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# On the regularization of linear and nonlinear Abel-type integral equations of the first kind

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## ABSTRACT

We consider the linear integral equation

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K(t,s)y(s)ds = f(t), 0 \le t \le T, \alpha \in (0,1),$$

and the nonlinear integral equation

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K(t,s)F(s,u(s))ds = f(t), 0 \le t \le T, \alpha \in (0,1),$$

with a continuous kernel $K(t,s)$ satisfying a Hölder condition in $t$ and with a continuous function $F(s,u)$ which as an operator superposition is continuous and bounded from $L_p(0,T)$ to $L_q(0,T)$, $1 \le p \le \infty$, $1 \le q \le \infty$. We derive stability estimates and discuss their consequences for the problem of regularization. Stability estimates for the solution of the linear equation are in exponentially weighted $L_q$ norms $(1 \le q \le \infty)$. For the nonlinear equation these estimates are established via the inverse modulus of continuity of the function $F(t,u)$ with respect to $u$.

**Keywords:** Abel integral equations; fractional differentiation and integration; ill-posed problems; nonlinear integral equations; regularization; stability estimates.

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## INTRODUCTION

Let $p \in [1,\infty]$, $q \in [1,\infty]$, $\alpha \in (0,1)$, $T \in \mathbf{R}^+ = (0,\infty)$, $V = \{(t,s)|0 \le s \le t \le T\}$, $U = \{(s,u)|0 \le s \le T, u\mathbf{R}\}$. We consider the linear Abel-type integral equation

$$(K_\alpha y)(t) = f(t) \tag{1}$$

with

$$(K_\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K(t,s) y(s) ds$$

and the nonlinear Abel-type integral equation

$$(A_\alpha u)(t) = f(t) \tag{2}$$

with

$$(A_\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K(t,s) F(s, u(s)) ds,$$

where $(t, s) \in V$, $\Gamma$ denotes Euler's gamma function, $K$, $F$ and $f$ are given functions defined in $V$, $U$ and $[0, T]$, respectively, and $y$ and $u$ are unknown functions in $L_q(0, T)$ and $L_p(0, T)$, respectively. Assume further that the following conditions (i), (ii) and (iii) are fulfilled.

(i) *Let* $K: V \to \mathbf{C}$ *be a continuous function, defined on the triangle* $V$ *and satisfying there with an exponent* $\lambda \in (\alpha, 1]$ *and a positive number* $m$, *a Hölder condition:*

$$|K(t_2, s) - K(t_1, s)| \leq m|t_2 - t_1|^\lambda, \quad 0 \leq s \leq \min\{t_1, t_2\} \leq T.$$

*Furthermore, assume* $K(t, t) = 1$ *for* $0 \leq t \leq T$.

(ii) *Let the function* $F: U \to \mathbf{C}$ *be continuous and define by it the operator superposition* $\tilde{F}: L_p(0, T) \to L_q(0, T)$ *via* $(\tilde{F}u)(s) = F(s, u(s))$. *Moreover, let there exist a continuous function* $\omega(\gamma)$, *for* $\gamma \geq 0$, *being the inverse continuity modulus of* $\tilde{F}$, *namely with* $u_1, u_2 \in L_p(0, T)$,

$$\omega(\gamma) = \sup ||u_2 - u_1||_p, \quad when \ ||\tilde{F}(u_2) - \tilde{F}(u_1)||_q \leq \gamma.$$

(iii) *Let* $f \in J^\alpha(L_q)$, $J^\beta$ *denoting the fractional integration operator, defined for all* $\beta > 0$ *by*

$$(J^\beta y)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds.$$

**Remark 1.** *If* $\tilde{F}$ *in condition* (ii) *is injective, then* $\omega(0) = 0$.

Obviously, the integral operator $K_\alpha: L_q(0, T) \to L_q(0, T)$ is linear and compact, hence the problem of solving the generalized first kind Abel integral equation (1), with $f \in L_q(0, T)$ given and $y \in L_q(0, T)$ unknown, is ill-posed in $L_q(0, T)$. Henceforth, we shall simply write $L_p$ for $L_p(0, T)$ and $L_q$ for $L_q(0, T)$.

