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Oscillation behavior of second order nonhomogenuous superlinear differential equations

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ABSTRACT

In this paper. we study the oscillation behavior of the second order suplinear differential equation of the form $(r(t)\Psi(x)\dot{x}(t)) + m(t)f(x)\dot{x}(t) + Q(t,x) = p(t)$. The results obtained improve and extend earlier results of particular cases of this equation.

1. INTRODUCTION

The aim of this paper is to study the sufficient conditions for the oscillation of the solutions of the second order superlinear differential equation of the form

$$(r(t)\Psi(x)\dot{x}(t)) + m(t)f(x)\dot{x}(t) + Q(t,x) = p(t) \qquad (E)$$

where m, p and $r : r : [t_0, \infty) \to R$ are continuous functions and $r(t)$ is positive for all $t \geq t_0$. Ψ and $f : R \to R$ are continuous functions, $\Psi(t) > 0$ and $Q(t,x) : [t_0, \infty) \times R \to R$ is a continuous function with $Q(t,x) \geq q(t)g(x)$ where $q(t)$ is a continuous function on $[t_0, \infty)$ and $g(x)$ is a continuous function on the real line R with $xg(x) > 0$ and $g'(x) \geq k > 0$ for all $x \neq 0$.

Throughout this paper, we restrict our attention to only the solutions of the differential equation (E) which exist on some ray $[T, \infty)$, where T may depend on the particular solution.

A solution $x(t)$ of the equation (E) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory. In the sequel we assume that: for $\varepsilon > 0$

$$(c_1) \int_{\pm\varepsilon}^{\pm\infty} \frac{1}{g(y)} dy < \infty. \quad (c_2) \int_{\pm\varepsilon}^{\pm\infty} \frac{f(y)}{g(y)} dy < \infty. \quad (c_3) \int_{\pm\varepsilon}^{\pm\infty} \frac{\Psi(y)}{g(y)} dy < \infty.$$

A function g that satisfies condition (c_1) is said to satisfy a superlinear condition.

It is of interest to discuss conditions on the alternating coefficient $q(t)$ which are sufficient for all solutions of (E) to be oscillatory.

An interesting case is that of establishing oscillation criteria for equation (E) and/or related equations, which involve the average behavior of the integral of the alternating coefficient q. As recent contributions to this study, we cite the papers of Butler (1977), Cecchi *et al.* (1992), Elabbasy (1994, 1995, 1997), Grace (1990, 1992), Philos & Purnaras (1992), Wong & Yeh (1992), Yeh (1982), and the references cited there. The results which are presented in this paper extend and improve Butler's result (1977), Yeh (1982), and Philos (1985).

2. MAIN RESULTS

Theorem 2.1. *Let ρ be a positive continuous function on $[t_0, \infty)$ with $\dot{\rho}(t) \geq 0$ and the function $\rho(t)r(t)$ is nonincreasing on $[t_0, \infty)$. Suppose that conditions (c_2) and (c_3) hold, and $m(t)$ is a positive nonincreasing function on $[t_0, \infty)$. If,*

$$(c_4) \limsup_{t\to\infty} \left[\int_{t_0}^{t} \rho(s)ds\right]^{-1} \int_{t_0}^{t} \rho(s) \int_{t_0}^{s} \left(q(t) - \frac{|p(\tau)|}{M}\right) d\tau\, ds = \infty,$$

where

$$M = g(\theta) \quad \text{and} \quad \theta = \begin{cases} \max_{t\geq t_0} x(t), & \text{if } x(t) < 0, \\ \min_{t\geq t_0} x(t), & \text{if } x(t) > 0, \end{cases}$$

then every solution of (E) satisfies

$$\liminf_{t\to\infty} |x(t)| = 0.$$

Proof. Let $x(t)$ be a solution of (E) such that $\liminf_{t\to\infty} |x(t)| > 0$. Then, clearly, $x(t)$ is nonoscillatory. Without loss of generality, we can assume that $x(t) \geq \theta > 0$ for all $t \in [t_0, \infty)$. We define the function $w(t)$ as

$$w(t) = \rho(t) \int_{t_0}^{t} \frac{r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))} ds, \quad t \geq t_0.$$

