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An integral of a complete elliptic integral

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ABSTRACT

A definite integral whose integrand contains the complementary complete elliptic integral of the first kind,

$$\int_{1/z}^{z} \frac{\mathbf{K}'(u)}{\sqrt{(z-u)(u-1/z)}}\,du,$$

which is similar to formulas tabulated in a previous compilation, is evaluated in closed form.

INTRODUCTION

Several years ago the author published a compilation of a large number of definite integrals, most of them not previously tabulated, containing the complete elliptic integral of the first kind $\mathbf{K}$ in the integrand. Since then, part of the collection has been reproduced in sections of more complete tables (see, e.g., Prudnikov *et al.* 1990) and the author has received queries about similar integrals. Thus it appears that these formulas are of more than purely academic interest. Indeed, the core of the compilation resulted from investigations in the physics of crystal lattices (Morita & Horiguchi 1971), neutron stars (Glasser & Kaplan 1975) and resistor networks (Doyle & Snell 1984), among other areas. The aim of this paper is to present the evaluation of a similar integral, where $\mathbf{K}$ is replaced by the complementary function $\mathbf{K}'(k) = \mathbf{K}(\sqrt{1-k^2})$. The integral in question is

$$I = \int_{1/z}^{z} \frac{\mathbf{K}'(u)}{\sqrt{(z-u)(u-1/z)}}\,du \qquad (z > 1). \tag{1}$$

EVALUATION

We begin by expanding the elliptic integral in (1) in powers of u and integrating term-by-term. Thus,

$$I = \frac{\pi}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k \frac{(1/2)_n^2}{(n!)^2}\binom{n}{k}\int_{1/z}^{z} \frac{u^{2k}}{\sqrt{(z-u)(u-1/z)}}\,du. \tag{2}$$

But, from the representation of the Legendre polynomial

$$P_{2k}(cosh\eta) = \frac{1}{\pi}\int_{e^{-\eta}}^{e^{\eta}} \frac{x^{2k}}{\sqrt{(e^{\eta}-x)(x-e^{-\eta})}}dx \tag{3}$$

we get

$$I = \frac{\pi^2}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{(1/2)_n^2}{(n!)^2}\binom{n}{k}P_{2k}\left(\frac{z^2+1}{2z}\right). \tag{4}$$

Now,

$$P_{2k}(x) = \sum_{i=0}^{\infty}\frac{(2k+1)_l(-2k)_l}{(l!)^2}\left(\frac{1-x}{2}\right)^l \tag{5}$$

so

$$I = \frac{\pi^2}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^k\frac{(1/2)_n^2(2k+1)_l(-2k)_l}{(n!)^2(l!)^2}\binom{n}{k}\left[-\frac{(z-1)^2}{4z}\right]^l. \tag{6}$$

Next, we note the integral representation

$$(-2k)_l(2k+1)_l = (-1)^l\frac{(1/2)_l l!}{4\pi i}\oint(1+t)^{2k+l}t^{-(2l+1)}dt, \tag{7}$$

where the contour is a small circle around $t=0$. After noting that the k-sum is

$$\sum_{k=0}^{\infty}(-1)^k\binom{n}{k}(1+t)^{2k} = (1-(1+t^2))^n, \tag{8}$$

we have

$$I = \frac{\pi}{8i}\oint\frac{dt}{t}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{(1/2)_n^2(1/2)_l}{(n!)^2 l!^4}(1-(1+t)^2)^n\left[\frac{(z-1)^2(1+t)}{4zt^2}\right]^l. \tag{9}$$

The n-sum yields a complete elliptic integral, which after utilizing Landen's transformation $\mathbf{K}(\sqrt{1-(1+t)^2}) = (1+t)^{-1/2}\mathbf{K}(it/2\sqrt{1+t})$, gives

$$I = \frac{1}{4i}\oint\frac{dt}{t\sqrt{1+t}}\mathbf{K}\left(i\frac{t}{2\sqrt{1+t}}\right)\sum_{l=0}^{\infty}\frac{(1/2)_l}{l!}\left[\frac{(z-1)^2(t+1)}{4zt^2}\right]^l. \tag{10}$$

The t-integral can now be evaluated by residues yielding

$$\frac{1}{2\pi i}\oint \frac{dt}{t\sqrt{1+t}}\left(\frac{1+t}{t^2}\right)^l \mathbf{K}\left(\frac{it}{2\sqrt{1+t}}\right)$$
$$=\frac{\pi}{2}\left(-\frac{1}{4}\right)^l \frac{(1/2)_l}{l!}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},-l;1,\frac{1}{2}-l;1\right). \tag{11}$$

Now, by the hypergeometric identity (Slater 1966),

$${}_3F_2\left(\frac{1}{2},\frac{1}{2},-l;1,\frac{1}{2}-l;1\right)=\frac{(1/4)_l(3/4)_l}{(1/2)_l^2}\,{}_4F_3\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}, & -l, & -l; \\ 1, & \frac{1}{4}-l, & \frac{3}{4}-l; & \end{matrix}\ 1\right], \tag{12}$$

we have

$$I=\frac{\pi^2}{2}\sum_{l=0}^{\infty}(-1)^l\frac{(\frac{1}{4})_l(\frac{3}{4})_l}{(l!)^2}\,{}_4F_3\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}, & -l, & -l; \\ 1, & \frac{1}{4}-l, & \frac{3}{4}-l; & \end{matrix}\ 1\right]\left[\frac{(z-1)^2}{4z}\right]^l. \tag{13}$$

Finally, by noting the hypergeometric summation (Prudnikov *et al.* 1990)

$$\sum_{l=0}^{\infty}\frac{(a)_l(b)_l z^l}{l!(c)_l}\,{}_4F_3\left[\begin{matrix}a', & b', & 1-c-l, & -l; \\ c', & 1-a-l, & 1-b-l; & \end{matrix}\ 1\right]={}_2F_1(a,b;c;z)\,{}_2F_1(a',b';c';z), \tag{14}$$

we can express the result as the square of a hypergeometric function,

$$I=\frac{\pi^2}{2}\left[{}_2F_1(1/4,3/4;1;-(z-1)^2/4z)\right]^2 \tag{15}$$

and by (Prudnikov *et al.* 1990), we obtain for $z>1$

$$\int_{1/z}^{z}\frac{\mathbf{K}'(u)}{\sqrt{(z-u)(u-1/z)}}du=\frac{4\sqrt{z}}{z+1}\mathbf{K}^2\left(\frac{\sqrt{z}-1}{\sqrt{2}\sqrt{z+1}}\right). \tag{16}$$

Note that the modulus of the elliptic integral in the integrand of (16) is imaginary over the range $1<u$, but for $x>1$, $\mathbf{K}(ix)=(1+x^2)^{-1/2}\mathbf{K}(x/\sqrt{1+x^2})$, so it is not suprising that the integral is real.

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خلاصة

نعتبر هنا أحد التكاملات المحددة الذي يشتمل مكامله على متمم التكامل الناقصي التام من النوع الأول ،

$$\int_{1/z}^{z} \frac{K'(u)}{\sqrt{(z-u)(u-1/z)}}\,du,$$

في هذا البحث نوجد قيمة هذا التكامل في شكل مغلق مماثلة للصيغ التي تم جدولتها في تصنيف سابق.