For the basic theory concerning existence and uniqueness of the solution $y$ of Eq. (1) we refer the reader to §31.3 and §31.4 of Samko, Kilbas and Marichev (1993), where it is shown that there exists exactly one solution $y$ in $L_q$ if $f \in J^\alpha(L_q)$. It is also shown that $K_\alpha(L_q) = J^\alpha(L_q)$. For discussing the problem of regularizing the solutions of Eqs. (1) and (2) however, we do not need a sharp characterization of the range of the operators $K_\alpha$ and $A_\alpha$. The above mentioned equality of the ranges of $K_\alpha$ and $J^\alpha$ suffices well for our purposes. Our model assumption for regularization is that there exists a *true* element $y \in L_q$ for the case of Eq. (1), $u \in L_p$ for the case of Eq. (2) that we want to *recover*, the better to *approximate*, from inaccurately given data $\phi(t), 0 \le t \le T$, deviating from $f = K_\alpha y$ for (1), from $f = A_\alpha u$ for (2), by an error at most $\varepsilon$:

$$||K_\alpha y - \phi||_q \le \varepsilon \text{ for Eq. (1)}, \quad ||A_\alpha u - \phi||_p \le \varepsilon \text{ for Eq. (2)}. \tag{3}$$

From these conditions and in some set $M$ of solutions $y$ and $u$ of problems (1) and (2), approximations $\tilde{y}$ in $L_q$ and $\tilde{u}$ in $L_p$ should be constructed and error estimates of type

$$||y - \tilde{y}||_q \le \delta(M, \varepsilon), \quad ||u - \tilde{u}||_p \le \delta(M, \varepsilon)$$

should be provided, such that $\delta(M, \varepsilon) \to 0$ as $\varepsilon \to 0$. The set of solutions $M$ is usually described by extra information in the form of a bound $E$ on a suitable nonnegative functional of the solutions $y$, $u$ (often of some stronger norm or semi-norm of $y, u$). Then $\delta$ depends on $E$ and $\varepsilon$, and we write $\delta = \delta(E, \varepsilon)$. Thus our approach is based on a regularization of the linear Eq. (1) with the operator $K_\alpha$ and uses well known properties of the inverse modulus of continuity for the regularization of the nonlinear Eq. (2). Of course, under these model assumptions, a *true* right hand side $f$ lying in the range of the operators $K_\alpha$ and $A_\alpha$ does exist, but we need not worry about the fine structure of this range.

Various methods of regularizing the linear Eq. (1) under differentiability assumptions on the kernel $K$, or even under the very special assumption $K(s, t) \equiv 1$ (then one has the classical Abel integral equation), have been described in the literature, see, e.g., Gorenflo and Vessella (1991), Ang, Gorenflo and Hai (1992), Gorenflo and Yamamoto (1995) and Bukhgeim (1999), but the study of this regularization problem under the weaker assumption of only Hölder continuity seems to have been neglected. Equation (2) is the particular case of the general nonlinear Abel-type integral equation (see Gorenflo and Vessella 1991). For the study of correctness and regularization of nonlinear Abel-type integral equations it is usually assumed that the kernel is differentiable with respect to $u$ and other arguments. See, *e.g.*, Branca (1978), Brunner and Van der Houwen (1986), Ang and Gorenflo (1991), Gorenflo and Vessella (1991), Janno and von Wolfersdorf (1994) and Bukhgeim (1999). Obviously, the conditions assumed here for Eq. (2) are weaker. For convenience we introduce a few more operators for whose theory and interconnections one may also consult Samko *et al.* (1993) or Gorenflo and Vessella (1991). We use the operator $J = J^1$ of integration, the operator $D$ of differentiation $(Dy)(t) = (dy(t)/dt)$, and the operator $D^\alpha = DJ^{1-\alpha}$ (for $0 < \alpha < 1$) of fractional differentiation:

$$(D^\alpha y)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} y(s) ds.$$

The operators $D$ and $D^\alpha$ are defined on the ranges of the operators $J$ and $J^\alpha$, respectively, and $D$ and $D^\alpha$ are left-inverse (but not right-inverse) to $J$ and $J^\alpha$, respectively, with $J$ and $J^\alpha$ being considered as operators from $L_q$ into $L_q$. With $I$ denoting the identity operator ($Iy = y$), we have $DJ = I$, $D^\alpha J^\alpha = I$.

