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On quasi HNN groups

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ABSTRACT

In this paper we extend the class of HNN groups to a new class of groups called quasi HNN groups. Also in this paper we formulate and prove an embedding theorem and version of Britton's Lemma for the new class of quasi HNN groups.

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1. INTRODUCTION

First we start by the following definition.

Let A be a nontrivial group, and $\phi : A \to A$ be an automorphism. We say that ϕ is an outer automorphism of rank n determined by $a \in A$, if $\phi(a) = a$, and n is the minimal integer such that $n \geq 2$, $\phi \notin Inn(A)$, $\phi^n \in Inn(A)$, and $\phi^n(x) = a\ a^{-1}$ for all elements x of A.

If A is trivial, we take the identity map to be outer automorphism of rank $n \geq 2$ determined by the identity element of A.

We note that if ϕ is an automorphism of a nontrivial group G such that, $\phi \notin Inn(G)$, $\phi^n \in Inn(G)$, $\phi(g) = g$ for a fixed element g of G, and for a positive integer n, such that $\phi^n(x) = g\ g^{-1}$ for all elements x of G, then it is clear that $n \geq 2$ and ϕ is an outer automorphism of G of rank $n \geq 2$ determined by g.

Now we define the quasi HNN groups as follows.

Let G be a group and I and J be index sets. Let $\{A_i; i \in I\}, \{B_i; i \in I\}$, and $\{C_j; j \in J\}$ be families of subgroups of G. For each $i \in I$, let $\phi_i : A_i \to B_i$ be an onto isomorphism and for each $j \in J$, let $\alpha_j : C_j \to C_j$ be an outer automorphism of rank $h_j \geq 2$ determined by $c_j \in C_j$.

The group G^* with the presentation

$G^* = \langle G, t_i, t_j \mid rel\ G, t_i A_i t_i^{-1} = B_i, t_j C_j t_j^{-1} = C_j, t_j^{h_j} = c_j, i \in I, j \in J \rangle$ is called a quasi HNN group with base G, stable letters t_i and t_j and associated pairs (A_i, B_i) and $(C_j, C_j), i \in I, j \in J$, of subgroups of G, where $t_i A_i t_i^{-1} = B_i$, stands for the set of relations $t_i\ a\ t_i^{-1} = \phi_i(a)$, for all $a \in A_i$ and $t_j\ C_j\ t_j^{-1} = C_j$ stands for the set of relations $t_j\ c\ t_j^{-1} = \alpha_j(c)$, for all $c \in C_j$. We note that if $J = \phi$, then G^* is an HNN group with base G and associated pairs $(A_i, B_i), i \in I$, of subgroups of G.

This paper is divided into four sections. In section two we prove an embedding theorem for quasi HNN groups. In section three Britton's Lemma for HNN groups is extended to quasi HNN groups. In section four we give examples of quasi HNN groups.

2. THE EMBEDDING THEOREM FOR QUASI HNN GROUPS

The main result of this paper states that if G^* is a quasi HNN group with base G, then G is embedded in G^*. The proof of the main result will follow from the following lemmas.

Lemma 1. Let A be a group and let $\alpha : A \to A$ be an outer automorphism of rank $n \geq 2$ determined by $a_0 \in A$.

Let $A^* = \langle A, t \mid rel\ A, t\,A\,t^{-1} = A, t^n = a_0 \rangle$ be a quasi HNN group with base A. Then A is embedded in A^*. Moreover, n is the smallest power of t in the image of A in A^*.

Proof. If A is trivial, the result is clear.

Let A be nontrivial and $S = A \cup At \ \cup\ At^2\ \cup \ldots \cup At^{n-1}$ be a union of cosets of A.

Let S^* denote the group of all permutations of S. Our aim is to construct a homomorphism from A^* into S^*. Clearly, it suffices to define this homomorphism on A and t, and then verify that all defining relations in A^* hold under the constructed homomorphism. We adopt the convention that $t^0 = 1_A$, the identity of A and α^0, the identity automorphism. We first define a homomorphism $\theta : A \to S^*$.

