM 6.2. *In the class of strictly convex fuctions having continuous third derivatives in the interval $[a, b]$, the quadratic function is the only one for which the optimal knots, for any subinterval of $[a, b]$ are equally spaced.*

*Proof*. Without loss of generality we may take the interval to be $[0, b]$. Now if $h$ is defined so that $\xi_{i,2} = \xi_{i,1} + 2h$, then $\xi_{i+1,1} - \xi_{i,2} = 2h$ and we find that the continuity condition (4.1) is just $\Delta^3 f(\xi_{i,1}) = 0$ for all $i$, where $\Delta$ is the difference operator with interval $2h$. The only continuous functions which have $3^{rd}$ differences equal to zero are of the form: $f(x) = ax^2 + \beta x + \gamma + \delta w(x)$ where $\alpha, \beta, \gamma, \delta$ are arbitrary constants and $w(x)$ is an arbitrary periodic function with periodicity $2h$ with $w(x) = 0$ at $h, 3h, 5h, \ldots$. Such a periodic function $w(x)$ is not convex over $[0, b]$, so that the arbitrary constant $\delta$ must equal zero, and $f(x)$ is a simple quadratic.

## 7. CONVERGENCE OF THE SEQUENCE $\{u_k^i\}$

THEOREM 7.1. $\underline{\lim}_{i} \left|u_{k+1}^i - u_k^i\right| > 0$

*Proof*. If this is not true, then there exists a subsequence $i_m$ such that $u_{k+1}^{i_m} - u_k^{i_m} \to 0$ but $u_{k+2}^{i_m} - u_{k+1}^{i_m} \not\to 0$, otherwise the whole interval becomes one point. There exists also $\{j_m\} \subset \{i_m\}$ such that $u_k^{j_m} \to \alpha$ , $u_{k+1}^{j_m} \to \alpha$ and $u_{k+2}^{j_m} \to \beta$ where $\beta \neq \alpha$.

Since $u_k^{j_m} \to \alpha$ and $u_{k+1}^{j_m} \to \alpha$, so also the interpolation points between $u_k^{j_m}$ and $u_{k+1}^{j_m}$ tend to $\alpha$. This means that the line between the knots $u_{k+1}^{j_m}$ and $u_{k+2}^{j_m}$ would interpolate the convex fucntion $f$ at three points ($\alpha$ is one of them). This is impossible, because of the convexity of $f$.

Now our aim is to prove that $\lim_{i\to\infty}\left|u_k^{i+1} - u_k^i\right| = 0$. To come to this end, we prove first the following two lemmas.

LEMMA 7.1. *Let $|A_n|$ denote the area of all triangles in the n-step of our procedure (mentioned in section 4), Then $|A_n| \to 0$.*

*Proof*. Let $\sigma_n$ denote the piecewise lines in the *n-step*, which interpolate the strictly convex function $f(x)$ at points one-quarter and three-quarters of the way between each successive pair of knots.

Let $s_n$ denote the spline raised from $\sigma_n$ by intersecting the piecewise lines of $\sigma_n$. Our procedure leads to

$$\int_a^b |f - \sigma_n| dx \geq \int_a^b |f - s_n| dx \geq \int_a^b |f - \sigma_{n+1}| dx$$

i.e. $\int_a^b |f - \sigma_n| dx \geq \int_a^b |f - \sigma_{n+1}| dx$.

Now

$$|A_n| = \int_a^b |f - \sigma_n| dx - \int_a^b |f - s_n| dx \leq \int_a^b |f - \sigma_n| dx - \int_a^b |f - \sigma_{n+1}| dx =: \delta_n - \delta_{n+1}$$

where $\int_a^b |f - \sigma_n| dx =: \delta_n$.

So we have $|A_n| \leq \delta_n - \delta_{n+1}$.

Now $\sum_{i=1}^k |A_i| \leq (\delta_1 - \delta_2) + (\delta_2 - \delta_3) + \ldots + (\delta_k - \delta_{k+1}) = \delta_1 - \delta_{k+1} \leq \delta_1$.

Since $\sum_{i=1}^k |A_i| \leq \delta_1$ so the series $\sum_{i=1}^\infty |A_i|$ is convergent and therefore $|A_n| \to 0$.