Thus,

$$\dot{w}(t) = \dot{\rho}(t) \int_{t_0}^{t} \frac{r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))} ds + \frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}. \tag{2.1}$$

Equation (E) implies

$$(r(t)\Psi(x(t))\dot{x}(t)) \leq p(t) - m(t)f(x(t))\dot{x}(t) - q(t)g(x(t)).$$

Dividing by $g(x(t))$ and then integrating from t_0 to t, we obtain

$$\int_{t_0}^{t} \frac{r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))} ds \leq \int_{t_0}^{t} \frac{p(s)}{g(x(s))} ds - \int_{t_0}^{t} \frac{m(s)f(x(s))\dot{x}(s)}{g(x(s))} ds - \int_{t_0}^{t} q(s)ds.$$

By Bonnet's theorem, there exists a $\lambda \in [t_0, t]$ such that

$$-\int_{t_0}^{t} \frac{m(s)f(x(s))\dot{x}(s)}{g(x(s))} ds = -m(t_0) \int_{x(t_0)}^{x(\lambda)} \frac{f(y)}{g(y)} dy$$
$$< \begin{cases} 0 & \text{if } x(\lambda) > x(t_0) \\ m(t_0) \displaystyle\int_{\theta}^{x} \frac{f(y)}{g(y)} dy & \text{if } x(\lambda) < x(t_0), \end{cases}$$

where θ is defined as above. Hence

$$\frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} \leq \int_{t_0}^{t} \left[\frac{|p(s)|}{M} - q(s) \right] ds + A,$$

where

$$A = m(t_0) \int_{\theta}^{\infty} \frac{f(y)}{g(y)} dy + \frac{r(t_0)\Psi(x(t_0))\dot{x}(t_0)}{g(x(t_0))}.$$

Hence,

$$\frac{\rho(t)r(t)\Psi(x(t))\dot{x}(t_0)}{g(x(t))} \leq A\rho(t) + \rho(t) \int_{t_0}^{t} \left[\frac{|p(s)|}{M} - q(s) \right] ds. \tag{2.2}$$

From (2.2) and (2.1), we obtain

$$\dot{w}(t) \leq \dot{\rho}(t) \int_{t_0}^{t} \frac{r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))} ds + A\rho(t) + \rho(t) \int_{t_0}^{t} \left[\frac{|p(s)|}{M} - q(s) \right] ds.$$

Integrating from t_0 to t, we obtain

$$w(t) \leq w(t_0) + \int_{t_0}^{t} \dot{\rho}(s) \int_{t_0}^{s} \frac{r(\tau)\Psi(x(\tau))\dot{x}(\tau)}{g(x(\tau))} d\tau\, ds + A \int_{t_0}^{t} \rho(s) ds$$
$$+ \int_{t_0}^{t} \rho(s) \int_{t_0} \left[\frac{|p(\tau)|}{M} - q(\tau) \right] d\tau\, ds.$$

This implies

$$\int_{t_0}^{t} \rho(s) \int_{t_0}^{s} \left[q(\tau) - \frac{|p(\tau)|}{M} \right] d\tau\, ds \leq w(t_0) + A \int_{t_0}^{t} \rho(s) ds - \int_{t_0}^{t} \frac{\rho(s)r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))} ds.$$

By Bonnet's theorem, there exists $\zeta \in [t_0, t]$ such that

$$-\int_{t_0}^{t} \frac{\rho(s)r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))} ds = -\rho(t_0)r(t_0) \int_{x(t_0)}^{x(\zeta)} \frac{\Psi(y)}{g(y)} dy.$$

