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A simple unreliable service model characterizes exponential distribution

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ABSTRACT

A new characterization of the exponential distribution via invariant property of probability distributions of the execution time in queuing models with an unreliable server, and no more than two interruptions of service are discussed. The original conditions of known similar characterizations have been weakened.

Keywords: Characterization theorem; lack of memory property; NBU and NWU families; occupation time; reliability; unreliable server.

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INTRODUCTION AND NOTATIONS

Characterization of exponential distribution has drawn a great deal of attention in the literature. Since Galambos and Kotz (1978), new properties based on processes and sampling features have been shown to be equivalent to the well-known lack of memory property. Recently Dimitrov and Khalil (1990) derived a new characterization using a queuing model with an unreliable server. They considered a single server queuing system in which, while the jobs are processed, the server is subject to failures. The failures are repaired instantaneously, and interrupted service is started anew and ends whenever a requested service period is found free from failures.

Let Y be the originally required service time of a job whose cumulative distribution function (c.d.f.) is denoted by $F_Y(y)$. That is, $F_Y(y) = P(Y \leq y)$. When breakdowns occur in $(0, t]$, Dimitrov and Khalil (1990) consider them occurring at times $0 < t_1 < t_2 < \ldots < t_n < t$, given from a point process, $\{N_t\}$, satisfying $P(N_t < \infty) = 1$ for each finite t. The variable N_t itself is the number of breakdowns on the interval $(0, t]$. The job enters service at $t_0 = 0$, and if interrupted, starts anew. This means that there is a sequence of independent identically distributed (iid) random variables (r.v.'s) $\{Y_n\}_{n=0}^{\infty}$ with the c.d.f. of Y. When the server breaks at t_k it is repaired instantaneously, the interrupted job is immediately restarted, and will require service time Y_k independently of what has happened before. This service will be completed without interruption if $Y_k < t_{k+1} - t_k$. Let $X_k = t_k - t_{k-1}$ denote the life time of the server between $(k-1)$-st and k-th breakdowns. If we assume that $\{X_k\}_{k=1}^{\infty}$ form a sequence of iid r.v.'s with c.d.f. given by $H(x)$, then $\{N_t\}$ will be the renewal process associated with the random variable X whose c.d.f. is $H(x)$, and $\{N_t\}$ is called also a counting process associated with the sequence $\{X_k\}$. The *total processing time* (or *occupation time*), denoted by T, for the job is defined as the time from the first start $t_0 = 0$ until the job is completed, i.e. $T = X_1 + X_2 + \ldots + X_{\nu-1} + Y_\nu$, where the variable $\nu = \min\{n : [Y_1 \geq X_1] \cap [Y_2 \geq X_2] \cap \ldots \cap [Y_{n-1} \geq X_{n-1}] \cap [Y_n < X_n]\}$. Let $G(t) = P(T > t)$ be the survival function of the r.v. T. Dimitrov and Khalil (1990) have shown, under the assumption that $\{N_t\}$ is a Poisson process with any rate $\lambda > 0$, that T and Y have the same probability distribution if and only if this distribution is the exponential. Van Harn and Steutel (1991) gave an explicit consideration for this characterization by relaxing the conditions from the stronger "for any $\lambda > 0$" to the weaker "a single value λ_0 of the failure rate, $\lambda_0 > 0$". Subsequently, Huang and Shoung (1993) have shown that this characterization is still true under the assumption that $\{N_t\}$ is a general renewal process (not necessarily Poisson), and its underlying $H(x)$ is a continuous c.d.f. Lin (1993) remarked that the characterization of Dimitrov and Khalil is also valid either when $\mathrm{E}(T) = \mathrm{E}(Y)$ or when $\mathrm{E}(T^2) = 2[\mathrm{E}(T)]^2$. Recently, Galambos and Hagwood (1994) established similar characterization of the exponential distribution under the following assumptions, named *failure scenarios*:

(FS)(i) The times $\{t_k\}$ are non-random, infinitely many, taken from the set of all nonrandom sequences (where t_1 may be arbitrarily close to 0) [the comment is ours], and $t_k \to \infty$ for each sequence in use.

(FS)(ii) The times $t_k = \tau_k$ are random, infinitely many, do not depend on service, τ_1 may be arbitrarily close to 0 with positive probability, and $P(N_t < \infty) = 1$ for each finite t.