For $a' \in A$ define $\theta(a') : S \to S$ as follows

$$(a\ t^r)\ \theta(a') = a\ \alpha^r(a')t^r\ ,\ 0 \leq r \leq n-1\ .$$

Now for any $a_1, a_2 \in A$ we have

$$\begin{aligned}(a\ t^r)\ \theta(a_1)\ \theta(a_2) &= a\ \alpha^r(a_1)\ t^r\ \theta(a_2)\\ &= a\ \alpha^r(a_1)\ \alpha^r(a_2)\,t^r\\ &= a\ \alpha^r(a_1a_2)\,t^r\ ,\ \text{ since } \alpha^r \in Aut(A)\\ &= (a\ t^r)\ \theta(a_1a_2)\ .\end{aligned}$$

Thus $\theta(a_1)\,\theta(a_2) = \theta(a_1 a_2)$. In particular $\theta(a_1)\,\theta(a_1^{-1}) = \theta(a_1^{-1})\,\theta(a_1)$ is a permutation of S. That is, relations of A hold under θ. Hence $\theta(a')$ is a permutation of S and θ is a homomorphism from A into S^*.

Since $(1_A)\,\theta(a) = a$ for every $a \in A$, therefore $\theta(a)$ is the identity on S if and only if $a = 1_A \in A$. Hence θ is a monomorphism form A into S^*.

Now we construct a homomorphism $\phi^* : A^* \to S^*$.

Define $\phi(a') : S \to S$ by $\phi(a') = \theta(a')$ and define $\phi(t) : S \to S$ as follows.

$$(a\ t^r)\ \phi(t) = a\ t^{r+1} \quad , \quad \text{for } 0 \le r \le n-2\ .$$

$$(a\ t^{n-1})\ \phi(t) = a\ a_0\ .$$

To show that $\phi(t)$ is also a permutation on S define $\Psi(t) : S \to S$ by

$$(a\ t^r)\ \Psi(t) \begin{cases} a\ t^{r-1} & \textit{if } 1 \le r \le n\text{-}1 \\ a\ a_0^{-1}\ t^{n-1} & \textit{if } r\ =\ 0 \end{cases}$$

Then we have

$$(a\ t^r)\,\phi(t)\,\Psi(t) = (a\ t^{r+1})\,\Psi(t) = at^r\ ,\ 0 \le r \le n-2$$

$$(a\ t^{n-1})\,\phi(t)\,\Psi(t) = a\ a_0\,\Psi(t) = a\ a_0\ a_0^{-1}\ t^{n-1} = a\ t^{n-1}$$

$$(a\ t^r)\,\Psi(t)\,\phi(t) = (a\ t^{r-1})\,\phi(t) = a\ t^r\ ,\ 1 \le r \le n-1$$

$$(a)\,\Psi(t)\,\phi(t) = (a\ a_0^{-1}\ t^{n-1})\,\phi(t) = a\ a_0^{-1}\ a_0 = a\ \text{ for }\ r = 0\ .$$

Thus $\phi(t)\,\Psi(t) = \Psi(t)\,\phi(t) =$ the identity on S. Hence $\phi(t)$ is a permutation on S. To show that $\phi(a')$ together with $\phi(t)$ induce a homomorphism ϕ^* from A^* into S^*, we only have to verify that the defining relations in A^* hold under ϕ.

Let $\phi(t^n)$ denote $\phi(t)\,\phi(t)\ldots\phi(t)$ to n factors. Then

$$\begin{aligned}(a\ t^r)\,\phi(t^n) &= (a\ t^{r+1})\,\phi(t^{n-1})\ ,\ 0 \le r \le n-2 \\ &= (a\ t^{r+2})\,\phi(t^{n-2}) \\ &\ \vdots \\ &= a\ a_0\ t^r \\ &= a\ \alpha^r(a_0)t^r\ ,\ \text{since } \alpha^r(a_0) = a_0 \\ &= (a\ t^r)\,\phi(a_0)\ .\end{aligned}$$