In the following sections we will make essential use of exponentially weighted norms in $L_q$ spaces. Define $e_{-r}(t) = e^{-rt}$ for $r > 0$, $0 \le t \le T$, and then the norm $||\cdot||_{q,r}$ by

$$||y||_{q,r} = ||e_{-r}y||_q \text{ for } y \in L_q. \tag{4}$$

This is a norm in $L_q$ which is equivalent to the usual $L_q$ norm $||\cdot||_q$. In fact, for all $y \in L_q$,

$$||y||_{q,r} \le ||y||_q \le e^{rT}||y||_{q,r}. \tag{5}$$

Our paper is organized as follows. We start with stability estimates for the linear Eq. (1) and use them to obtain stability estimates for the nonlinear Eq. (2). Thereby we exploit properties of the inverse modulus of continuity. We discuss the regularization algorithm and an example of regularization based on stability estimates for Eqs. (1) and (2).

## STABILITY ESTIMATES FOR THE LINEAR EQUATION

Assuming $f$ to lie in the range of $K_\alpha$ (or of $J^\alpha$ what amounts to the same) we produce an explicit positive constant $C$ such that the solution $y$ of Eq. (1) obeys the inequality $||y||_q \le C||D^\alpha f||_q$. Actually, we shall first derive analogous but sharper estimates in exponentially weighted norms. Here the constant $C$ does not depend on $f$.

From §31.3 of Samko *et al.* (1993) we take our basic premises.

**Lemma 1.** *Let there be satisfied the conditions* (i) *and* (iii). *Then the integral equation of the first kind* (1) *is equivalent to the integral equation of the second kind*

$$(I + B_\alpha)y = D^\alpha f \tag{6}$$

*with the operator* $B_\alpha: L_q \to L_q$ *given by*

$$(B_\alpha y)(t) = \int_0^t H(t,s)y(s)ds, \quad H(t,s) = \frac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_s^t \frac{K(t,s) - K(\tau,s)}{(t-\tau)^{1+\alpha}(\tau - s)^{1-\alpha}} d\tau,$$

*and* $I + B_\alpha$ *is an isomorphism of* $L_q$ *onto itself.*

**Comment:** *The essence of Lemma 1 is the factorization* $K_\alpha = J^\alpha(I + B_\alpha)$.

**Lemma 2:** *For $(t, s) \in V$ we have the inequality*

$$|H(t,s)| \le \frac{m\alpha\Gamma(\lambda-\alpha)}{\Gamma(1-\alpha)\Gamma(\lambda)}(t-s)^{\lambda-1}. \tag{7}$$

**Proof.** By the Hölder condition imposed on $K$ we have

$$\begin{aligned} |H(t,s)| &\le \frac{\alpha m}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_s^t \frac{(t-\tau)^\lambda}{(t-\tau^{1+\alpha}(\tau-s)^{1-\alpha}} d\tau \\ &= \frac{\alpha m}{\Gamma(\alpha)\Gamma(1-\alpha)}(t-s)^{\lambda-1} \int_0^1 (1-\xi)^{\lambda-\alpha-1}\xi^{\alpha-1} d\xi \\ &= \frac{\alpha m}{\Gamma(\alpha)\Gamma(1-\alpha)}(t-s)^{\lambda-1} B(\alpha, \lambda-\alpha) = \frac{\alpha m\Gamma(\lambda-\alpha)}{\Gamma(1-\alpha)\Gamma(\lambda)}(t-s)^{\lambda-1}. \end{aligned}$$

The proof of the lemma is completed.

We now ask the reader to recall our definition (4) of the norms $||\cdot||_{q,r}$.

**Lemma 3.** *With $e_{-r}(t) = e^{-rt}$ we have, for $r > 0, \beta > 0$ and $y \in L_q$, the inequality*

$$||e_{-r}J^\beta y||_q \le r^{-\beta}||e_{-r}y||_q \tag{8}$$

*that can also be written in the form*

$$||J^\beta y||_{q,r} \le r^{-\beta}||y||_{q,r}. \tag{9}$$

**Proof.** With $g(t) = (1/\Gamma(\beta))e^{-rt}t^{\beta-1}$, $h(t) = e^{-rt}y(t)$, we have $e^{-rt}(J^\beta y)(t) = \int_0^t g(t-s)h(s)ds$, hence, by Young's inequality for convolutions, $||e_{-r}J^\beta y||_q \le ||g||_1 \, ||e_{-r}y||_q$, and (8) follows from $||g||_1 \le \int_0^\infty |g(t)|dt = r^{-\beta}$.

Let us now consider the equation

$$(I + B_\alpha)w = g \tag{10}$$

for $g$ given in $L_q$, $w$ to be determined in $L_q$. By Lemma 1 there exists exactly one solution $w$.