An additional assumption that $G(t) = 1 - F_Y(t)$ has right derivative at zero is also made. In other words, they assume that the process $\{N_t\}$ is generated by any arbitrary sequence $\{X_n\}_{n=0}^{\infty}$, non-random for (FS)(i) and random for (FS)(ii), which may have infinitely many points located arbitrarily close to zero (no need for $\{N_t\}$ to be a renewal process). These conditions imply that the server may have arbitrarily short life times, and this is the key to their proofs.

In this note we show that similar characterization of the exponential distribution is true by relaxing the assumptions of Galambos and Hagwood (1994). It is enough to assume, in their scenario (FS)(i), a point process with just two points of specific nature $0 < t_1 < t_2 < \infty$, or one random point (one continuous variable, X_1) in their scenario (FS)(ii), where $P(X_2 = \infty) = 1$. Our additional assumption is that the function $F_Y(t)$ is right continuous at zero.

To avoid possible misinterpretation for the involved time points, we will use notations a and b instead of t_1 and t_2, and specify the models considered here of the point process $\{N_t\}$ in the following cases:

Case 1. The process $\{N_t\}$ contains just two points, formed by deterministic variables $X_1 = a$, $X_2 = b - a$ (since $X_3 = \infty$), where a and b are two incommensurable numbers. That is, a and b are two positive numbers such that the ratio a/b is an irrational number, and these are only points of the process $\{N_t\}$.

Case 2. The process $\{N_t\}$ contains only one random point X_1, since the second variable is $X_2 = \infty$ with probability 1.

More specifically, we assume that the service process for any new job obeys one of the following two failure scenarios:

FS1: The job processing starts at $t_0 = 0$. The server failures (or interruptions) occurring on $(0, \infty)$ are only at the two non-random times, $t_1 = a$ and $t_2 = b$ after the process starts, where $0 < a < b < \infty$, with the property that a and b are incommensurable numbers. If an interruption occurs, the service is immediately restarted as a new job. One job processing may have at most two interruptions. Occupation time ends when the job is successfully completed without interruption.

FS2: The job processing starts at $t_0 = 0$. The server may fail just once during the total processing time. The time to failure $X_1 > 0$, has a continuous distribution function $H(x)$ strictly increasing on $[0, \infty)$, and X_1 is independent of the service times before and after the failure. If interrupted, the service is restarted as for a new job and no more interruptions will occur for this particular service. Occupation time ends when the service is successfully done.

The practical motivation for the above two scenarios is as follows: in the first case, we pick two time points, a and b, with the property to be incommensurable, and give two chances for the job to be finished. At these two time points the server will be refreshed instantaneously (as another identical server) and the service (if not completed prior to this failure) will start anew. After the second interruption the service will be completed without any interruptions (no more hazards). In the second scenario, at the start of processing a new job, a clock with a random life time (given by X_1) is also started. If the clock fails before the processing is over, the ongoing service is interrupted, the server is refreshed instantaneously, and the job processing is restarted with no further interruptions.

We show that the total processing time is not affected by the possible service interruptions under either one of the two failure scenarios if and only if the originally requested service time Y is exponentially distributed.

THE RESULTS

Our analysis is based on a long-acknowledged fact from Carl G. Jacobi (1804–1851), concerning a real-valued periodic function. (Its original formulation and proof can be found in Jacobi (1882), II, 25–26). Here we state this main proposition in the terms needed for our purposes.

Proposition 1. Let $a > 0, b > 0$ be two incommensurable real numbers. If a real-valued function $g(t)$ defined on $[0, \infty)$ is right-continuous at the zero and is periodic of period a as well as of period b, then $g(t)$ is a constant function $g(t) = g(0)$ for all $t \geq 0$.

Proof. It is worthy to note some steps of the proof. First, observe that if $g(t)$ is periodic of period a, it is also periodic of period ka for any choice of the integer $k \geq 1$. Second, observe that if $g(t)$ is periodic of period ka, and on the other hand periodic of period mb with an arbitrary choice of integers $k \geq 1$, $m \geq 1$, then $g(t)$ is also periodic of period $ka - mb$, and the latter could be made arbitrarily small, due to the properties of the incommensurable real numbers.

Now we turn to our specific considerations. We write $T =^d Y$ to state that two r.v.'s T and Y have the same probability distribution, i.e., that $P(T \leq t) = P(Y \leq t)$ for any $t \in (-\infty, \infty)$.

Case 1. Consider the first scenario FS1. During a service at most two failures may occur at certain non-random times a and b, where $a < b$.