Thus $\phi(t^n) = \phi(a_0)$.

$$(a\ t^r)\,\phi(t)\,\phi(a')(\phi(t))^{-1} = (a\ t^{r+1})\,\phi(a')(\phi(t))^{-1}\ ,\ 0 \le r \le n-2$$
$$= (a\ \alpha^{r+1}(a')t^{r+1})\,\Psi(t)$$
$$= a\ \alpha^{r+1}(a')\,t^r$$
$$= a\ \alpha^r(\alpha(a'))\,t^r$$
$$= (a\ t^r)\,\phi(\alpha(a'))$$

Now for $r = n-1$ we have

$$(a\ t^{n-1})\,\phi(t)\,\phi(a')(\phi(t))^{-1} = (a\ a_0)\,\phi(a')(\phi(t))^{-1}$$
$$= (a\ a_0\ a')\,\Psi(t)$$
$$= a\ a_0\ a'\ a_0^{-1}\ t^{n-1}$$
$$= a\ \alpha^n(a')\,t^{n-1}$$
$$= a\ \alpha^{n-1}(\alpha(a')\,t^{n-1}$$
$$= (a\ t^{n-1})\,\phi(\alpha(a'))$$

Thus $\phi(t)\,\phi(a')(\phi(t))^{-1} = \phi(\alpha(a'))$, and the defining relations in A^* hold under ϕ.

Hence ϕ induces a homomorphism ϕ^* from A^* into S^*. Since ϕ is a monomorphism A is embedded in A^*. Moreover, identifying A with its image in A^*, we have $(1_A)\,\phi(t^r) = t^r$, $1 \le r \le n-1$, $(1_A)\,\phi(a) = a$, for every $a \in A$. Thus $\phi(t^r) \ne \phi(a)$ in S^*. Hence $t^r \ne a$ in A^* for $1 \le r \le n-1$. That is, no nontrivial power of t smaller than n is in the image of A. This completes the proof.

Lemma 2. Let G be a group, and A be a subgroup of G. Let $\alpha : A \to A$ be an outer automorphism of rank $n \ge 2$ determined by $a_0 \in A$.

Let $G^* = \langle G, t \,|\, rel\ G, t\,A\,t^{-1} = A, t^n = a_0\rangle$ be the quasi HNN gourp with base G. Then G is embedded in G^*, Moreover, the smallest power of t in the image of A in G^* is n.

Proof. If A is trivial, then $G^* = \langle G, t \,|\, rel\ G, t^n = 1\rangle$. Then G^* is the free product of G and of a cyclic group of order n generated by t. Then it is clear that G is embedded in G^*.

Now let A be nontrivial.

If $G = A$, we have Lemma 1. If $A \neq G$, let A^* be the group

$A^* = \langle A, t \,|\, rel\ A, t\ A\ t^{-1} = A, t^n = a_0 \rangle$. By Lemma 1, A is embedded as a subgroup A' in A^*. We consider the amalgamated free product $H = G \underset{A}{*} A^*$ in which we amalgamate the subgroup A of G with the subgroup A' of A^* by an isomorphism from A onto A'. Then H has the presentation

$H = \langle G, A', t \,|\, rel\ G, rel\ A', t\ A'\ t^{-1} = A', t^n = a_0, a' = a, a \in A \rangle$. Hence the presentation $H = \langle G, t \,|\ rel\ G, t\ A\ t^{-1} = A, t^n = a_0 \rangle$. Thus $H \cong G^*$, and hence $G^* \cong G \underset{A}{*} A^*$. By Theorem 2.6 (page 187 of Lyndon and Schupp 1977) G is embedded in G^*. This completes the proof.