LEMMA 7.2. $|u_k^{i+1} - u_k^i|^2 \leq \frac{|A_n|}{\tan \frac{\theta^{(i)}}{2}}$ *where* $\theta^{(i)}$ *is the angle between the lines* $l_k$ *and* $l_{k+1}$ *at the i-setp.*

*Proof*. The area of the triangle, which has already been mentioned in the procedure is:

$$|A| = \frac{1}{2} \frac{|u_k^{i+1} - u_k^i|}{\cos \varphi^{(i)}} \frac{|u_k^{i+1} - u_k^i|}{\cos (\varphi^{(i)} + \theta^{(i)})} \sin \theta^{(i)}$$

where $\varphi^{(i)}$ is the angle between $l_k$ and the $x$-axis. Now

$$\cos \varphi^{(i)} \cos(\varphi^{(i)} + \theta^{(i)}) = \frac{1}{2}[\cos (2 \varphi^{(i)} + \theta^{(i)}) + \cos \theta^{(i)}] .$$

This expression takes its maximum when $\varphi^{(i)} = -\frac{\theta^{(i)}}{2}$.

So $|A| \geq \frac{1}{2} \frac{|u_k^{i+1} - u_k^i|^2}{\cos \frac{\theta^{(i)}}{2} \cos \frac{\theta^{(i)}}{2}} \; 2 \sin \frac{\theta^{(i)}}{2} \cos \frac{\theta^{(i)}}{2}$

Therefore $|u_k^{i+1} - u_k^i|^2 \leq \frac{|A|}{\tan \frac{\theta^{(i)}}{2}} \leq \frac{|A_n|}{\tan \frac{\theta^{(i)}}{2}}$ i.e. $|u_k^{i+1} - u_k^i|^2 \leq \frac{|A_n|}{\tan \frac{\theta^{(i)}}{2}}$.

THEOREM 7.2. $\lim_{i \to \infty} |u_k^{i+1} - u_k^i| = 0$ .

*Proof*. Assume that $|u_k^{i+1} - u_k^i| \not\to 0$, then there exists a subsequence $\{n_i\}$ such that $u_k^{ni+1} - u_k^{n_i} \to \alpha \neq 0$ and there exists $\{m_i\} \subset \{u_i\}$ such

that $u_k^{m_i} \to \beta$ and $u_k^{m_i+1} \to \gamma$ where $\gamma = \begin{cases} \beta + \alpha \\ \text{or} \\ \beta - \alpha \ . \end{cases}$

Now $\lim_{i\to\infty}\left[f'(u_k^{m_i+1}) - f'(u_k^{m_i})\right] = f'(\gamma) - f'(\beta) \neq 0.$

Let $\theta$ be the angle between $l_k$ and $l_{k+1}$,

$\theta_1$ be the angle between $l_k$ and the x-axis,

$\theta_2$ be the angle between $l_{k+1}$ and the x-axis.

We have the following cases:

a) $\theta_1 > \frac{\pi}{2}, \theta_2$ acute $\Rightarrow \theta = \theta_2 + \pi - \theta_1$ ,

b) $\theta, \theta_1, \theta_2$ are acute $\Rightarrow \theta = \theta_2 - \theta_1$ ,

c) $\theta_1 > \frac{\pi}{2}$ and $\theta_2 > \frac{\pi}{2} \Rightarrow \theta = \theta_2 - \theta_1$ .

In all cases we have

$$|\tan\ \theta^{(i)}| = \left|\frac{\tan\ \theta_2^{(i)} - \tan\ \theta_1^{(i)}}{1 + \tan\ \theta_2^{(i)}\ \tan\ \theta_1^{(i)}}\right| .$$

We assume that $|f'| < M$ in $[a, b]$, therefore $|1 + \tan\ \theta_2^{(i)}\ \tan\ \theta_1^{(i)}| \leq 1 + M^2$.

Now $\lim_{i\to\infty}\ |\tan\ \theta^{(i)}| \geq \frac{|f'(\gamma) - f'(\beta)|}{1+M^2}$. Hence there is $N$ such that $\forall i \geq n$ and we have

$$\tan\ \theta^{(i)} \geq \frac{\delta}{1+M^2}\ , \text{ where } \delta \text{ may be taken as } \frac{1}{2}|f'(\gamma) - f'(\beta)|\ .$$

Therefore $\theta^{(i)} \neq 0$ and $\tan\ \frac{\theta^{(i)}}{2} \neq 0$ .

But since $|u_k^{i+1} - u_k^i|^2 \leq \frac{|A_n|}{\tan\ \frac{\theta^{(i)}}{2}}$ and $|A_n| \to 0$, therefore $|u_k^{i+1} - u_k^i| \to 0$, a contradiction.

We raise now the question of convergence of the sequence $\{u_k^{(i)}\}$. We begin with the following definition.

DEFINITION 7.1. The knots $\{u_k\}_{k=0}^n$, $u_0 = a$, $u_n = b$ are called locally optimal if the polygonal function, formed by locally optimal segments, is continuous.