Consequently,

$$\int_{t_0}^{t} \rho(s) \int_{t_0}^{s} \left[q(\tau) - \frac{|p(\tau)|}{M} \right] d\tau\, ds \leq B + A \int_{t_0}^{t} \rho(s) ds,$$

where

$$B = w(t_0) + \rho(t_0)r(t_0)\int_{\theta}^{\infty}\frac{\Psi(y)}{g(y)}dy.$$

Then

$$\left[\int_{t_0}^{t}\rho(s)ds\right]^{-1}\int_{t_0}^{t}\rho(s)\int_{t_0}^{s}\left[q(\tau)-\frac{p(\tau)}{M}\right]d\tau\, ds \le A + \frac{B}{\displaystyle\int_{t_0}^{t}\rho(s)ds}.$$

The right-hand side is bounded as $t \to \infty$ which contradicts condition (c_4).

Example 2.1. Consider the differential equation

$$\left(\frac{1}{3t}x(t)\right) + \frac{1}{t^2}x(t) + tx^5(t) + \frac{x^2(t)+1}{t^2(t^2+1)} = \frac{2}{t^4}, \qquad t > 0.$$

By choosing $\rho(t) = 1$, all hypotheses of Theorem 2.1 hold. For example, such a solution of the given equation is $x(t) = 1/t$ that is not oscillatory with $\liminf_{t\to\infty}|x(t)| = 0$.

Theorem 2.2. *Assume that $\Psi'(x) \le 0$, if (c_2) and the following conditions hold:* *$(c_5)m(t)$ is a positive nondecreasing function on $[t_0, \infty)$,*

$$(c_6)\int_{t_0}^{\infty}\frac{1}{r(s)}ds = \infty,$$

$$(c_7)\liminf_{t\to\infty}\int_{t_0}^{t}\left[q(s)-\frac{p(s)}{M}\right]ds \ge -\lambda > -\infty, \qquad \lambda > 0,$$

$$(c_8)\limsup_{t\to\infty}\frac{1}{t}\left[\int_{t_0}^{t}r(s)ds\right]^{-1/2}\int_{t_0}^{t}\int_{t_0}^{s}\left[q(\tau)-\frac{|p(\tau)|}{M}\right]d\tau\, ds = \infty,$$

where M is defined in Theorem 2.1. Then the solutions of (E) are such that

$$\liminf_{t\to\infty}|x(t)| = 0.$$

Proof. On the contrary, let us assume that (E) has a solution $x(t)$ such that $t \to \infty \liminf |x(t)| \ne 0$. This means that $x(t)$ is nonoscillatory. Without loss of generality, we can assume that $x(t) \ge \theta > 0$ for all $t \in [t_0, \infty)$. Define

$$w(t) = \frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad \text{for } t \ge t_0.$$

Thus equation (E) implies that

$$\dot{w}(t) \leq \frac{p(t)}{g(x(t))} - \frac{m(t)f(x(t))\dot{x}(t)}{g(x(t))} - \frac{w^2(t)g'(x(t))}{r(t)\Psi(x(t))} - q(t).$$

By integrating from $t_1 \geq t_0$ to $t(> t_1)$, we obtain

$$w(t) \leq w(t_1) + \int_{t_1}^{t} \left(\frac{p(s)}{g(x(s))} - q(s) \right) ds - \int_{t_1}^{t} \frac{m(s)f(x(s))\dot{x}(s)}{g(x(s))} ds - \int_{t_1}^{t} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))} ds. \tag{2.3}$$

By Bonnet's theorem, there exists $\lambda \in [t_1, t]$ such that

$$-\int_{t_1}^{t} \frac{m(t)f(x(t))\dot{x}(s)}{g(x(s))} ds = -m(t_1) \int_{x(t_1)}^{x(\lambda)} \frac{f(u)}{g(u)} du = m(t_1) \int_{x(\lambda)}^{x(t_1)} \frac{f(u)}{g(u)} du.$$