**Lemma 4.** *With $m_1 = m\alpha\Gamma(\lambda-\alpha)/\Gamma(1-\alpha)$ take an arbitrary number $r > m_1^{1/\lambda}$ and let $w$ be the solution of Eq.* (10). *Then we have*

$$||w||_{q,r} \le (1 - m_1 r^{-\lambda})^{-1}||g||_{q,r}. \tag{11}$$

**Proof.** Using Lemma 2 we see that for $v(t) = |w(t)|$ *we have*

$$v(t) \le |g(t)| + \frac{m\alpha\Gamma(\lambda-\alpha)}{\Gamma(1-\alpha)\Gamma(\lambda)} \int_0^s (t-s)^{\lambda-1} v(s)ds = |g(t)| + m_1(J^\lambda v)(t).$$

Now (8), (4) and $m_1 r^{-\lambda} < 1$ imply (11). The proof is completed.

**Remark 2.** The same technique of estimation yields $||B_\alpha y||_{q,r} \le m_1 r^{-\lambda}||y||_{q,r}$ for all $y \in L_q$, and we recognize that the operator norm of $B_\alpha: L_q \to L_q$, where $L_q$ is normed by $||\cdot||_{q,r}$, is $||B_\alpha||_{q,r} \le m_1 r^{-\lambda} < 1$. As a consequence, the series $\sum_{n=0}^\infty (-1)^n B_\alpha^n$, formally representing $(I+B_\alpha)^{-1}$, is in the norm majorized by the convergent series $\sum_{n=0}^{\infty}(m_1 r^{-\lambda})^n$. This shows (what we already know) that $I+B_\alpha$ is an isomorphism of $L_q$ onto itself.

Reformation of Lemma 4 yields:

**Theorem 1.** *Take an arbitrary positive number c and put* $r=(m_1+c)^{1/\lambda}$. *Then for the solution w of Eq.* (10) *and the solution y of Eq.* (1) *we have*

$$\|w\|_{q,r} \le \Big(1+\frac{m_1}{c}\Big)\|g\|_{q,r}, \quad \|y\|_{q,r} \le \Big(1+\frac{m_1}{c}\Big)\|D^\alpha f\|_{q,r}. \tag{12}$$

From (12), by using $e^{-rT} \le e^{-rt} \le 1$, we obtain

$$\|w\|_{q} \le \Big(1+\frac{m_1}{c}\Big)e^{rT}\|g\|_{q}, \quad \|y\|_{q} \le \Big(1+\frac{m_1}{c}\Big)e^{rT}\|D^\alpha f\|_{q}. \tag{13}$$

By specifying $c$ we can get more handsome estimates. The choice $c=m_1$ gives $r=(2m_1)^{1/\lambda}$, hence

$$\|w\|_{q,r} \le 2\|g\|_{q,r}, \quad \|y\|_{q,r} \le 2\|D^\alpha f\|_{q,r}, \tag{14}$$

from which we get

$$\|w\|_{q} \le 2e^{rT}\|g\|_{q}, \quad \|y\|_{q} \le 2e^{rT}\|D^\alpha f\|_{q}. \tag{15}$$

We keep these results as:

**Corollary 1.** *With* $r=(2m_1)^{1/\lambda}$ *the estimates* (14) *and* (15) *hold for the solutions of w of Eq.* (10) *and y of Eq.* (1).

**Remark 3.** These estimates are stability estimates in the following sense. They exhibit the sensitivity of the solution with respect to a perturbation of the data function $f$ within the range of the operator $K_\alpha$. To see this, simply consider linear equations $K_\alpha y_j = f_j$ with $f_j \in K_\alpha(L_q)$, $j \in \{1,2\}$. Because $K_\alpha$ is a linear operator we conclude that

$$\|y_2-y_1\|_{q,r} \le 2\|D^\alpha(f_2-f_1)\|_{q,r}, \quad \|y_2-y_1\|_{q} \le 2e^{rT}\|D^\alpha(f_2-f_1)\|_{q}.$$

**Remark 4.** Use of exponentially weighted norms is not unusual in the treatment of Volterra integral equations. Often it is quite natural: the simple Volterra equation of second kind $u(t)-1=\int_0^t u(s)ds$ has the solution $u(t)=e^t$. See, e.g., Janno and von Wolfersdorf (1994) and Bukhgeim (1999). However, in integral equations of type (1) it is usually assumed that $K$ is Lipschitz-continuous or satisfies a condition stronger than Hölder continuity, such as $|K(t_2,s)-K(t_1,s)| \le m|t_2-t_1| \max\{|\log|t_2-t_1||, 1\}$ (See Lavrentiev *et al.* (1986)).