THEOREM 7.3. *Optimal knots are locally optimal.*

*Proof*. Assume conversely that optimal knots are not locally optimal, i.e. the piecewise linear function $\sigma$ is not continuous, then by applying our procedure we will make it continuous and at the same time decrease the overall $L_1$-norm.

This contradicts the assumption that the knots are optimal.

Now we make the following assumption.

Assumption A: For the given function $f$, the interval $[a, b]$ and the number $n$, there is a unique system of locally optimal knots.

THEOREM 7.4. *Under assumption A, the sequence* $\{u_k^{(i)}\}$ *converges for any* $k$*, where* $k = 0, 1, ..., n$*. The limits are different for different* $k$*, and they are the optimal ones.*

*Proof*. First we prove that the sequences are convergent. Assume that $\{u_k^{(i)}\}$ does not converge for some $k$. We can find two subsequences $\{u_j^{(n_i)}\}$ and $\{u_j^{(m_i)}\}$ such that these subsequences converge, say $\lim_{i\to\infty} u_j^{(n_i)} = \bar{u}_j$ and $\lim_{i\to\infty} u_j^{(m_i)} = \hat{u}_j, j = 0, ..., n$. Since $\{u_k^{(i)}\}$ does not converge, we can assume $\bar{u}_k \neq \hat{u}_k$. According to Theorem 7.1, we have $\bar{u}_i \neq \bar{u}_j$ and $\hat{u}_i \neq \hat{u}_j$ for $i \neq j$.

We consider now the sequence of the splines $S_{n_i}$. Since $\left\{u_j^{(n_i)}\right\}$ converges, therefore $\lim_{i\to\infty} S_{n_i}$ exists, and it is again a spline.

Let us prove now that the limiting knots are locally optimal i.e. they have the $\frac{1}{4}$ and $\frac{3}{4}$ property. It is clear that:

$$\xi_{1,k}^{n_i+1} = \frac{3}{4} u_k^{n_i} + \frac{1}{4} u_{k+1}^{n_i} - \frac{3}{4} (u_k^{n_i+1} - u_k^{n_i}) - \frac{1}{4} (u_{k+1}^{n_i+1} - u_{k+1}^{n_i}) \, ,$$

$$\xi_{2,k}^{n_i+1} = \frac{1}{4} u_k^{n_i} + \frac{3}{4} u_{k+1}^{n_i} - \frac{1}{4} (u_k^{n_i+1} - u_k^{n_i}) - \frac{3}{4} (u_{k+1}^{n_i+1} - u_{k+1}^{n_i}) \, .$$

Now if $i \to \infty$, we have, according to Theorem 7.2

$$\xi_{1,k+1}^{n_i} \to \frac{3}{4} \bar{u}_k + \frac{1}{4} \bar{u}_{k+1} \text{ and } \xi_{2,k+1}^{n_i} \to \frac{1}{4} \bar{u}_k + \frac{3}{4} \bar{u}_{k+1} \, .$$

In the same way, we prove for $S_{m_i}$ and obtain two different systems of locally optimal knots. This gives a contradiction, because of assumption A.

Thus the sequences $\{u_k^i\}$ are convergent. Using the same arguments as above, we obtain that the limits here also are distinct and locally optimal. By assumption A, they are optimal.

It is to be noted that our algorithm and Theorem 7.4 are based on the assumption that there is a unique system of locally optimal knots (assumption A). We do believe that Assumption A holds for any monotonic convex function $f \in C[a, b]$. It is, however, well-known that Assumption A does not hold always. If we are approximating, for example, the curve of $f(x) = x^4$ on the interval $[-1, 1]$ by a first degree spline with two segments, then there exists locally optimal knots sequences at $[-1, -0.4895, 1]$, $[-1, 0, 1]$, $[-1, 0.4895, 1]$. This example was given by the referee. See also Nürnberger and Braess (1982) for examples in more general settings.

### Determination of the knot $u_{i+1}$

THEOREM 7.5. *Knot $u_{i+1}$ is uniquely determined by the knot $u_i$, where $i = 1, \ldots, n-2$.*

*Proof*. It is clear, for the optimal line in the subinterval $[u_i, u_{i+1}]$, that we have

$$|\xi_{i+1,1} - \xi_{i+1,2}| = 2|\xi_{i+1,1} - u_i| \ .$$

This cannot hold for any other line coincides with the optimal line at $u_i$, because of the strict convexity of $f$. Therefore $u_{i+1}$ is uniquely determined by $u_i$.

## 8. OPTIMAL SITES AND THE CURVATURE OF $f(x)$

The sites of optimal knots depend on the curvature of $f(x)$. To locate the optimal knots by our procedure, we better start with equally spaced knots. As we know, the quadratic function is the only one, in the class of strictly convex functions, for which the optimal knots are equally spaced. If the curvature of our function $f$ is less (greater) than that of the quadratic function, then we notice that the optimal knots of $f$ are to the right (left) of the optimal knots of the quadratic function

Ex.: $y = e^x, [0, 1]$ .