Then

$$w(t) \leq A + \int_{t_1}^{t} \left(\frac{|p(s)|}{M} - q(s) \right) ds - \int_{t_1}^{t} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))} ds$$

where

$$A = w(t_1) + m(t_1) \int_{x(\lambda)}^{\infty} \frac{f(u)}{g(u)} du.$$

Thus, we obtain

$$\int_{t_1}^{t} \left(q(s) - \frac{|p(s)|}{M} \right) ds \leq A - w(t) - \int_{t_1}^{t} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))} ds. \tag{2.4}$$

Now we consider the following three cases of the behavior of $\dot{x}(t)$:

Case 1. $\dot{x}(t)$ is oscillatory. Then there exists a sequence $\{t_\nu\}_{\nu=1}^{\infty}$ in $[t_1, \infty)$ with $\lim_{\nu\to\infty} t_\nu = \infty$ such that $\dot{x}(t_\nu) = 0$, $\nu = 1, 2, 3, \ldots$ therefore, we obtain from the inequality (2.4)

$$k \int_{t_1}^{t} \frac{w^2(s)}{r(s)\Psi(x(s))} ds \leq \int_{t_1}^{t} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))} ds \leq A - \int_{t_1}^{t} \left(q(s) - \frac{|p(s)|}{M} \right) ds.$$

Then, by the condition (c_7), we have

$$\int_{t_1}^{\infty} \frac{w^2(s)}{r(s)\Psi(r(s))} ds \leq B,$$

for some positive constant B.

By the Schwarz inequality for $t \geq t_1$ we obtain

$$\left| -\int_{t_1}^{t} w(s)ds \right|^2 = \left| \int_{t_1}^{t} \sqrt{r(s)\Psi(x(s))} \frac{w(s)}{\sqrt{r(s)\Psi(x(s))}} ds \right|^2 \leq B \int_{t_1}^{t} r(s)\Psi(x(s))ds.$$

Since $\Psi(x(t))$ is nonincreasing, we have

$$\left| -\int_{t_1}^{t} w(s)ds \right|^2 \leq B\Psi(x(t_1)) \int_{t_1}^{t} r(s)ds. \tag{2.5}$$

Thus,

$$-\int_{t_1}^{t} w(s)ds \leq B_1 \left[\int_{t_1}^{t} r(s)ds \right]^{1/2}$$

where $B_1 = \sqrt{B\Psi(x)(t_1)}$. So, from (2.4), we have

$$\int_{t_1}^{t} \left(q(s) - \frac{|p(s)|}{M} \right) ds \leq A - w(t).$$

Thus,

$$\int_{t_1}^{t} \int_{t_1}^{s} \left(q(\tau) - \frac{|p(\tau)|}{M} \right) d\tau ds \leq A(t - t_1) - \int_{t_1}^{t} w(s)ds.$$

Then

$$\frac{1}{t} \left[\int_{t_1}^{t} r(s)ds \right]^{1/2} \int_{t_1}^{t} \int_{t_1}^{s} \left(q(\tau) - \frac{|p(\tau)|}{M} \right) d\tau ds \leq \frac{A\left(1 - \frac{t_1}{t}\right)}{\left[\int_{t_1}^{t} r(s)ds \right]^{1/2}} + \frac{B_1}{t}.$$

Hence, as $t \to \infty$, we get a contradiction to (c_8).

Case 2. $\dot{x}(t)$ is positive for all $t \in [T_1, \infty)$, $T_1 \geq t_0$. In this case the inequality (2.4) takes the form

$$\int_{T_1}^{t} \left(q(s) - \frac{|p(s)|}{M} \right) ds \leq A - \int_{T_1}^{t} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))} ds \leq A.$$

Then, we have a contradiction as in Case 1.

Case 3. $\dot{x}(t)$ is negative on $[T_2, \infty)$, $T_2 \geq t_0$. If the integral

$$\int_{T_2}^{\infty} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))} ds$$

is finite, then by using $g'(x) \geq k > 0$, we have a contradiction similar to Case 1.

