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On a generalized gamma-type distribution with τ-confluent hypergeometric function

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ABSTRACT

A new gamma-type distribution involving the τ-confluent hypergeometric function is introduced. A generalized form of incomplete gamma function and its complementary form are introduced in order to obtain some statistical functions. Some basic functions, associated with the new density, specifically, moment, mean, expected value, hazard rate function and mean residue life function, are derived. Several special cases such as the generalized Weibull, gamma and confluent densities are mentioned. The behaviour of probability density function and hazard rate functions are depicted graphically for certain values of the parameters.

INTRODUCTION

Recently Virchenko (1999) and Virchenko, Kalla & Al-Zamel (2000) have defined and developed a new class of functions which may be called τ-hypergeometric and τ-confluent hypergeometric functions. These functions are natural generalizations of classical hypergeometric functions (Lebedev 1972). The success of a particular special function depends on its usefulness in applications and effective computability.

Probability density functions have been used in a variety of situations including, but not limited to, reliability theory, the theory of demographic rates, biomedicine, and failures of electrical equipment (Jorgensen 1982).

Agarwal & Kalla (1996) have defined and studied a generalized gamma distribution using a generalized form of gamma function given by Kobayashi (1991). The Kobayashi generalized gamma function is essentially a confluent hypergeometric function of the second kind (Abramowitz & Stegun 1972, Lebedev 1972).

Recently Kalla and others (2000) have given a unified form of gamma-type distributions, using a generalized gamma function defined by Al-Musallam and Kalla (1997, 1998).

In this paper, first we mention some properties of the τ-hypergeometric and confluent hypergeometric functions and then we define a generalized form of incomplete gamma functions. We define a density function associated with the τ-confluent hypergeometric function. The gamma, generalized gamma, Weibull and other gamma-type distributions follow as particular cases of our generalized density function. Some useful common properties associated with this density function are derived, namely, the k-th moment, expected value, the hazard rate function and the mean residue life function. Comparative graphs of the density function are given to show the role of the parameters involved. Mathematical models developed here might have some applications in the theory of probability.

τ-HYPERGEOMETRIC FUNCTIONS

The τ-hypergeometric function was defined (Virchenko *et al.* 2000) as

$$ {}_2R_1^{\tau}(z) = {}_2R_1(a, b; c; \tau; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+\tau k)}{\Gamma(c+\tau k)} \frac{z^k}{k!} \tag{1} $$

where a, b, c are complex parameters, $\tau > 0$, $b + \tau k$, and $a + k \neq 0, -1, -2, \ldots$.

The series is uniformly convergent in the unit circle $|z| = 1$. Moreover as was shown in Verchenko (1999):

$$ \begin{aligned} {}_2R_1^{\tau}(z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt^{\tau})^{-a}\, dt \\ &= \frac{\Gamma(c)}{\tau\Gamma(b)\Gamma(c-b)} \int_0^1 t^{\frac{b}{\tau}-1}(1-t^{\frac{1}{\tau}})^{c-b-1}(1-zt)^{-a}\, dt, \end{aligned} \tag{2} $$

$\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $|\arg(1-z)| < \pi$.

One can, by the use of either (1) or (2), derive the following:

$$ \frac{d^n}{dz^n}\, {}_2R_1^{\tau}(z) = \frac{\Gamma(c)\Gamma(a+n)\Gamma(b+\tau n)}{\Gamma(a)\Gamma(b)\Gamma(c+\tau n)}\, {}_2R_1(a+n, b+\tau n; c+\tau n; \tau; z) \tag{3} $$

$$ (b - a\tau)\, {}_2R_1(a, b; c; \tau; z) = b\, {}_2R_1(a, b+1; c; \tau; z) - a\, {}_2R_1(a+1, b; c; \tau; z). \tag{4} $$

A related function, the τ-confluent hypergeometric function, is defined as follows

$$ \Phi^{\tau}(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+\tau k)}{\Gamma(c+\tau k)} \frac{z^k}{k!} \tag{5} $$

where a and c are complex parameters, $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, $\tau > 0$, and $a + \tau k \neq 0, -1, -2, \ldots$. The series is uniformly convergent for all finite z.