The optimal knots are: 0; 0.2829; 0.5414; 0.7794; 1 .

## 9. ALGORITHMS AND NUMERICAL EXAMPLES

ALGORITHM 9.1. *Given a strictly convex function $f[a, b]$ and $U = \{u_0, u_1, \ldots, u_n\}$ a set of arbitrary points such that*

$$a = u_0 < u_1 < \ldots < u_n = b \ ,$$

the algorithm constructs the optimal $L_1$ approximation lines in the subintervals as follows:

for $i = 1$ to $n$ do

- Compute the best interpolating points on the $i^{th}$ interval $[u_{i-1}, u_i]$ as follows

$$\xi_{i1} = \frac{3}{4}u_{i-1} + \frac{1}{4}u_i \ ,$$

$$\xi_{i2} = \frac{1}{4}u_{i-1} + \frac{3}{4}u_i \ .$$

- Calculate the approximating straight lines

$$l_i(x) = \frac{f(\xi_{i2}) - f(\xi_{i1})}{\xi_{i2} - \xi_{i1}}(x - \xi_{i1}) + f(\xi_{i1}) \ .$$

ALGORITHM 9.2. • *Set* $c := 0$

- *Set* $U^{*(0)} = U^{(0)}$
- *Repeat until convergence*
  - *for* $i = 1$ *up to* $n$ .
    - *Use algorithm 9.1 to calculate the set of straight lines* $L^{(c)}$
  - *for* $i = n$ *down to 1.*
    - *Calculate the set of knots* $U^{*(c)}$
- *Set* $c := c + 1$.

## Numerical examples

EXAMPLE 9.1. $f(x) = x^m, m = 2, 3, 4, 5$ were considered on $[0, 1]$. The exact optimal knots are as follows

| $m$ | $p(x)$ | $u$ |
|---|---|---|
| 2 | $x - 0.5$ | 0.5 |
| 3 | $3x^2 - 1$ | 0.577350 |
| 4 | $52x^3 - 18x^2 - 12x - 13$ | 0.635642 |
| 5 | $10x^4 + 6x^3 - 3x - 2$ | 0.680851 |

It is worth noting the movement of the knots to the right as $m$ increases.

EXAMPLE 9.2. $f(x) = x^2, x \in [0, 2]$ with $u^0 := \{0, .7, 1.4, 2\}$.

The exact optimal knots for this problem are $\frac{2}{3}, \frac{4}{3}$. The following table indicates the convergence of the algorithm to the exact solution.

| Iter | $U^*$ | |
|---|---|---|
| 20 | .667087 | 1.333806 |
| 40 | .666670 | 1.333337 |
| 60 | .666666 | 1.333333 |

EXAMPLE 9.3. $f(x) = \exp\ x, x \in [0, 1]$ with $U^0 := \{0, .1, .3, .7, 1\}$

| Iter | $U^*$ | | |
|---|---|---|---|
| 20 | .2674425 | .521360 | .766370 |
| 40 | .281424 | .539513 | .778208 |
| 60 | .282785 | .541278 | .779360 |

## CONCLUSION

The central point in spline approximation with free knots is the better placement of the knots. This is a nonlinear problem which is of interest. This paper presents a novel algorithm for finding a best $L_1$ approximation for approximating strictly convex curves by first degree splines with free knots. The results presented in this paper add up to the results of Kioustelidis and Spyropoulos (1978). The analytical results have been put in an algorithm to obtain the optimal knots of the $L_1$ approximation, and it has been shown how the numerical results confirm the analytical ones.

The algorithm presented in this paper has some advantages as compared to that of Kioustelidis and Spyropoulos. It does not require the use of derivatives, nor does it need to solve a nonlinear system of equations. It is a simple iterative one.

The conversions of our algorithm are well established, while their algorithm is based, partially, on numerical experience.

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# حول العقد الأمثلية لدالة السبلاين من الدرجة الأولى

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## خلاصة

نتناول في هذا البحث تقريب الدوال بواسطة دوال سبلاين من الدرجة الأولى. ولقد تمكنا من إيجاد أفضل تقريب $L_1$ لتلك الدوال وذلك بأسلوب بناء. وكانت طريقتنا في الوصول للحل الأمثل هي طريقة تكرارية. وقد تميزت ببساطتها وقدرتها في إيجاد الحل الأمثل تدريجياً. وكان الخطأ بين الدالة المحدبة ودالة السبلاين يتناقص لدى كل خطوة.