Proposition 2. Let a single server queuing system have a job requiring service time Y, whose probability distribution is right continuous at 0. The server is non-reliable according to FS1. The server's occupation time T by the job coincides in distribution with the required processing time Y (i.e., the equality $T =^d Y$ holds) if and only if one of the following two cases holds:

(i) Either $P(Y \leq a) = 1$, or $P(Y \leq a) = 0$, and then $P(Y = \infty) = 1$,

or

(ii) $0 < P(Y \leq a) < 1$, and then Y is exponentially distributed.

Proof. *Necessity:* First we prove that when (i) or (ii) holds, then we have $T =^d Y$.

Denote by Y_0, Y_1 and Y_2 the requested processing times at the start, and the possible first and second interruptions correspondingly. The value $a > 0$ is fixed in FS1. Thus, if $P(Y \leq a) = 1$, then the service will always be completed before the first breakdown of the server occurs, and hence $T =^d Y$. If $P(Y = \infty) = 1$, same as Y_0, Y_1, and Y_2, any service will need an infinite amount of time. The first two processes will be interrupted and the third one, started at point b, will be not. Adding an infinite amount of time Y_2 to the already used time, namely b, we get the total service time $T + Y_2 + b = \infty$. Thus, we have $P(T > t) = P(Y > t) = 1$ for all $t \geq 0$, without the exponential assumption.

Let us now assume that the required service time has exponential distribution. Then the service times of any other attempt for service Y_0, Y_1 and Y_2, are also exponentially distributed with the same parameter, say μ. Therefore, we have

$$P(Y_i > t) = e^{-\mu t}, \quad t \geq 0, \quad i = 0, 1, 2. \tag{1}$$

We are interested in the distribution of the r.v. T, the total processing time of the job, under the FS1 working conditions. When there are no failures in $(0, t], (t \leq a)$ we simply have

$$P(T > t) = P(Y > t) = \exp(-\mu t), \quad 0 < t \leq a. \tag{2}$$

When server failures may occur in $(0, t]$ for the failure scenario FS1, the non-random failure mode, we consequently have

$$P(T > t) = \begin{cases} P(Y_0 > t) = e^{-\mu t}, & \text{if} \quad t \leq a; \\ P(Y_0 > a,\, Y_1 > t - a) = e^{-\mu t}, & \text{if} \quad t \in (a, b]; \\ P(Y_0 > a,\, Y_1 > b - a,\, Y_2 > t - b) = e^{-\mu t}, & \text{if} \quad t > b. \end{cases} \tag{3}$$

We show the third case of the above equation. We use (1) and the independence. If

$t > b$, then

$$\begin{aligned} P(T > t) &= P(Y_0 > a, Y_1 > b - a, Y_2 > t - b) \\ &= P(Y_0 > a)P(Y_1 > b - a)P(Y_2 > t - b) \\ &= e^{-\mu a}e^{-\mu(b-a)}e^{-\mu(t-b)} = e^{-\mu t}. \end{aligned}$$

The other proof is similar. That is, for any $t \geq 0$, e have $P(T > t) = P(Y > t)$ for whichever of the cases (i), or (ii) is valid.

Sufficiency: Let it be fulfilled that $T =^d Y$ under the FS1 model. The inequalities

$$0 < P(Y > a) < 1 \tag{4}$$

mean that with positive probability at least one interruption will be observed in this case, and also that there is a positive chance for the service to be completed on the first (or second) attempt. Then this will correspond to case (ii) in our statement. In view of (4), we will prove that any service duration also can be observed, and also we show what the two extreme cases $P(Y \leq a) = 1$, and $P(Y > a) = 1$ produce case (i) as a consequence.

Since Y_i's are independent, and the interruptions do not prolong the service, we have $T =^d Y_i$, $i = 0, 1, 2$. Detailed probability analysis of "what may happen beyond first interruption" shows that we also have the equation

$$P(T > t) = P(Y_0 > a, Y_1 > t - a) = P(Y_0 > a)P(T > t - a), \tag{5}$$

which holds for any $t > 0$. Notice that Eq. (5) is equivalent to

$$G(a)G(t - a) = G(t) \text{ for any } t > 0. \tag{6}$$

By setting in (6) $t = 2a, 3a, \ldots$, we obtain (by induction, and in view of (4)) that it is true

$$0 \leq G(na) = [G(a)]^n \leq 1. \tag{7}$$

It follows from the equations in (7), that if $G(a) = 0$, then $P(Y \leq a) = 1$ and the first case of statement (i) holds. If $G(a) = 1 = P(Y > a)$, then also $P(Y > na) = 1$ for any arbitrary n. This implies that $P(Y = \infty) = 1$, and the second case of statement (i) is then true.