For $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, and $\tau > 0$ we have

$$\Phi^{\tau}(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt^{\tau}}\, dt$$
$$= \frac{\Gamma(c)}{\tau\Gamma(a)\Gamma(c-a)} \int_0^1 t^{\frac{a}{\tau}-1}(1-t^{\frac{1}{\tau}})^{c-a-1} e^{zt}\, dt.$$

Using (5), we can easily derive the following two relations:

$$(i) \qquad \frac{d^n}{dz^n}\Phi^{\tau}(a; c; z) = \frac{\Gamma(c)\Gamma(a+\tau n)}{\Gamma(a)\Gamma(c+\tau n)}\Phi^{\tau}(a+\tau n; c+\tau n; z), \tag{6}$$

and

$$(ii) \qquad \Phi^{\tau}(a; c; z) = \frac{a}{c}\Phi^{\tau}(a+1; c+1; z) + \frac{c-a}{c}\Phi^{\tau}(a; c+1; z). \tag{7}$$

More properties and other related functions are found in Virchenko *et al.* (2000), and Virchenko (1999).

GENERALIZED INCOMPLETE GAMMA FUNCTION

We use the τ-confluent hypergeometric function to define the following generalized incomplete gamma function:

$$_{\lambda}\gamma^{\tau}_{w}(p, \delta; a; c; v) = \int_0^{w} x^{\lambda-1} e^{-px^{\delta}} \Phi^{\tau}(a; c; vx^{\delta}) dx \tag{8}$$

where $\mathrm{Re}(x) > 0$, $\mathrm{Re}(p) > 0$, $\tau > 0$, a and c are complex parameters, and $a + \tau k \neq 0, -1, -2, \ldots$.

The complimentary generalized gamma function is defined as

$$_{\lambda}\Gamma^{\tau}_{w}(p, \delta; a; c; v) = \int_{w}^{\infty} x^{\lambda-1} e^{-px^{\delta}} \Phi^{\tau}(a; c; vx^{\delta}) dx. \tag{9}$$

We observe that

$$_{\lambda}\gamma^{\tau}_{w}(1, 1; 0; c; v) = \int_0^{w} x^{\lambda-1} e^{-x}\, dx = \gamma(\lambda, w),$$
$$_{\lambda}\gamma^{\tau}_{w}(p, \delta; 0; c; v) = \int_0^{w} x^{\lambda-1} e^{-px^{\delta}}\, dx = \frac{1}{\delta p^{\frac{\lambda}{\delta}}}\gamma\left(\frac{\lambda}{\delta}, pw^{\delta}\right), \quad \text{and}$$
$$_{\lambda}\Gamma^{\tau}_{w}(1, 1; 0; c; v) = \int_{w}^{\infty} x^{\lambda-1} e^{-x^{\delta}}\, dx = \Gamma(\lambda, w)$$

where $\gamma(\lambda, w)$ and $\Gamma(\lambda, w)$ are the classical generalized incomplete and complimentary incomplete gamma functions, respectively.

The generalized incomplete gamma function (8) can also be expressed as a series of classical incomplete gamma functions, specifically,

$$ {}_{\lambda}\gamma_{w}^{\tau}(p,\delta;a;c;v) = \frac{\Gamma(c)}{\delta p^{\frac{\lambda}{\delta}}\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+\tau k)}{\Gamma(c+\tau k)} \frac{\left(\frac{v}{p}\right)^{k}}{k!} \gamma\left(\frac{\lambda}{\delta}+k, pw^{\delta}\right). $$

From (8) and (9), one obtains the following derivative formulas

$$ \frac{d}{dw}\,{}_{\lambda}\gamma_{w}^{\tau}(p,\delta;a;c;v) = w^{\lambda-1}e^{-pw^{\delta}}\Phi^{\tau}(a;c;vw^{\delta}), \quad \text{and} $$

$$ \frac{d}{dw}\,{}_{\lambda}\Gamma_{w}^{\tau}(p,\delta;a;c;v) = -w^{\lambda-1}e^{-pw^{\delta}}\Phi^{\tau}(a;c;vw^{\delta}). $$

Using (7) yields the relation

$$ {}_{\lambda}\gamma_{w}^{\tau}(p,\delta;a;c;v) = \frac{a}{c}\,{}_{\lambda}\gamma_{w}^{\tau}(p,\delta;a+1;c+1;v) + \frac{c-a}{c}\,{}_{\lambda}\gamma_{w}^{\tau}(p,\delta;a;c+1;v). $$

A PROBABILITY DENSITY FUNCTION

In this section we use the τ-confluent hypergeometric function (5) to define a probability density function.