## STABILITY ESTIMATES FOR THE NONLINEAR EQUATION

In order to obtain stability estimates we first state:

**Lemma 5.** *The inverse continuity modulus* $\omega(\gamma)$, $\gamma \geq 0$, *has the following properties.*

(a) $\omega(\gamma_1) \leq \omega(\gamma_2)$ *for* $\gamma_1 \leq \gamma_2$,
(b) $\omega(\gamma_1 + \gamma_2) \leq \omega(\gamma_1) + \omega(\gamma_2)$,
(c) $\omega(c_1\gamma) \leq (c_1 + 1)\omega(\gamma)$ *for every real number* $c_1 \geq 0$.

**Proof.** Inequality (a) follows directly from the definition of $\omega$ (See condition (ii) of the Introduction), (b) is trivial if $\gamma_1 = 0$ or $\gamma_2 = 0$. Otherwise let $u_2 - u_1 = h$, $h_j = \gamma_j h/(\gamma_1 + \gamma_2)$ and $u_3 = u_1 + h_1$ Then $h = h_2 + h_1$, $u_2 = u_1 + h = u_1 + h_1 + h_2$. Then, according to the definition,

$$\begin{aligned}\omega(\gamma_1 + \gamma_2) &= \sup ||u_2 - u_1||_p \\ &= \sup ||h_2 + h_1||_p \leq \sup ||(u_3 + h_2) - u_3||_p + \sup ||(u_1 + h_1) - u_1||_p\end{aligned}$$

when

$$\begin{aligned}||\tilde{F}(u_2) - \tilde{F}(u_1)||_q &= ||\tilde{F}(u_1 + h_1 + h_2) - \tilde{F}(u_1)||_q \\ &= ||\tilde{F}(u_3 + h_2) + \tilde{F}(u_3) - \tilde{F}(u_3) - \tilde{F}(u_1)||_q \leq \gamma_1 + \gamma_2,\end{aligned}$$

and simultaneously the conditions $||\tilde{F}(u_3 + h_2) - \tilde{F}(u_3)||_q \leq \gamma_2$, $||\tilde{F}(u_1 + h_1) - \tilde{F}(u_1)||_q \leq \gamma_1$ are satisfied. Consequently, (b) is valid. Then, by induction, $\omega(n\gamma) \leq n\omega(\gamma)$ for an arbitrary nonnegative integer $n$. With $[c_1]$ as the integer part of a real number $c_1$, using the inequality (b), we get (c), namely $\omega(c_1, \gamma) \leq ([c_1] + 1)\omega(\gamma) \leq (c_1 + 1)\omega(\gamma)$. The lemma is proved.

Let us now consider the nonlinear Eq. (2). Writing it in the form $A_\alpha u = f, f \in L_q$ given and $u \in L_p$ unknown, and substituting $y(s) = F(s, u(s))$ in (2) we get the linear Eq. (1) for the now unknown function $y$, namely the equation

$$K_\alpha y = f,\ f \in L_q \text{ given},\ y \in L_q \text{ unknown}. \tag{16}$$

This equation has been studied in the preceding section, and for its solution we have proved Theorem 1. From this theorem and Remark 2 it follows that

$$||y_2 - y_1||_q \leq e^{rT}\left(1 + \frac{m_1}{c}\right)||D^\alpha(f_2 - f_1)||_q, \tag{17}$$

where $y_1, y_2$ are solutions of Eq. (16) with right hand sides $f_1, f_2$, respectively, and $r = (m_1 + c)^{1/\lambda}$ with an arbitrary positive number $c$.