Suppose that the above integral is infinite. From (2.4), and (c_7) we have

$$-w(t) \geq -A + \int_{T_1}^{t} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))} ds = c + \int_{T_2}^{t} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))} ds, \qquad (2.6)$$

where $c = -\lambda - A$. Since the integral in (2.6) is infinite as $t \to \infty$, we see that $w(t)$ is negative on $[T_3, \infty)$, $T_3 \geq t_2$. We choose $T \geq T_3$ such that

$$c + \int_{T}^{t} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))} ds = c_1 > 0.$$

Then for $t \geq T$, we have

$$w(t)\left\{c + \int_{Y}^{t} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))}\right\}^{-1} \leq -1.$$

Thus,

$$\frac{w^2(t)g'(x(t))}{r(t)\Psi(x(t))}\left\{c + \int_{Y}^{t} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))} ds\right\}^{-1} \geq -\frac{g'(x(t))\dot{x}(t)}{g(x(t))}.$$

By integrating from T to t, we get

$$\log \frac{c + \int_{T}^{t} w^2(s)g'(x(s))r(s)\Psi(x(s))ds}{c_1} \geq \log \frac{g(x(T))}{g(x(t))}, \qquad t \geq T.$$

Therefore,

$$c + \int_{T}^{t} \frac{w^2(s)g'(x(s))}{r(s)\Psi(x(s))} ds \geq \frac{c_2}{g(x(t))}$$

where $c_2 = c_1 g(x(T)) > 0$. Then from (2.6),

$$-w(t) \geq \frac{c_2}{g(x(t))}.$$

From the definition of $w(t)$, we obtain

$$\dot{x}(t) \leq \frac{-c_2}{r(t)\Psi(x(t))},$$

and integrating, we have

$$x(t) \leq x(T) - c_2 \int_{T}^{t} \frac{ds}{r(s)\Psi(x(s))} \leq x(T) - \frac{c_2}{\Psi(x(T))} \int_{T}^{t} \frac{ds}{r(s)}.$$

By condition (c_6) as $t \to \infty$, we have $x(t) \to -\infty$ contradicting that $x(t)$ is positive. This completes the proof.

If $m(t)$ fails to satisfy the condition $\dot{m}(t) \geq 0$, but is bounded, then we have the following results.

Corollary 2.1. *Let the condition* (c_5) *in Theorem 2.2 be replaced by* $(c_{5'})$ $m(t)$ *is bounded on* $[t_0, \infty)$.

Then the conclusion of Theorem 2.2 is still true.

Example 2.2. Consider the differential equation

$$\ddot{x}(t) + \cos t x(t)\dot{x}(t) + (t + \cos t)x^3(t) = \frac{2+t}{t^3}, \quad t > 0.$$

One such solution of the given equation is $x(t) = 1/t$ that is non-oscillatory with $\liminf_{t\to\infty} |x(t)| = 0$. We can also see that the conditions of Theorem 2.2 are satisfied but $m(t)$ is not monotonic for all $t > 0$. This means that Theorem 2.2 can not be applied.

Theorem 2.3. *Let* ρ *be a positive continuously differentiable function on the interval* $[t_0, \infty)$ *such that* $-\dot{\rho}(t) \geq 0$ *and* $\dot{\rho}(t)r(t)$ *is nonincreasing, and the functions* $(\rho(t)r(t))$ *and* $(\rho(t)m(t))$ *are positive nondecreasing on* $[t_0, \infty)$. *Assume that the conditions* (c_2) *and* (c_3) *hold, and* $p(t)$ *is bounded on* $[t_0, \infty)$. *If*

$$(c_9)\ \limsup_{t\to\infty} \frac{1}{t}\left[\int_{t_0}^{t}\int_{t_0}^{s} \rho(\tau)dt\, ds\right]^{-1} \int_{t_0}^{t}\int_{t_0}^{s} \rho(\tau)q(\tau)d\tau\, ds = \infty,$$

then (E) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (E). Assume that $x(t) \geq \theta > 0$ (the case $x(t) < 0$ can be treated similarly). We define the function $w(t)$ as

$$w(t) = \rho(t)\int_{t_0}^{t} \frac{r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))} ds, \quad t \geq t_0.$$