To avoid these trivial cases in our further considerations we assume that strict inequalities in (4) hold. Then we show that the service time Y may have arbitrarily large values with positive probability, since always $G(na) = P(T > na) = P(Y > na) > 0$. Moreover, from Eqs. (6) and (7) we obtain the equalities.

$$G(t) = G\left(t - \left[\frac{t}{a}\right]a\right)(G(a))^{\left[\frac{t}{a}\right]}, \text{ for any } t > 0, \tag{8}$$

where $[x]$ denotes the integer part of number x.

Now we consider the case when two breakdowns occur. In this case, from the last equality in (3) we see that the function $G(t)$ also satisfies the equation

$$G(t) = G(t - b)G(b - a)G(a), \text{ for any } t > b. \tag{9}$$

If we put $t = b$ in (6) we get $G(b) = G(b-a)G(a)$. Substitute this expression into the right-hand side of (9) and obtain that the following equation is true

$$G(t) = G(t-b)G(b) \text{ for any } t > b. \tag{10}$$

Similarly to the derivation of (8), we obtain that for any $t > 0$ the equality

$$G(t) = G\left(t - \left[\frac{t}{b}\right]b\right)[G(b)]^{[\frac{t}{b}]} \tag{11}$$

is also true. From Eqs. (8) and (11) we conclude that the equalities

$$G(ka) = [G(a)]^k, \text{ and } G(mb) = [G(b)]^m$$

are true for any choice of the positive integers k and m. From the latter we obtain

$$\frac{\ln G(a)}{a} = \frac{\ln G(ka)}{ka}; \quad \frac{\ln G(b)}{b} = \frac{\ln G(mb)}{mb}. \tag{12}$$

For any $t > 0$ there exist two numbers k and m such that it is fulfilled

$$ka \le t < (k+1)t, \text{ and } mb \le t < (m+1)b.$$

Since the left-hand sides for each of the equations in (12) are finite, and $G(t) > 0$ for any t is monotonic (therefore $\ln G(t)$ is monotonic), we get the limits

$$\lim_{t\to\infty} \frac{\ln G(t)}{t} = \lim_{k\to\infty} \frac{\ln G(ka)}{ka} = \lim_{m\to\infty} \frac{\ln G(mb)}{mb}.$$

Moreover, in view of (12) we have

$$\lambda = \frac{\ln G(a)}{a} = \frac{\ln G(b)}{b} < 0. \tag{13}$$

Consider the function

$$\alpha(t) = \ln G(t) - \lambda t.$$

Using Equation (6), it can easily be verified that

$$\begin{aligned}\alpha(t+a) &= \ln G(t) + \ln G(a) - \lambda t - \lambda a \\ &= \alpha(t) + a\left(\frac{\ln G(a)}{a} - \lambda\right) = \alpha(t),\end{aligned}$$

for any $t > 0$. That is, the function $\alpha(t)$ is periodic with period a. Similarly, using Eq. (10) we obtain

$$\begin{aligned}\alpha(t+b) &= \ln G(t) + \ln G(b) - \lambda t - \lambda b \\ &= \alpha(t) + b\left(\frac{\ln G(b)}{b} - \lambda\right) = \alpha(t),\end{aligned}$$

therefore, $\alpha(t)$ is also periodic with period b. The two numbers a and b are incommensurable, and $G(t)$ is supposed right continuous at 0. Thus, from

Proposition 1, it follows that $\alpha(t)$ is a constant function. That is,

$$\alpha(t) = \alpha(a) = \alpha(b) = \alpha(0) = 0,$$

implying that $\ln G(t) = \lambda t$, or $G(t) = e^{\lambda t}, \lambda < 0$, i.e., Y has an exponential distribution.

Remark. The following example shows that the requirement that a and b are incommensurable is essential. We will show that there are continuous non-exponential distributions of Y, for which $P(T > t) = P(Y > t)$ for the model scenario FS1.