Let

$$ f(x) = \frac{\delta p^{\frac{\lambda}{\delta}} x^{\lambda-1} e^{-px^{\delta}} \Phi^{\tau}(a;c;vx^{\delta})}{\Gamma\left(\frac{\lambda}{\delta}\right) {}_{2}R_{1}\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{v}{p}\right)}, \quad x \geq 0 \tag{10} $$

where $\lambda, p, \tau > 0$, v and p are scalar parameters such that $p > |v|$ and λ and δ are shape parameters.

Since

$$ \int_{0}^{\infty} x^{\lambda-1} e^{-px^{\delta}} \Phi^{\tau}(a;c;vx^{\delta}) dx = \frac{\Gamma\left(\frac{\lambda}{\delta}\right)}{\delta p^{\frac{\lambda}{\delta}}}\, {}_{2}R_{1}\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{v}{p}\right), $$

it follows that

$$ \int_{0}^{\infty} f(x)dx = 1, $$

and therefore $f(x)$ is a probability density function.

Furthermore, it is immediate that

$$ \text{(i) } f(0) = \begin{cases} 0 & \lambda > 1 \\ \dfrac{\delta p^{\frac{\lambda}{\delta}}}{\Gamma\left(\frac{\lambda}{\delta}\right) {}_{2}R_{1}\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{v}{p}\right)}, & \lambda = 1 \end{cases}. $$

(ii) $\lim_{x \to 0^{+}} f(x) = \infty$, provided that $\lambda < 1$.

(iii) $\lim_{x\to\infty} f(x) = 0$.

It is not difficult to verify that

$$f'(x) = f(x)\left[\frac{\lambda-1}{x} - p\delta x^{\delta-1} + \frac{\delta\nu\Gamma(c)\Gamma(a+\tau)}{\Gamma(a)\Gamma(c+\tau)} x^{\delta-1}\frac{\Phi^{\tau}(a+\tau; c+\tau; \nu x^{\delta})}{\Phi^{\tau}(a; c; \nu x^{\delta})}\right]$$
$$= x^{\delta-1} f(x)\left[\frac{\lambda-1}{x^{\delta}} - p\delta + \frac{\delta\nu\Gamma(c)\Gamma(a+\tau)}{\Gamma(a)\Gamma(c+\tau)}\frac{\Phi^{\tau}(a+\tau; c+\tau; \nu x^{\delta})}{\Phi^{\tau}(a; c; \nu x^{\delta})}\right].$$

Special cases of the density function $f(x)$ give well-known functions. For example,

1. Setting $\tau = 1$ gives the confluent hypergeometric density function

$$f(x) = \frac{\delta p^{\frac{\lambda}{\delta}} x^{\lambda-1} e^{-px^{\delta}} \Phi(a; c; \nu x^{\delta})}{\Gamma\left(\frac{\lambda}{\delta}\right) F\left(a; \frac{\lambda}{\delta}; c; \frac{\nu}{p}\right)}, \quad x \geq 0 \tag{11}$$

   where λ, $p > 0$, and $F(a, \frac{\lambda}{\delta}; c; \frac{\nu}{p})$ is Gauss hypergeometric function.

2. The exponential generalized density function (Kalla *et al.* 2000) is recovered from (11) by putting $\nu = \delta = 1$, $c = \lambda = p = \alpha + 1$, and recognizing that $F(a, b; b; x) = (1-x)^{-\alpha}$ $|\arg(1-x)| < IT$, for any value of b (Lebedev 1972),

$$f(x) = \frac{\alpha^{a}(\alpha+1)^{p-a}}{\Gamma(p)} x^{p-1} e^{-(\alpha+1)x} \Phi(a; p; x).$$

3. If $\nu = 0$, the density function becomes

$$f(x) = \frac{\delta p^{\frac{\lambda}{\delta}}}{\Gamma\left(\frac{\lambda}{\delta}\right)} x^{\lambda-1} e^{-px^{\delta}} \tag{12}$$

   which may be called the generalized Weibull density function.