Lemma 3. Let G be a group and $\{A_i; i \in I\}$ be a family of subgroups of G. For each $i \in I$, let $\alpha_i : A_i \to A_i$ be an outer automorphism of rank $n_i \geq 2$ determined by $a_i^{'} \in A_i$. Then G is embedded in the quasi HNN group

$$G^* = \langle G, t_i \,|\ rel\ G, t_i A_i t_i^{-1} = A_i, t_i^{n_i} = a_i^{'}, i \in I \rangle\ .$$

Proof. If A_i is trivial, $i \in I$, then $G^* = \langle G, t_i \mid rel\ G, t_i^n = 1 \rangle$. Then G^* is the free product of G and of a cyclic group of order n_i generated by t_i. Then it is clear that G is embedded in G^*.

Now let $A_i, i \in I$ be nontrivial. For each $i \in I$, let G_i be an isomorphic copy of G. Then G contains an isomorphic copy $\overline{A_i}$ of the subgroup A_i of G. For each $i \in I$, let

$$H_i = \langle G_i, t_i \,|\ rel\ G_i, t_i\ \overline{A_i}\ t_i^{-1} = \overline{A_i}, t_i^{n_i} = \hat{a}_i, \hat{a}_i \in \overline{A_i} \rangle\ .$$

Then H_i is a quasi HNN group with base G_i. By Lemma 2, G_i is embedded in H_i for each $i \in I$. Take $H = \, {}_G^* H_i$ for each $i \in I$, amalgamating the subgroups G_i. Then H_i is embedded in H for each $i \in I$. It follows that G_i and G are embedded in H. Now a presentation for H is $HH = \langle G, G_i, t_i \mid rel\ G, t_i\ \overline{A_i}\ t_i^{-1} = \overline{A_i},\ t_i^{n_i} = \hat{a}_i,\ G_i = G, i \in I \rangle$. Hence H has the presentation $H = \langle G, t_i \mid rel\ G, t_i\ A_i\ t_i^{-1} = A_i,\ t_i^{n_i} = a_i^{'},\ a_i^{'} \in A_i,\ i \in I \rangle$. Thus $H \cong G^*, G^* = {}_G^*\ HH_i$, for each $i \in I$, and hence G is embedded in G^*.

This completes the proof.

The main result of this section is the following theorem.

Theorem 1. Let $\{A_i, i \in I\}, \{B_i, i \in I\}$, and $\{C_j, j \in J\}$ be families of subgroups of a group G. For each $i \in I$ let $\phi_i : A_i \to B_i$ be an onto isomorphism and for each $j \in J$ let $\alpha_j : C_j \to C_j$ be an outer automporphism of rank $h_j \geq 2$ determined by $c_j \in C_j$.

Let $G^* = \langle G, t_i, t_j \mid rel\ G, t_i\ A_i\ t_i^{-1} = B_i,\ t_j\, C_j\, t_j^{-1} = C_j,\ t_j^{h_j} = c_j,\ i \in I,\ j \in J\rangle$ be a quasi HNN group. Then G is embedded in G^*.

Proof. If the subgroups $C_j, j \in J$ are trivial, then G^* is the free product of the HNN group $\langle G, t_i \mid rel\ G, t_i\ A_i\ t_i^{-1} = B_i, i \in I\rangle$ and cyclic groups of orders $h_j \geq 2$ generated by $t_j, j \in J$. By the embedding theorem for HNN groups, G is embedded in $\langle G, t_i \mid rel\ G, t_i\ A_i\ t_i^{-1} = B_i, i \in I\rangle$. This implies that G is embedded in G^*

Now assume that the subgroups $C_j, j \in J$ are nontrivial.

Let $G_1 = \langle G, t_i \mid rel\ G, t_i\ A_i\ t_i^{-1} = B_i, i \in I\rangle$.

Then G_1 is an HNN group with base G, stable letters t_i and associated pairs $(A_i, B_i), i \in I$ of subgroups of G_1. Then by Theorem 2.1 (page 182 of Karrass and Solitar 1970), G is embedded in G_1. Now take

$$G^* = \langle G_1, t_j \mid rel\ G_1, t_j\ C_j\ t_j^{-1} = C_j, t_j^{h_j} = c_j, j \in J\rangle\ .$$

Then G^* is a quasi HNN group with base G_1, stable letters t_j, and associated pairs $(C_j, C_j), j \in J$, of subgroups of G_1 . By Lemma 3, G_1 is embedded in G^*.