Example. Let the p.d.f. of Y be given by the expression

$$g(t) = \beta^{[\frac{t}{a}]}(1-\beta)g_0\left(t - \left[\frac{t}{a}\right]a\right), \quad 0 < \beta < 1,$$

where $g_0(y)$ is the p.d.f. of any random variable Y_0 satisfying the condition $P(0 \leq Y_0 < a) = 1$. Then the survival function of Y is

$$\begin{aligned} G(t) &= 1 - \int_0^t g(t)dt = 1 - \sum_{k=0}^{[\frac{t}{a}]-1} \int_{ka}^{(k+1)a} \beta^k (1-\beta) g(t-ka)dt \\ &\quad + \int_{[\frac{t}{a}]a}^{t} \beta^{[\frac{t}{a}]}(1-\beta)g_0\left(t - \left[\frac{t}{a}\right]a\right)dt \\ &= 1 - \sum_{k=0}^{([\frac{t}{a}]-1)a} \beta^k(1-\beta) + \beta^{[\frac{t}{a}]}(1-\beta)\int_0^{t-[\frac{t}{a}]a} g_0(u)du \\ &= \beta^{[\frac{t}{a}]}\left(\beta + (1-\beta)\left(1 - G_0\left(t - \left[\frac{t}{a}\right]a\right)\right)\right) \\ &= \beta^{[\frac{t}{a}]}\left(G_0\left(t - \left[\frac{t}{a}\right]a\right)\right), \quad 0 < \beta < 1. \end{aligned}$$

Here $G_0(t) = P(Y_0 > t)$ stays for the survival function of Y_0 that satisfies the conditions $G_0(0) = 1, G_0(a) = 0$, and may be continuous on $[0, a)$.

Let the failures occur at times $t_1 = a$ and $t_2 = b = 3a$. Then, using the fact that

$$k + \left[\frac{t - ka}{a}\right] = \left[\frac{t}{a}\right], \text{ for any } t \geq ka, \quad k = 1, 2, \ldots,$$

it can readily be verified that Eq. (3) is also satisfied by $G(t)$.

Case 2. Now consider the second scenario FS2. Suppose we have one failure that occurs at random time X. It holds.

Proposition 3. Suppose that in a single server queuing system a job requires service time Y with continuous probability distribution. The service is non-reliable and follows FS2. The total occupation time by the job, T, may have the same probability distribution as the required processing time, Y, (i.e., $T =^d Y$) if and only if Y has an exponential distribution.

Proof. *Necessity*. Denote by Y_1 and Y_2 the requested service times in the first, and the possible second start of the job processing. Both variables are iid, with exponential distribution of some parameter $\mu > 0$ as the distribution of Y, and are independent of the time to failure, X. Conditioning on the values of X, by the total probability rules, we get

$$\begin{aligned} P(T > t) &= P(Y_1 > t, X > t) + P(Y_1 > X, X + Y_2 > t) \\ &= P(Y_1 > t)P(X > t) + \int_0^t P(Y_1 > x)P(Y_2 > t - x)dH(x) \\ &= e^{-\mu t}P(X > t) + \int_0^t e^{-\mu x}e^{-\mu(t-x)}dH(x) \\ &= e^{-\mu t}P(X > t) + e^{-\mu t}P(X \le t) = e^{-\mu t}. \end{aligned} \tag{14}$$

Thus, the "if" part of the proposition is proved.

Sufficiency: Let it be true that $T =^d Y$, i.e., $G(t) = P(Y > t) = P(T > t)$ for arbitrary $t > 0$. With the assumption that X has an absolutely continuous c.d.f on $[0, \infty)$, one can easily see, from the first equation in (14), that the following relation

$$\int_0^t [G(u)G(t-u) - G(t)]dH(u) = \int_0^t [G(u)G(t-u) - G(t)]h(u)du = 0 \tag{15}$$

must be true for any $t > 0$. With the assumption that X has a continuous distribution which is strictly increasing on the interval $[0, \infty)$, we have that $h(t) = H'(t) > 0$ holds for all $t > 0$. We will prove that (15) implies

$$G(u)G(t-u) = G(t) \text{ for all } 0 \le u \le t, \text{ and all } t > 0.$$

This is the usual *lack of memory* property that characterizes the exponential distribution.

Let

$$\theta = inf\{t;\ t \ge 0,\ G(t) \ne G(t-u)G(u),\ \text{for all } u,\ 0 \le u \le t\},$$

and assume that $\theta < \infty$. Then, by the continuity of $G(t)$, there exists some $\varepsilon > 0$ such that $r(t, u) := G(t) - G(t-u)G(u)$ is either constantly positive or constantly negative for any $t \in (\theta, \theta + \varepsilon)$. Thus, for any value of $t \in (\theta, \theta + \varepsilon)$ we have (since there is at least one u for which $h(u) > 0$)

$$\begin{aligned} \textstyle\int_0^t r(t, u)h(u)du &= \textstyle\int_0^\theta r(t, u)h(u)du + \int_\theta^t r(t, u)h(u)du \\ &= \textstyle\int_\theta^t r(t, u)h(u)du \ne 0, \end{aligned}$$

which contradicts Eq. (15).