4. Letting $\lambda = \delta$ in (12) yields the well-known Weibull distribution,

$$f(x) = \lambda p x^{\lambda-1} e^{-px^{\delta}}.$$

5. The Gamma density is recovered from (12) when $\delta = 1$,

$$f(x) = \frac{p^{\lambda}}{\Gamma(\lambda)} x^{\lambda-1} e^{-px}, \quad x > 0, \ p, \lambda > 0.$$

6. Letting $a = c$ in (11) leads to the density function

$$f(x) = \frac{\delta(p-c)^{\frac{\lambda}{\delta}}}{\Gamma\left(\frac{\lambda}{\delta}\right)} x^{\lambda-1} e^{-(p-\nu)x^{\delta}}, \quad x > 0, \ p > \nu > 0, \ \lambda > 0.$$

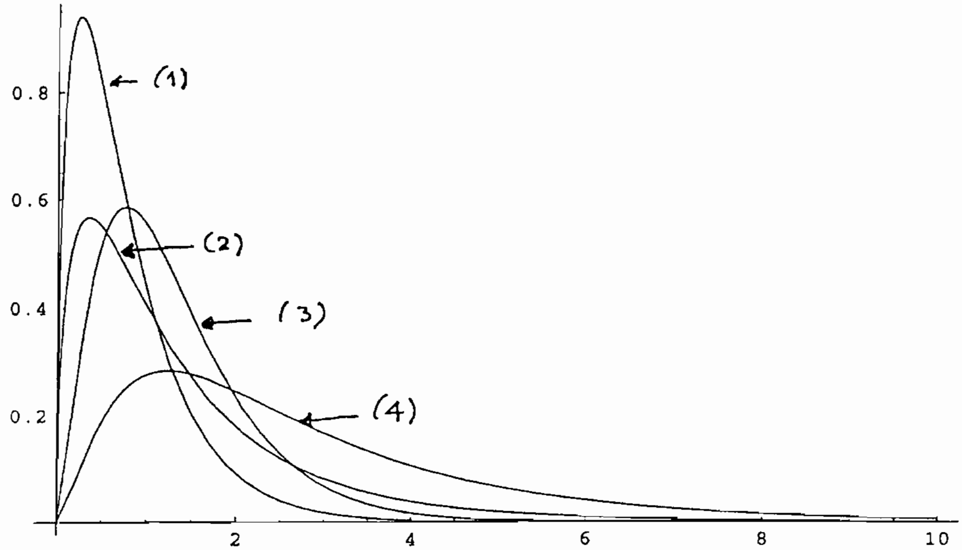

Fig. 1. Representation of $f(x)$ for different values of the parameters. Graph (1) for $\lambda = 1.5$, and $v = 1$; graph (2) for $\lambda = 1.5$, and $v = 2$; graph (3) for $\lambda = 2.5$, and $v = 1$; graph (4) for $\lambda = 2.5$, and $v = 2$.

Figure 1 depicts the graph of the probability density function $f(x)$ when $\delta = 1$, $a = 1.5$, $c = 3$, and $p = 2.5$.

STATISTICAL FUNCTIONS

In this section we derive basic statistical functions associated with the density function (10).

Moment

As in (Kalla *et al.* 2000), the k-th moment about the origin μ'_k, of a continuous real random variable x with density function $f(x)$ is

$$\mu'_k = \int_0^\infty x^k f(x)dx.$$

Expressing the τ-confluent hypergeometric function as a series, we find for the density defined in (10) that the k-th moment about the origin is

$$\begin{aligned}\mu'_k &= A\int_0^\infty x^{\lambda+k-1}e^{-px^\delta}\Phi^\tau(a;c;vx^\delta)dx\\ &= A\int_0^\infty x^{\lambda+k-1}e^{-px^\delta}\frac{\Gamma(c)}{\Gamma(a)}\sum_{n=1}^\infty\frac{\Gamma(a+\tau n)}{\Gamma(c+\tau n)}\frac{v^n x^{\delta n}}{n!}dx\\ &= \frac{A\Gamma\left(\frac{\lambda+k}{\delta}\right)}{p^{\frac{k}{\delta}}}\,{}_2R_1\left(\frac{\lambda+k}{\delta}, a; c; \tau; \frac{v}{p}\right) \end{aligned} \quad (13)$$

where

$$A = \frac{1}{\Gamma\left(\frac{\lambda}{\delta}\right) {}_2R_1\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{v}{p}\right)}.$$