Hence G is embedded in G^*. This completes the proof.

3. VERSIONS OF BRITTON'S LEMMA FOR QUASI HNN GROUPS

We turn now to a discussion of the word problem in quasi HNN groups.

Britton's Lemma (Lyndon and Schupp 1977, page 1881) settle the problem for HNN groups. We are seeking some versions of this lemma which will hold in quasi HNN groups.

Recall that if $G = \langle X|R\rangle$ is a presentation of the group G, then a reduced word w on the generating set X is a word on X which contains no subword of the form $x\ x^{-1}$ or $x^{-1}, x \in X$.

Lemma 4. Let A and A^* be as in Lemma 1. If w is a reduced word on $\{A, t\}$, involving t, and if $w = 1$ in A^*, then w contains a subword of the form $t^e\, a\, t^\delta$, or $t^{\pm n}$, where $a \in A$, and $e, \delta = \pm 1$.

Proof. By Lemma 1, A is embedded in A^*. We identify A with its image in A^*. We need only consider the word $w = a'\, t^{e_1}\, a_1 \ldots t^{e_k}\, a_k, k \geq 1$, where $e_i \neq 0$, are integers, $-n < e_i < n, i = 1, \ldots, k$, and $a', a_1, \ldots, a_k$ are in A of which only a' and a_k may be trivial. Let $Z_n =< z >$ denote the cyclic group of order n generated by z and consider the map $\phi : A^* \to Z_n$ defined by $\phi(a) = 1$ for all $a \in A$, and $\phi(t) = z$. Then ϕ is a homomorphism because all relations in A^* are mapped to the identity in Z_n.

Now $\phi(w) = \phi(a' t^{e_1} a_1 \dots t^{e_k} a_k) = 1$ in Z_n. That is, $z^{e_1+e_2+\dots+e_k} = 1$ in Z_n. Hence $\sum_{i=1}^{k} e_i \equiv 0(n)$ and $k \neq 1$. That is, $k \geq 2$ and hence w contains a subword $t^e\, a\, t^\delta$, where $a \in A$, and $e, \delta = \pm 1$. this completes the proof.

Lemma 5. Let G, A, α, and G^* be as in Lemma 2. If w is a reduced word on $\{G, t\}$ involving t, and if $w = 1$ in G^*, then w contains a subword of the form $t^e\, a\, t^\delta$, or $t^{\pm n}$, where $a \in A$, and $e, \delta = \pm 1$.

Proof. By Lemma 2, G is embedded in $G^* = G \underset{A}{*} A^*$, where

$$A^* = \langle A, t \mid rel\ A, t\, A\, t^{-1} = A, t^n = a_0 \rangle\ .$$

We identify G with its image in G^*, and consider the word $w = g_0\, t^{e_1}\, g_1 \dots t^{e_k} g_k$, $k \geq 1$, where $e_i \neq 0$, are integers, $-n < e_i < n, i = 1, \dots, k$, and $g_0, g_1, \dots, g_k$ are in G of which g_0 and g_k may be trivial. Now $w = g_0\, t^{e_1}\, g_1 \dots t^{e_k}\, g_k = 1$ in $G^* = G \underset{A}{*} A^*$. Since t^{e_i} is not in A, by Lemma 1, at least $g_i, 1 \leq i \leq k-1$ is in the amalgamated subgroup A. We observe that $k \geq 2$, since otherwise $t^{e_1} = g_0^{-1}\, g_1^{-1}$ in G and hence in A because of the amalgamation. This contradicts Lemma 1. Hence w contains no subword of the claimed form. This completes the proof.

Lemma 6. Let $G, A_i, \alpha_i, i \in I$, be as in Lemma 3. Let

$$G^* = \langle G, t_i \mid rel\ G, t_i\, A_i\, t_i^{-1} = A_i, t_i^{n_i} = a_i', i \in I \rangle\ .$$

Let w be a reduced word on $\{G, t_i\}$ involving $t_i, i \in I$. If $w = 1$ in G^*, then w contains a subword of the form $t_i^e\, a_i\, t_i^\delta$, or $t_i^{\pm n_i}$, where $a_i \in A_i$, and $e, \delta = \pm 1$.