**Theorem 2.** *Assume the conditions* (i), (ii) *and* (iii) (of the Introduction) *to be satisfied. Then for the solution of Eq.* (2) *the stability estimate*

$$||u_2 - u_1||_p \leq \left(1 + e^{rT}\left(1 + \frac{m_1}{c}\right)\right)\omega(||D^\alpha(f_2 - f_1)||_q) \tag{18}$$

*holds. Here* $u_1$, $u_2$ *are solutions of Eq.* (2) *with right hand side* $f_1, f_2$, *respectively, and* $c$ *is an arbitrary positive number.*

**Proof.** By the definition of the inverse continuity modulus (see (ii)), we have $||u_2 - u_1||_p \leq \omega(\gamma)$ when $||F(t, u_2) - F(t, u_1)||_q \leq \gamma$. We recall $y(t) = F(t, u(t))$. Then $F(t, u_j) = y_j$ for $j = 1$ and $j = 2$, and $||u_2 - u_1||_p \leq \omega(\gamma)$ when $||y_2 - y_1||_q \leq \gamma$. Taking

$\gamma = ||y_2 - y_1||_q$ we get $||u_2 - u_1||_p \leq \omega(||y_2 - y_1||_q)$. Inserting here the estimate (17) in the right hand side we obtain

$$||u_2 - u_1||_p \leq \omega\Big(e^{rT}\Big(1 + \frac{m_1}{c}\Big)||D^\alpha(f_2 - f_1)||_q\Big). \tag{19}$$

Now, using Lemma 5 and applying the inequality $\omega(c_1\gamma) \leq (c_1 + 1)\omega(\gamma)$ in the right hand side of the estimate (19) we obtain (18), and Theorem 2 is proved.

**Remark 5.** By specifying the positive number $c$ we can obtain estimates with different constants. The choice $c = m_1$ gives $r = (2m_1)^{1/\lambda}$ and $||u_2 - u_1||_p \leq (2e^{rT} + 1)\omega(||D^\alpha(f_2 - f_1)||_q)$.

**Remark 6.** Let $F(t, u)$ be continuously differentiable with respect to $u$, assume the condition (ii) to be fulfilled, and let there exist the inverse of this function with respect to $u$. Then this inverse function is continuous and also differentiable, and for its modulus of continuity we have $\omega(\gamma) \leq c_2\gamma$, the positive constant $c_2$ not depending on $\gamma$. Then the stability estimate (18) takes the form

$$||u_2 - u_1||_p \leq c_2\Big(e^{rT}\Big(1 + \frac{m_1}{c}\Big) + 1\Big)||D^\alpha(f_2 - f_1)||_q.$$

The choice $c = m_1$ gives $r = (2m_1)^{1/\lambda}$ and $||u_2 - u_1||_p \leq (2e^{rT} + 1)c_2||D^\alpha(f_2 - f_1)||_q$.

## REGULARIZATION

In the problem of regularization, as shortly described in the Introduction, it is assumed that we have in hand an element $\phi \in L_q$ and a positive number $\varepsilon$ such that $||f - \phi||_q \leq \varepsilon$ and that for the *true* solution of the considered problem we have information that it belongs to some set $M$. This set is usually described by extra information provided by a bound $E$ on a suitable nonnegative functional defined in the set of solutions we want to admit. We start from the problem of regularizing Eq. (1). In view of Theorem 1 it now suffices to have a method of finding a good approximation $v$ to $D^\alpha f$ and then to take $(I + B_\alpha)^{-1}v$ as an approximation to the *true* solution $y$. We assume that we dispose on a linear bounded operator $D_{\alpha,\rho}: L_q \to L_q$, depending on a positive parameter $\rho$, and that we have an estimate of the form $||D^\alpha f - D_{\alpha,\rho}\phi||_q \leq \sigma(M, \varepsilon, \rho)$. We are lucky if by appropriate choice of $\rho = \rho(M, \varepsilon)$, putting $\sigma(M, \varepsilon, \rho(M, \varepsilon)) = \tau(M, \varepsilon)$ we have $\tau(M, \varepsilon) \to 0$ as $\varepsilon \to 0$. Then we have successfully regularized the *approximate* integral equation $K_\alpha y \approx \phi$. In fact, the operator $R_{M,\varepsilon} = (I + B_\alpha)^{-1}D_{\alpha,\rho(M,\varepsilon)}: L_q \to L_q$ is bounded and injective from $L_q$ into itself. Then the error of the regularized solution $R_{M,\varepsilon}\phi$ of problem (1) with noisy data $\phi$ in place of $f$ *is* $y - R_{M,\varepsilon}\phi = (I + B_\alpha)^{-1}(D^\alpha f - D_{\alpha,\rho(M,\varepsilon)}\phi)$. Taking $g = D^\alpha f - D_{\alpha,\rho}\phi$ and noting that $w = (I + B_\alpha)^{-1}g$ is the solution of Eq. (10), we inherit from Theorem 1 with $r = (m_1 + c)^{1/\lambda}$ the estimate

$$||y - R_{M,\varepsilon}\phi||_{q,r} \leq \Big(1 + \frac{m_1}{c}\Big)||D^\alpha f - D_{\alpha,\rho(M,\varepsilon)}\phi||_{q,r}. \tag{20}$$