Put $k = 1$ in (13) to obtain the "mean"

$$\mu_1' = E_x = \frac{\Gamma\left(\frac{\lambda+1}{\delta}\right) {}_2R_1\left(\frac{\lambda+1}{\delta}, a; c; \tau; \frac{v}{p}\right)}{p^{\frac{1}{\delta}}\Gamma\left(\frac{\lambda}{\delta}\right) {}_2R_1\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{v}{p}\right)}. \tag{14}$$

A special case of (14) follows by taking $a = 0$, namely,

$$\mu_1' = Ex = \frac{\Gamma\left(\frac{\lambda+1}{\delta}\right)}{p^{\frac{1}{\delta}}\Gamma\left(\frac{\lambda}{\delta}\right)}.$$

Expected value

Let $\psi(x)$ be a function of a continuous random variable x with density function $f(x)$. The expected value of $\psi(x)$, denoted by $E[\psi(x)]$, is (Mathai 1993):

$$E[\psi(x)] = \int_0^\infty \psi(x) f(x) dx.$$

The existence of the expected value $E[\psi(x)]$ depends on the nature of the functions $f(x)$ and $\psi(x)$ in the prescribed interval. It can also be interpreted as an "integral transform" of the density function $f(x)$ with respect to the kernel $\psi(x)$ (Debnath 1995).

As an example, suppose that x is a positive real random variable with density function $f(x)$, $x \geq 0$ ($f(x) = 0$ for $x < 0$), then

$$E(x^{s-1}) = \int_0^\infty x^{s-1} f(x) dx = M[f(x); s].$$

Here, $M[f(x); s]$ is the "Mellin transform" of $f(x)$.

Thus if $f(x)$ is the density function given in (10), then by straight-forward integration, the expected value of x^{s-1} is

$$E(x^{s-1}) = \frac{p^{-\left(\frac{s-1}{\delta}\right)}\Gamma\left(\frac{\lambda+s-1}{\delta}\right) {}_2R_1\left(\frac{\lambda+s-1}{\delta}, a; c; \tau; \frac{v}{p}\right)}{\Gamma\left(\frac{\lambda}{\delta}\right) {}_2R_1\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{v}{p}\right)}, \quad \mathrm{Re}(s) > 0.$$

Moreover, the moment generating function

$$E[e^{xt}] = M_x(t) = \int_0^\infty e^{tx} f(x)$$
$$= \frac{\delta p^{\frac{\lambda}{\delta}}}{\Gamma\left(\frac{\lambda}{\delta}\right) {}_2R_1\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{v}{p}\right)} \int_0^\infty x^{\lambda-1} e^{-px^\delta + tx} \Phi^\tau(a; c; vx^\delta) dx \tag{15}$$

where $p > t \geq 0$.

Setting $\delta = 1$ in (15), we obtain

$$E[e^{xt}] = M_x(t) = \frac{p^\lambda \, {}_2R_1\left(\lambda, a; c; \tau; \frac{v}{p-t}\right)}{(p-t)^\lambda \, {}_2R_1\left(\lambda, a; c; \tau; \frac{v}{p}\right)}$$

and when taking the derivative of order r of the above function we get

$$M_x^{(r)}(t) = \frac{p^\lambda \Gamma(\lambda + r) \, {}_2R_1\left(\lambda + r, a; c; \tau; \frac{v}{p-t}\right)}{\Gamma(\lambda)(p-t)^{\lambda+r} \, {}_2R_1\left(\lambda, a; c; \tau; \frac{v}{p}\right)}. \tag{16}$$