Since $p(t)$ is bounded: $|p(t)| < \delta$, and $g(x)$ is increasing, Eq. (E) implies

$$\begin{aligned}\ddot{w}(t) \leq \ddot{\rho}(t)\int_{t_0}^{t} \frac{r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))} ds + \frac{2\dot{\rho}(t)r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} - \rho(t)q(t)\\ + \frac{\delta\rho(t)}{g(\theta)} + \frac{\rho(t)m(t)f(x(t))\dot{x}(t)}{g(x(t))} - \frac{\rho(t)r(t)\Psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}.\end{aligned}$$

Hence,

$$\begin{aligned}\dot{w}(t) \leq \dot{w}(t_0) + \int_{t_0}^{t}\ddot{\rho}(s)\int_{t_0}^{s} \frac{r(t)\Psi(x(t))\dot{x}(t)}{g(x(t))} dt\, ds + 2\int_{t_0}^{t} \frac{\dot{\rho}(s)r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))} ds\\ + M_1\int_{t_0}^{t}\rho(s)ds - \int_{t_0}^{t}\rho(s)q(s)ds - \int_{t_0}^{t}\rho(s)m(s)\frac{f(x(s))}{g(x(s))}\dot{x}(s)ds\\ - k\int_{t_0}^{t} \frac{\rho(s)r(s)\Psi(x(s))\dot{x}^2(s)}{g^2(x(s))} ds \end{aligned} \tag{2.7}$$

where $M_1 = \delta/M$ and M is as defined in Theorem 2.1.

Now, by Bonnet's theorem, there exists $\lambda \in [t_0, t]$ such that

$$-\int_{t_0}^{t} \rho(s)m(s)\frac{f(x(s)\dot{x}(s)}{g(x(s))}ds = -\rho(t_0)m(t_0)\int_{x(t_0)}^{x(M)}\frac{f(y)}{g(y)}dy$$

where $A_1 = \int_\theta^\infty f(y)/g(y)dy$, $\theta = \min x(t)$, $t \geq t_0$. Since $(\dot{\rho}(t)r(t))$ is nonincreasing by Bonnet's theorem,

$$\int_{t_0}^{t}\frac{\dot{\rho}(s)r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))}ds = \dot{\rho}(t_0)r(t_0)\int_{x(t_0)}^{x(\zeta)}\frac{\Psi(y)}{g(y)}dy \leq \dot{\rho}(t_0)r(t_0)\int_{x(t_0)}^{\infty}\frac{\Psi(y)}{g(y)}dy = B_2 < \infty.$$

Since $(\rho(t)r(t))$ is nondecreasing on $[t_0, \infty)$ we get

$$-\int_{t_0}^{t}\frac{\rho(s)r(s)\Psi(x(s))\dot{x}^2(s)}{g^2(x(s))}ds \leq -\rho(t_0)r(t_0)\int_{t_0}^{t}\frac{\dot{x}^2(s)\Psi(x(s))}{g^2(x(s))}ds.$$

Since

$$\left[\int_{t_0}^{t}\frac{\dot{x}(s)\sqrt{\Psi(x(s))}}{g(x(s))}ds\right]^2$$

is positive, we choose $T \geq t_0$ such that

$$\left[\int_{T}^{t}\frac{\dot{x}(s)\sqrt{\Psi(x(s))}}{g(x(s))}ds\right]^2 \geq B > 0.$$