Proof. For each $i \in I$, let $H_i = \langle G, t_i \mid rel\ G, t_i\, A_i\, t_i^{-1} = A_i, t_i^{n_i} = a_i', a_i' \in A_i \rangle$.

Then H_i is a quasi HNN group with base G and stable letters $t_i, i \in I$.

For this i, let $A_i^* = \langle A_i, t_i \mid rel\ A_i, t_i\, A_i\, t_i^{-1} = A_i, t_i^{n_i} = a_i', a_i' \in A_i \rangle$.

By Lemma 2, $H_i = G \underset{A_i}{*} A_i^*$, and by the proof of Lemma 3, $G^* = \underset{G}{*} H_i$, for each $i \in I$. Then G^* is generated by an isomorphic copies of the H_i. Thus an element of G^* can be expressed as a word $w = h_{i_1} \dots h_{i_r}$, where $h_{i_j} \in H_{i_j}, j = 1, \dots, r$, and $H_{i_j} \neq H_{i_{j+1}}$, for $j = 1, \dots, r-1$. Now suppose that $w = h_{i_1} \dots h_{i_r}$, represents the identity in G^*. Then at least one h_{i_j} for $1 < j < r$ is in the amalgamated subgroup G. Then h_{i_j} can be expressed as a reduced word on $\{G, t_i\}$, for some $i : h_{i_j} = g_0\, t_i^{e1}\, g_1 \dots t_i^{em}\, g_m$, where $g_0, g_1, \dots, g_m$ are in G and for this $i, -n_i < e_k < n_i, k = 1, \dots, m$. Now since the word h_{i_j} represents an element of G, there is a word g on hte generators of G such that $h_{i_j}\, g^{-1} = g_0\, t_i^{ei}\, g_1 \dots t_i^{em}\, g_m\, g^{-1}$ representing the identity in H_i. But $H_i = G \underset{A_i}{*} A_i^*$, and by Lemma 2, t_i^{ek} is not in G for $k = 1, \dots, m$. Since g does not involve t_i, we have at least one

$g_s, 1 \leq s \leq m-1$ in the amalgamated subgroup A_i. Hence $h_{i_j} g^{-1}, h_{i_j}$, and w contain a subword of at least one of the claimed forms. This completes the proof.

We now consider words representing the identity in the most general quasi HNN groups of Theorem 1. This general question is answered in the following.

Theorem 2. Let $G, A_i, B_i, \phi_i, C_j, \alpha_j, i \in I$ and, $j \in J$ be as in Theorem 1. Let

$$\pi^* = \langle G, t_i, t_j \mid rel\ G, t_i A_i t_i^{-1} = B_i, t_j C_j t_j^{-1} = C_j, t_j^{h_j} = c_j, i \in I, j \in J \rangle \ .$$

Let w be a reduced word on $\{G, t_i, t_j\}$ involving some t_i or some t_j or both.

If $w = 1$ in π^*, then w contains a subword of the forms

(1) $t_i a t_i^{-1}, a \in A_i$, for some $i \in I$,

or (2) $t_i^{-1} b t^i, b \in B_i$, for some $i \in I$,

or (3) $t_j^e c t_j^\delta, c \in C_j$ for some $j \in J$, and $e, \delta = \pm 1$.

or (4) $t_j^{\pm h_j}$, for some $j \in J$.

Proof. Let $\pi_1 = \langle G, t_i \mid rel\ G, t_i A_i t_i^{-1} = B_i, i \in I \rangle$. Then π_1 is an HNN group with base G, stable letters t_i and associated pairs $(A_i, B_i), i \in I$. Therefore G and hence all subgroups of G are embedded in π_1.