As a special case of (16), let $v = 0$ and obtain the r-th derivative of the moment generating function for the generalized Weibull density function (12), (Mathai, 1993)

$$M_x^{(r)}(t) = \frac{p^\lambda \Gamma(\lambda + r)}{\Gamma(\lambda)(p-t)^{\lambda+r}}.$$

The Hazard Rate Function

The hazard rate function (failure rate) $h(x)$ is defined through

$$h(x) = \frac{f(x)}{s(x)}$$

where $s(x)$ is called the survival or reliability function and given by

$$s(x) = 1 - F(x) > 0, \quad x > 0$$

and $F(x)$ is the cumulative density function (c.d.f) defined as

$$F(x) = \int_0^x f(t) dt.$$

For the density function $f(x)$ in (10) we have

$$F(x) = \frac{\delta p^{\frac{\lambda}{\delta}}}{\Gamma\left(\frac{\lambda}{\delta}\right) {}_2R_1\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{v}{p}\right)} \int_0^x t^{\lambda-1} e^{-pt^{\delta}} \Phi^{\tau}(a; c; vt^{\delta}) dt$$

$$= \frac{\delta p^{\frac{\lambda}{\delta}} {}_{\lambda}\gamma_x^{\tau}(p, \delta; a; c; v)}{\Gamma\left(\frac{\lambda}{\delta}\right) {}_2R_1\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{v}{p}\right)},$$

τ, $\mathrm{Re}(\lambda)$, $\mathrm{Re}(p) > 0$. Thus the survival function in this case becomes

$$s(x) = \frac{\delta p^{\frac{\lambda}{\delta}} {}_{\lambda}\Gamma_x^{\tau}(p, \delta; a; c; v)}{\Gamma\left(\frac{\lambda}{\delta}\right) {}_2R_1\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{v}{p}\right)}. \tag{17}$$

Consequently, the hazard rate function is

$$h(x) = \frac{f(x)}{s(x)} = \frac{x^{\lambda-1} e^{-px^{\delta}} \Phi^{\tau}(a; c; vx^{\delta})}{{}_{\lambda}\Gamma_w^{\tau}(p, \delta; a; c; v)}.$$

A particular case results when $v = 0$ and gives the hazard rate function for the density defined in (12)

$$h(x) = \frac{x^{\lambda-1} e^{-px^{\delta}}}{\Gamma\left(\frac{\lambda}{\delta}, px^{\delta}\right)}.$$

Further, for $\delta = p = 1$, we obtain the corresponding result for the gamma density

$$h(x) = \frac{x^{\lambda-1} e^{-x}}{\Gamma(\lambda, x)}.$$

It is obvious that $h(0) = 0$, $(\lambda > 1)$ and the asymptotic behaviour of $h(x)$ as $x \to \infty$, for $a = c$, is

$$\lim_{x\to\infty} h(x) \simeq \delta(p - v)x^{\delta-1}$$

since

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{f(x)}{1 - F(x)} = -\lim_{x\to\infty} \frac{f'(x)}{f(x)} = -\lim_{x\to\infty} \left[\frac{\lambda - 1}{x} - \delta(p - v)x^{\delta-1}\right].$$

The hazard rate function $h(x)$, for specific values of the parameters, is represented in Fig. 2.

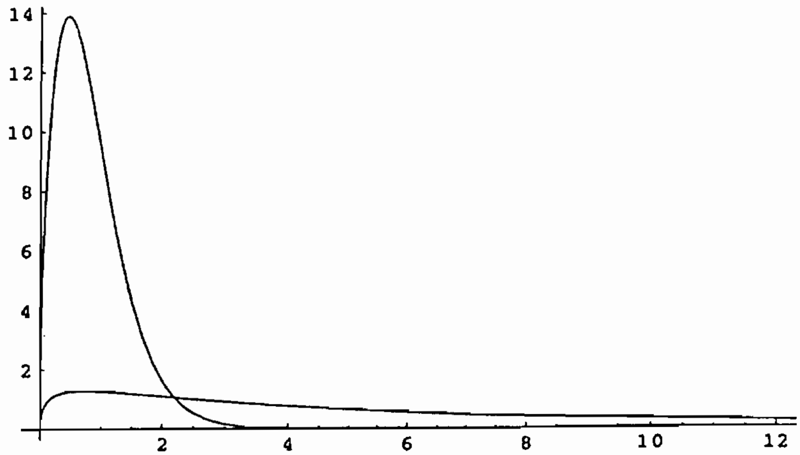

Fig. 2. Represents the hazard rate function $h(x)$ when $a = 1.5, c = 3, p = 2.5, \tau = 0.5, \lambda = 1.5$ and $\nu = 2$. The lower graph represents $h(x)$ when $\delta = 0.6$. The upper graph represents $h(x)$ when $\delta = 1.75$.