Now, by Schwarz inequality, we have

$$\left[\int_{T}^{t}\frac{\dot{x}(s)\sqrt{\Psi(x(s))}}{g(x(s))}ds\right]^2 \leq (t-T)\int_{T}^{t}\frac{\dot{x}^2(s)\Psi(x(s))}{g^2(x(s))}ds.$$

Then

$$B \leq \left[\int_{x(T)}^{x(t)}\frac{\sqrt{\Psi(y)}}{g(y)}ds\right]^2 \leq (t-T)\int_{T}^{t}\frac{\dot{x}^2(s)\Psi(x(s))}{g^2(x(s))}ds,$$

and this implies

$$-k\rho(t_0)r(t_0)\int_{T}^{t}\frac{\dot{x}^2(s)\Psi(x(s))}{g^2(x(s))}ds \leq \frac{-k\rho(t_0)r(t_0)B}{t-T}.$$

Consequently,

$$-k\int_{t_0}^{t}\frac{\rho(s)r(s)\Psi(x(s))\dot{x}^2(s)}{g^2(x(s))}ds \leq -k\rho(t_0)r(t_0)\int_{t_0}^{t}\frac{\dot{x}^2(s)\Psi(x(s))}{g^2(x(s))}ds \leq -\frac{B_3}{t-T} \leq -\frac{B_3}{t-t_0}$$

where $B_3 = B\rho(t_0)r(t_0)k$.

Hence (2.7) reduces to

$$\dot{w}(t) \leq \dot{w}(t_0) + \dot{\rho}(t)\int_{t_0}^{t}\frac{r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))}ds + B_2 + M_1\int_{t_0}^{t}\rho(s)ds - \int_{t_0}^{t}\rho(s)q(s)ds - \frac{B_3}{t-t_0} + B_1.$$

Hence,

$$w(t) \leq w(t_0) + \int_{t_0}^{t}\dot{\rho}(s)\int_{t_0}^{s}\frac{r(\tau)\Psi(x(\tau))\dot{x}(\tau)}{g^2(x(s))}d\tau\, ds + A_1(t-t_0) + M_1\int_{t_0}^{t}\int_{t_0}^{s}\rho(\tau)d\tau\, ds - \int_{t_0}^{t}\int_{t_0}^{s}\rho(\tau)q(\tau)d\tau\, ds - B_3\log(t-t_0),$$

where $A_1 = \dot{w}(t_0) + B_2 + B_1$. By integrating the first integral by parts, we have

$$\begin{aligned}\rho(t)\int_{t_0}^{t}\frac{r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))}ds \leq{}& w(t_0) + \left[\rho(s)\int_{t_0}^{s}\frac{r(\tau)\Psi(x(\tau))\dot{x}(\tau)}{g(x(\tau))}d\tau\right]_{s=t_0}^{t}\\ &- \int_{t_0}^{t}\frac{\rho(s)r(s)\Psi(x(s))\dot{x}(s)}{g(x(s))}ds + A_1(t-t_0)\\ &+ M_1\int_{t_0}^{t}\int_{t_0}^{s}\rho(\tau)d\tau\, ds - \int_{t_0}^{t}\int_{t_0}^{t}\rho(\tau)q(\tau)d\tau\, ds\\ &- B_1\log(t-t_0).\end{aligned}$$

Then, by Bonnet's theorem, for $\lambda \in [t_0, t]$, we have

$$\begin{aligned}\int_{t_0}^{t}\int_{t_0}^{s}\rho(\tau)q(\tau)d\tau\, ds \leq{}& w(t_0) + A_1(t-t_0) + M_1\int_{t_0}^{t}\int_{t_0}^{s}\rho(\tau)d\tau\, ds\\ &- \rho(t_0)r(t_0)\int_{x(t_0)}^{x(\lambda)}\frac{\Psi(u)}{g(u)}du - B_1\log(t-t_0)\\ \leq{}& B_4 + A_1(t-t_0) + M_1\int_{t_0}^{t}\int_{t_0}^{s}\rho(\tau)d\tau\, ds,\end{aligned}$$