This inequality and Eq. (5) imply, again with $r = (m_1 + c)^{1/\lambda}$, the estimate

$$||y - R_{M,\varepsilon}\phi||_q \leq e^{rT}\Big(1 + \frac{m_1}{c}\Big)||D^\alpha f - D_{\alpha,\rho(M,\varepsilon)}\phi||_q \leq e^{rT}\Big(1 + \frac{m_1}{c}\Big)\tau(M, \varepsilon). \qquad (21)$$

The choice $c = m_1$ in (20) gives $r = (2m_1)^{1/\lambda}$, hence

$$||y - R_{M,\varepsilon}\phi||_{q,r} \leq 2||D^\alpha f - D_{\alpha,\rho(M,\varepsilon)}\phi||_{q,r} \leq 2\tau(M, \varepsilon). \qquad (22)$$

From this inequality we easily obtain

$$||y - R_{M,\varepsilon}\phi||_q \leq 2e^{rT}||D^\alpha f - D_{\alpha,\rho(M,\varepsilon)}\phi||_q \leq 2e^{rT}\tau(M, \varepsilon). \qquad (23)$$

The inequalities (20)–(23) are different forms of error estimates for the regularized solution $R_{M,\varepsilon}\phi$ of the linear problem (1) with noisy data.

We now consider the regularization algorithm and error of estimation for the nonlinear Eq. (2). The error estimates are now based on Theorem 2 and properties of the inverse continuity modulus of $\tilde{F}(u)$. Let $F(t, u) = y$ and $R_{M,\varepsilon}\phi = y_{\rho,\varepsilon}$ be the regularized solution of the linear problem (1). Let $u_{\rho,\varepsilon}$ be a solution of $F(t, u_{\rho,\varepsilon}) = y_{\rho,\varepsilon}$. We regard $u_{\rho,\varepsilon}$ as a regularized solution of problem (2). Then $y - y_{\rho,\varepsilon} = F(t, u) - F(t, u_{\rho,\varepsilon})$, and according to the definition of $\omega(\gamma)$ (see in (ii) in the Introduction), we have $||u - u_{\rho,\varepsilon}||_p \leq \omega(||y - y_{\rho,\varepsilon}||_q)$ Then, in terms of Lemma 5, the estimates (21) and (23) imply

$$||u - u_{\rho,\varepsilon}||_p \leq \Big(1 + e^{rT}\Big(1 + \frac{m_1}{c}\Big)\Big)\omega(||D^\alpha f - D_{\alpha,\rho(M,\varepsilon)}\phi||_q) \text{ with } r = (m_1 + c)^{1/\lambda}, \qquad (24)$$

$$||u - u_{\rho,\varepsilon}||_p \leq (1 + 2e^{rT})\omega(||D^\alpha f - D_{\alpha,\rho(M,\varepsilon)}\phi||_q) \text{ with } r = (2m_1)^{1/\lambda}. \qquad (25)$$

Denoting the right hand sides of (24) and (25) by $\delta(M, \varepsilon)$ we have $||u - u_{\rho,\varepsilon}||_p \leq \delta(M, \varepsilon)$. Now $\tau(M, \varepsilon) \geq ||D^\alpha f - D_{\alpha,\rho(M,\varepsilon)}||_q \to 0$ as $\varepsilon \to 0$. Thus we have successfully regularized the linear Eq. (1). If the inverse continuity modulus $\omega$ has the property $\omega(0) = 0$, then also $\delta(M, \varepsilon) \to 0$ as $\varepsilon \to 0$, and we have successfully regularized the nonlinear Eq. (2). The inequalities (24) and (25) are error estimates for the regularized solution of Eq. (2).

**Remark 7.** Of course, the optimal choice of $\rho = \rho(M, \varepsilon)$ is the one that minimizes $\sigma(M, \varepsilon, \rho)$ relative to $\rho$. However, it is not always possible to find the strict minimizer.