The Mean Residue Life Function

For a continuous real random variable x with a density function $f(x)$, the residue life function is defined as

$$K(x) = \frac{1}{s(x)} \int_x^\infty (y - x) f(y) dy.$$

Observe that

$$\int_x^\infty y f(y) dy = \frac{\delta p^{\frac{\lambda}{\delta}}}{\Gamma\left(\frac{\lambda}{\delta}\right) {}_2R_1\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{\nu}{p}\right)} \int_x^\infty y^\lambda e^{-py^\delta} \Phi^\tau(a; c; \nu y^\delta) dy$$

$$= \frac{\delta p^{\frac{\lambda}{\delta}} {}_{1+\lambda}\Gamma_x^\tau(p, \delta; a; c; \nu)}{\Gamma\left(\frac{\lambda}{\delta}\right) {}_2R_1\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{\nu}{p}\right)}$$

and

$$x \int_0^\infty f(y) dy = \frac{\delta p^{\frac{\lambda}{\delta}} x \, {}_\lambda\Gamma_x^\tau(p, \delta; a; c; \nu)}{\Gamma\left(\frac{\lambda}{\delta}\right) {}_2R_1\left(\frac{\lambda}{\delta}, a; c; \tau; \frac{\nu}{p}\right)}. \tag{18}$$

Hence by virtue of (17), and (18) we get

$$K(x) = \frac{{}_{1+\lambda}\Gamma_x^\tau(p, \delta; a; c; \nu)}{{}_\lambda\Gamma_x^\tau(p, \delta; a; c; \nu)} - x.$$

For $\nu = 0$, we obtain the mean residue life function for the generalized Weibull density given in (12)

$$K(x) = \frac{\Gamma\left(\frac{\lambda+1}{\delta}, px^{\delta}\right)}{p^{\frac{1}{\delta}}\Gamma\left(\frac{\lambda}{\delta}, px^{\delta}\right)} - x.$$

DISCUSSION

In this paper we have defined and developed a probability density function using the τ-confluent hypergeometric function $\Phi^{\tau}(z)$ and ${}_2R_1^{\tau}(z)$. In order to get some statistical functions, such as the k-th moment, hazard rate function and the mean residue life function, etc. we introduced a generalized incomplete gamma function and its complementary form. Figures are plotted to show the behavior of the density function. The density function considered here contains a number of parameters (scale, shape, ... etc.) and generalize the well-known distribution: gamma, generalized gamma, Weibull and confluent hypergeometric. As the τ-hypergeometric functions and the density considered here are not familiar in the domain of probability and statistics, we hope that researchers working in these fields will find them useful in their theory and applications. Our aim is to develop the mathematical model with possible applications.

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ص.ب. 5969 – الصفاة 13060 – الكويت

خلاصة

نعرض في هذا البحث توزيعاً جديداً من النمط الكمّاوي يشتمل على دالـة فوهندسـية ح - مندمجة ، حيث نُدخل شكلاً معمماً لدالة كمّاوية غير تامة ، كما نُدخل شكلها المتتام وذلك مـن أجل الحصول على دوال إحصائية ، ونتيجة لذلك تم الحصول على بعض الـدوال الأساسـية المرافقة للكثافة الجديدة ، وبالتحديد : العزم ، الوسطي ، القيمة المتوقعة، دالة معدّل المجازفـة، والراسب الوسطي لدالة الحياة . نأتي كذلك على ذكر عدة حالات خاصة ، مثل كثافات فـايبول المعممة الكمّاوية المندمجة ، كما نُجري وصفاً بيانياً لسلوك دالة كثافة الاحتمال ودوال معـدّل المجازفة وذلك من أجل قيم معينة للوسطاء.