where

$$B_4 = w(t_0) + \rho(t_0)r(t_0)\int_{0}^{\infty}\frac{\Psi(u)}{g(u)}du.$$

This implies

$$\frac{1}{t}\left[\int_{t_0}^{t}\int_{t_0}^{s}\rho(\tau)d\tau\, ds\right]^{-1}\int_{t_0}^{t}\int_{t_0}^{s}\rho(\tau)q(\tau)d\tau\, ds \leq \frac{B_4}{t\int_{t_0}^{t}\int_{t_0}^{s}\rho(\tau)d\tau\, ds} - \frac{A_1\left(1-\frac{t_0}{t}\right)}{\int_{t_0}^{t}\int_{t_0}^{s}\rho(\tau)d\tau\, ds} - \frac{B_1\log(t-t_0)}{t\int_{t_0}^{t}\int_{t_0}^{s}\rho(\tau)d\tau\, ds}.$$

The right-hand side of the above inequality becomes bounded as $t \to \infty$. That contradicts condition (c_9), and then the proof is completed.

Example 2.3. Consider the differential equation

$$(t^{-1/2}x^{3/4}\dot{x}(t)) + \left(e^t - \frac{1}{t}\right)\frac{x^{1/3}e^{x^2}\cos x}{2+\sin x}\dot{x}(t) + \frac{(te^t + 2e^t + t^2x^{1/3})x^{1/3}e^{x^2}}{t}$$
$$= \sin 2t + \frac{t}{1+t} \quad (D).$$

Choosing $\rho(t) = t$, thus all solutions of the equation (D) are oscillatory.

REFERENCES

Butler, G.J. 1977. The oscillatory behavior of a second order nonlinear differential equation with damping. Journal of Mathematics and Applications **57**: 273–289.

Cecchi, M., Marini, M. & Villari, G. 1992. Integral criteria for a classification of solutions of linear differential equations. Journal of Differential Equations **99**: 381–397.

Elabbasy, E. M. 1994. Oscillatory behavior of solutions of forced second order differential equation with alternating coefficients. Applied Mathematics and Computation **6**: 1–13.

Elabbasy, E. M. 1995. Oscillation criteria for second order superlinear differential equations. FAR EAST Journal of Mathematics **3(1)**: 210–225.

Elabbasy, E.M. 1997. Oscillation theorems for perturbed second order nonlinear differential equations with damping. SERDICA Mathematical Journal **23**: 1–14.

Grace, S.R. 1990. Oscillation criteria for second order differential equations with damping. Journal of the Australian Mathematical Society **49**: 43–54.

Grace, S.R. 1992. Oscillation theorems for nonlinear differential equations of second order. Journal of Mathematical Analysis and Application **171**: 220–241.

Philos, C.G. 1985. Integral averages and second order superlinear oscillation. Mathematische Nachrichten **120**: 127–138.

Philos, C.G. & Purnaras, I.K. 1992. On the oscillation of second order nonlinear differential equations. Archivum Mathematicum **159**: 260–271.

Wong, F.H. & Yeh, C.C. 1992. Oscillation criteria for second order superlinear differential equations. Mathematical Japonica **37**: 573–584.

Yeh, C.C. 1982. Oscillation theorems for nonlinear second order differential equations with damped term. Proceeding of American Mathematical Society **84**: 397–402.

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السلوك التذبذبي للمعادلات التفاضلية فوق الخطية الغير متجانسة

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خلاصة

تناولنا في هذا البحث دراسة السلوك التذبذبي للمعادلات الغير خطية (فوق خطيـــة) وهـــي على الصورة العامة ، وقد حصلنا على العديد من النتائج التي تحسن بعض النتائج السابقة فـــي هذا الموضوع ، وتعمم بعض النتائج التي تم الحصول عليها لمعادلات تفاضلية أقل شـــمولية من المعادلات التي تمت دراستها.