Let $\pi_2 = \langle G, t_j \mid rel\ G, t_j C_j t_j^{-1} = C_j, t_j^{h_j} = c_j, j \in J \rangle$. Then π_2 is a quasi HNN group with base G, stable letters t_j and associated pairs $(C_j, C_j), j \in J$. By Lemma 3, G and hence all subgroups of G are embedded in π_2.

Take $\pi_1^* = \langle \pi_1, t_j \mid rel\ \pi_1, t_j C_j t_j^{-1} = C_j, t_j^{h_j} = c_j, j \in J \rangle$, and $\pi_2^* = \langle \pi_2, t_i \mid rel\ \pi_2, t_i A_i t_i^{-1} = B_i, i \in I \rangle$. Then π_1, G and all subgroups of G are embedded in π_1^*, while π_2, G and all subgroups of G are embedded in π_2^*. Clearly $\pi_1^* \cong \pi_2^* \cong \pi^*$. Now if w involves only t_i, then $w = 1$ in π_1. By Britton's Lemma for HNN groups w contains a subword of the form $t_i a t_i^{-1}, a \in A_i$, for some $i, i \in I$, or $t_i^{-1} b t_i, b \in B_i$, for some $i, i \in I$. If w involves only t_j, then $w = 1$ in π_2. By Lemma 6, w contains a subword of the form $t_j^e c t_j^\delta, c \in C_j$ for some $j, j \in J$, and $e, \delta = \pm 1$.

Consider now the general case in which w contains some t_i and some t_j. By the first part of the proof we have the embedding $G \subseteq \pi_2 \subseteq \pi^*$. then π^* is an HNN group with base the quasi HNN group with base π_2, stable letters t_i, and associated pairs $(A_i, B_i), i \in I$. By Britton's Lemma for HNN groups, w contains a subword of the form $t_i^e u t_i^{-e}, e = \pm 1$, where u is a word on $\{G, t_j, j \in J\}$ representing an element of A_i or B_i for some $i, i \in I$ whether $e = 1$ or $e = -1$.

Assume that $e = 1$. If u is a word on the generators of G alone then u

represents an element of A_i and $t_i\,u\,t_i^{-1}$ is the desired subword.

If u involves some of the t_j, then u is a word in π_2. Since u represents an element of A_i, that is, an element of G, there is a word v on the generators of G such that $u\,v^{-1}$ represents the identity in π_2. But π_2 is a quasi HNN group with base G, hence by Lemma 3, $u\,v^{-1}$ contains a subword of the form $t_j^e\,c\,t_j^\delta$, or $t_j^{\pm h_j}$, for some $j,j \in J$, and $e, \delta = \pm 1$, where c is a word on the generators of G representing an element of C_j for some $j \in J$. Since v contains no t_j, u and hence w contains a subword of at least one of the claimed froms.

The case $e = -1$ is similar. This completes the proof.

Remark. Consider the HNN group $G^* = \langle G, t_i \mid rel\ G, t_i\,A_i\,t_i^{-1} = B_i, i \in I \rangle$.

Karrass and Solitar (1977, Lemma 3) proved that if G^* is finitely generated and $A_i, i \in I$ are finitely generated, then G is finitely generated. Mahmood (1993) proved that if G^* and G are finitely presented, then $A_i, i \in I$ is finitely generated. Now we can generalize these results to quasi HNN groups as in the following lemma. The proof is similar.

Lemma 7. Let G^* be the quasi HNN group

$$G^* = \langle G, t_i, t_j \mid rel\ G, t_i\,A_i\,t_i^{-1} = B_i, t_j\,C_j\,t_j^{-1} = C_j, t_j^{h_j} = c_j, i \in I, j \in J \rangle\ .$$

Then

(1) If G^* is finitely generated, then $I \cup J$ is finite. Moreover, if $I \cup J$ is finite and G is finitely generated then G^* is finitely generated. Also, if $I \cup J$ is finite, A_i and $C_j, i \in I, j \in J$ are finitely generated and G is finitely presented, then G^* is finitely presented.

(2) If G^* is finitely generated and A_i and $C_j, i \in I, j \in J$ are finitely generated, then G is finitely generated.