## AN EXAMPLE OF REGULARIZATION

In view of the comment to Lemma 1 it suffices to use a method of regularizing the equation

$$J^\alpha w = f \qquad (26)$$

with $\alpha \in (0, 1)$. For the regularization of Eq. (1) we have to construct the operator $D_{\alpha,\rho}$ such that $||f - \phi||_q \leq \varepsilon$ implies $||D^\alpha f - D_{\alpha,\rho}\phi||_q \leq \sigma(M, \varepsilon, \rho)$. Therefore, we have to construct approximations $w_\rho = D_{\alpha,\rho}\phi$ to $w = D^\alpha f$ in solving Eq. (26) with *noisy* data $\phi$ in place of the *true* data $f$. Taking $y_\rho = (I + B_\alpha)^{-1}w_\rho$ we have the regularizing approximation to the *true* solution $y = (I + B_\alpha)^{-1} w$ of Eq. (1). The regularizing solution $u_\rho$ of Eq. (2) we have found from the condition $F(t, u_\rho) = y_\rho$. The extra condition required on the smoothness (or regularity) of $u$ will be expressed indirectly as one for $w = D^\alpha f$. We can exploit via formula (26) the available literature for regularizing Abel's classical equation. See, *e.g.*, Tikhonov and Arsenin (1977), Gorenflo and Vessella (1991), Vu Kim Tuan and Gorenflo (1994), Dinh Nho Hao *et al.* (1994) and Gorenflo and Yamamoto (1995). We present one example on how to do this.

Take the operator complementation method described by Gorenflo and Yamamoto (1995). Then according Theorem 5.1 of these authors the following statement is true.

*Let $w$ be absolutely continuous on* $[0, T], Dw \in L_q(0, t), \alpha \in (0, 1), q \in [1, \infty]$, $E > 0$, *and*

$$||J^\alpha w - \phi||_q \leq \varepsilon, \quad \frac{1}{T}||w||_q + ||Dw||_q \leq E.$$

*Then the regularization method described in* §5 *of Gorenflo and Yamamoto (1995) yields $w_\rho$ such that*

$$||w - w_\rho||_q \leq 2E\rho + \frac{8T^{1-\alpha}\varepsilon}{\Gamma(2-\alpha)\rho}.$$

*The choice* $\rho = 2T^{(1-\alpha)/2}(\Gamma(2-\alpha))^{-1/2}(\varepsilon/E)^{1/2}$ *here minimizes the right hand side and gives*

$$||w - w_\rho||_q \leq 8T^{(1-\alpha)/2}(\Gamma(2-\alpha))^{-1/2}(E\varepsilon)^{1/2} = \tau(E, \varepsilon).$$

*Therefore, taking* $y_\rho = (I + B_\alpha)^{-1} w_\rho$ *as regulation of* $y$ *in Eq.* (1), *we have, according to* (23), *the estimate*

$$||y - y_\rho||_q \leq 2e^{rT}\tau(E, \varepsilon) \tag{27}$$

*if* $(1/T)||D^\alpha f||_q + ||D^{1+\alpha} f||_q \leq E$ *and* $||f - \phi||_q \leq \varepsilon$. *Finally, we find the regularizing solution of Eq.* (2) *from the condition* $F(t, u_\rho) = y_\rho$. *Then the inequalities* (25) *and* (27) *yield the estimate* $||u - u_{\rho,\varepsilon}||_p \leq (1 + 2e^{rT})\omega(\tau(E, \varepsilon))$.

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# حول تنظيم معادلات أبل التكاملية الخطية والغير خطية من النوع الأول

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## الخلاصه

نحن نعتبر المعادلة التكاملية الخطية

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K(t,s)y(s)ds = f(t), 0 \le t \le T, \alpha \in (0,1),$$

والمعادلة التكاملية الغير خطية

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K(t,s)F(s,u(s))ds = f(t), 0 \le t \le T, \alpha \in (0,1),$$

بحيث أن $k\ (t, s)$ نواه متصلة تحقق شرط هولدر في الزمن وتكون الدالة المتصلة $F\ (s, u)$ مؤثر تراكب متصل محدود من $L_p\ (0, T)$ إلى $L_q\ (0, T)$ حيث $0 \le p \le \infty$ ، $0 \le q \le \infty$ إننا نصيغ تقديرات الستقرار ونناقش نتائجهم علـــى مســـائلة التنظيـم. إن تقديــرات الاستقرار لحل المعادلة التكاملية الخطية تكون في وزن إسي من معيارات $L_p$. أما تقديـــرات الستقرار للمعادلة التكاملية الغير خطية فقد أوجدت باستخدام المطلق العكسي للدالـــة المتصلـــة $F\ (s, u)$ طبقا للمتغير $u$.