We conclude with a result, that is in some sense an improvement of Theorem 3 in Huang and Shoung (1993). They proved that if the required service time Y belongs to the class of NBUE (or NWUE) distributions, i.e., when $\int_t^\infty (1 - F_Y(t)dt \le (\ge) \mathrm{E}(Y)(1 - F_Y(t)$ for any $t \ge 0$, then under a scenario of failures that form a renewal process, the equality $\mathrm{E}(T) = \mathrm{E}(Y)$ implies that Y has an exponential distribution. We drop the requirement for the possible infinitely many breakdowns during a service, but narrow the class to NBU (or NWU) distributions, where the property

$$P(Y \geq s + y) \leq (\geq) P(Y \geq x) P(Y \geq y)$$

holds for any non-negative values of x and y.

Proposition 4. Assume that the service follow FS2, and the server's life-time distribution function $H(x)$ contains at least two incommensurable points of increase, $a > 0, b > 0$, i.e., $dH(a) > 0$ and $dH(b) > 0$ hold. Furthermore, assume that the survival function $G(y)$ of the required service time Y is continuous with $\mathrm{E}(Y) < \infty$ and belongs to either one of the classes NBU, or NWU, of probability distributions. Then the total occupation time T by the job has the same average as the originally requested service time Y, i.e., $\mathrm{E}(T) - \mathrm{E}(Y)$ if and only if Y is exponentially distributed.

Proof. The sufficiency portion is already proved in Proposition 3, since when Y has exponential distribution then the same distribution has T. We prove the sufficiency, that $\mathrm{E}(T) = \mathrm{E}(Y)$ implies exponential distribution of Y. From the definition of T in the FS2 mode, we obtain (please, refer to the second line of Eq. (14))

$$P(T > t) = G(t)(1 - H(t)) + \int_0^t [G(u)G(t-u) - G(t)]dH(u) = 0. \tag{16}$$

Since T and Y are non-negative random variables, using the expressions

$$\mathrm{E}(T) = \int_0^{\infty} P(T > t)dt \text{ and } \mathrm{E}(Y) = \int_0^{\infty} P(Y > t)dt = \int_0^{\infty} G(t)dt$$

in an integration of Equation (16) from 0 to ∞, we get

$$\mathrm{E}(T) = \mathrm{E}(Y) + \int_0^{\infty} \int_0^t G(t)G(t-u)dH(u)dt - \int_0^{\infty} \int_0^t G(t)dH(u)dt.$$

If $\mathrm{E}(T) = \mathrm{E}(Y)$, then the above equality implies that the equation

$$\int_0^{\infty} \left(\int_0^t (G(t)G(t-u) - G(t))dH(u) \right) dt = 0 \tag{17}$$

holds. Notice that the difference $(G(u)G(t-u) - G(t))$ is either non-negative (for $G(t)$ in the NBU class) or non-positive (for $G(t)$ in the NWU class). Thus, Eq. (17) implies that

$$G(t) = G(u)G(t-u) \text{ for any } u \text{ with } dH(u) > 0, \text{ for all } t > u.$$

According to our assumption, there are at least two incommensurable points $a > 0$ and $b > 0$, where

$$G(t) = G(a)G(t-a) \text{ and } G(t) = G(b)G(t-b), \text{ for any } t > \max(a, b).$$

By using the statement of Proposition 2 for the FS1 scenario, the stated result follows as per the concluding remarks.

CONCLUDING REMARKS

Characterizing properties of the exponentially distributed r.v. may have many faces. The non-reliable service under specific conditions provides simple feasible features which can be used for recognition of exponentially distributed random variables. In this paper, by using a single server queue with simple failure modes (with one, or at most two failures), we obtain another characterization of the exponent. Similar characterization can also be carried out for the geometric distribution.

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نموذج مبسط للخدمة الغير معتد عليها لتمييز التوزيع الأسي

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قسم الرياضيات والعلوم– جامعة كيترنج

خلاصة

يتم مناقشة تمييز جديد للتوزيع الأسي بواسطة خاصية ثابتة لتوزيعـــات الاحتمـــال لزمـــن التنفيذ في نماذج الطوابير وذلك لخدمة غير موثوق بها ولم يناقش أكثر من قاطعين للخدمـــــة. وقد تم إضعاف الشروط الأصلية للمميزات المشابهة المعروفة.