(3) If G^* and G are finitely presented, then A_i and $C_j, i \in I, j \in J$ are finitely generated.

4. EXAMPLES OF QUASI HNN GROPUS

We say that a group H is a quasi free group if H is a free product of cyclic groups (finite or infinite), i.e. H is a free product of a free group and finite cyclic groups.

The next theorem and corollary give more examples of quasi free groups.

Theorem 3. A quasi HNN group is a quasi free group if and only if its base is trivial.

Proof. Let G^* be the quasi HNN group

$$G^* = \langle G, t_i, t_j \mid rel\ G, t_i A_i t_i^{-1} = B_i, t_j C_j t_j^{-1} = C_j, t_j^{h_j} = c_j, i \in I, j \in J \rangle \ .$$

Since G is trivial, therefore the subroups $A_i, C_j, i \in I, j \in J$ are trivial. This implies that $G^* = \langle t_i, t_j \mid t_j^{h_j} = 1, i \in I, j \in J \rangle$. Then G^* is the free product of the free group generated by $t_i, i \in I$ and cyclic groups of orders h_j generated by $t_j, j \in J$. Hence G^* is a quasi free group

Conversely, let G^* be a quasi free group. Then G^* is a free product of a free group generated by $t_i, i \in I$ and of finite cyclic groups of orders h_j generated by $t_j, j \in J$. Then G^* has the presentation $\langle t_i, t_j \mid t_j^{h_j} = 1, i \in I, j \in J \rangle$ which is a quasi HNN group with trivial base and trivial pairs of subgroups. This completes the proof.

The proof of the following corollary is clear.

Corollary 1. Let G^* be the quasi HNN group

$$G^* = \langle G, t_i, t_j \mid rel\ G, t_i A_i t_i^{-1} = B_i, t_j C_j t_j^{-1} = C_j, t_j^{h_j} = c_j, i \in I, j \in J \rangle \ .$$

Then

(1) If G is trivial, $I = \varphi$, $|J| = 2$, and $h_1 = h_2 = 2$, then G^* is the infinite dihedral group $\langle t_1, t_2 \mid t_1^2 = 1, t_2^2 = 1 \rangle$.

(2) If G is trivial, $I = \varphi$, and $|J| = 1$, then G^* is the cyclic group $\langle t \mid t^h = 1 \rangle$ of order h.

Now we conclude this section by the following lemma which gives more examples of quasi HNN groups. The proof is similar to the proof of Lemma 12 of Baumslag (1993).

Lemma 8. Let G^* be the quasi HNN group

$$G^* = \langle G, t_i, t_j \mid rel\ G, t_i A_i t_i^{-1} = B_i, t_j C_j t_j^{-1} = C_j, t_j^{h_j} = c_j, i \in I, j \in J \rangle \ .$$

Suppose that H is a subgroup of G such that

$$t_i(A_i \cap H)t_i^{-1} = B_i \cap H, t_j(C_j \cap H)t_j^{-1} = C_j \cap H, c_j \in H, i \in I, j \in J \ .$$

Then the subgroup $gp(H, t_i, t_j)$ of G^* generated by H, t_i and t_j is the quasi HNN group

$$\langle H, t_i, t_j \mid rel(H), t_i H_i t_i^{-1} = \overline{H}_i, t_j H_j t_j^{-1} = H_j, t_j^{h_j} = c_j, i \in I, j \in J \rangle$$

with base H, stable letters t_i, and t_j, and associated pairs $(H_i, \overline{H}_i)$ and (H_j, H_j), $i \in I, j \in j$, of subgroups of H, where

$$A_i \cap H = H_i, B_i \cap H = \overline{H}_i, \text{and } C_j \cap H = H_j, I \in I, j \in J \ .$$

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أشباه الزمر HNN

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خلاصة

لقد تم في هذا البحث توسيع فئات الزمر HNN إلى فئات جديدة من الزمر تسمى أشباه الزمر HNN. كذلك تم تعميم مبرهنة Britton للزمر HNN إلى أشباه الزمر HNN.