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A method on solving irregular boundary value problems with transmission conditions

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ABSTRACT

In this paper, we consider the boundary value problem with transmission conditions. The boundary value problem studied in this work includes eigenvalue parameter which is second degree in equation and first degree in one of the boundary conditions. We consider a different approach for the investigation of this boundary value problem which constructed on Regge problem.

Keywords: Boundary value problem, eigenvalue parameter, Regge problem, transmission conditions.

INTRODUCTION

The investigation of the boundary value problems for which the eigenvalue parameter appears in both the differential equation and the boundary condition originate from the works of Birkhoff (1908). The studies on the boundary value problem with eigenvalue parameter appear in many papers and books (see, for example, Akdogan, Demirci and Mukhtarov (2007), Kandemir, Mukhtarov and Yakubov Ya. (accepted 2009), Kandemir and Yakubov Ya. (accepted 2009), Mukhtarov (1994), Mukhtarov and Demir (1999), Mukhtarov , Kandemir and Kuruoğlu (2002), Mukhtarov and Kandemir (2002), Mukhtarov and Yakubov S. (2002), Yakubov S. (1994,1998, 2000), Yakubav Ya. (1998), Yakubov S. and Yakubav Ya(1999)).

Many important problems of mathematical physics deal with Birkhof-irregular (1908) boundary value problems. For example, the well-known Birkhof- irregular Regge problems (see, Regge (1958, 1963)) arises in the field of the quantum theory of scattering. We want to mention here the paper A. O. Krawiski (1968), Shkalikov (1983, 2001), S. Yakubov (2000) and Ya. Yakubov (1998) in which was investigated the irregular Regge Problem.

In the series of S. Yakubov and Y. Yakubov's, works have been constructed an abstract theory of boundary value problems with an eigenvalue parameters in the boundary condition (see, for example, Yakubov S. (1994,1998, 2000), Yakubav Ya. (1998), Yakubov S. and Yakubav Ya(1999)).

In this study we investigated the solution of the boundary value problem with discontinuous coefficient and transmission conditions at point zero in [-1,1] for Regge problem on which S. Yakubov has suggestion a new method about the solution in [0,1] (see, Yakubov S. (1998)).

The boundary value problem studied here differs from the standard boundary value problems, such that the studied boundary value problem contains eigenvalue parameter in one of the boundary conditions and two new conditions called transmission conditions.

It must be noted some special cases of the considered problem (1)-(5) arise after an application of the method of separation of variables to the varied assortment of physical problems. For examples, some boundary-value problems with transmission conditions arise in heat and mass transfer problems (see, for example, Livkov (1963), in vibrating string problems when the string loaded additionally with point masses (see, for example, Tikhonov (1963) and diffraction problems (see, for example, Voitovich, Katsenebaum and Sivov (1997)). Also, some problems with transmission conditions which arise in mechanics (thermal conduction problem for a thin laminated plate) were studied in the article Titeux and Yakubov Ya.(1997).

Some boundary value problems with discontinuous coefficients and transmission conditions dealing with spectral properties are investigated in the papers (Kandemir, Mukhtarov and Yakubov Ya. (accepted 2009), Kandemir and Yakubov Ya. (accepted 2009), Mukhtarov (1994), Mukhtarov and Demir (1999), Mukhtarov , Kandemir and Kuruoğlu (2002), Mukhtarov and Kandemir (2002)).

Statement of the problem

In this paper, we consider boundary value problem with transmission conditions constructed on Regge problem

$$L(\lambda, D)u := \lambda^2 u(x) - u''(x) + Q(x)u(x) = f(x), \quad x \in (-1,0) \cup (0,1) \tag{1}$$

$$L_1(\lambda)u := \lambda(u(-1) + u(+1)) + u'(-1) + u'(1) = f_1 \tag{2}$$

$$L_2 u := u(-0) = f_2 \tag{3}$$

$$L_3 u := u(+0) = f_3 \tag{4}$$

$$L_4 u := u(-1) + u(1) = f_4 \tag{5}$$

where, $f(x) = f_1(x)$, $Q(x) = Q_1(x)$ at $x \in [-1, 0)$ and $f(x) = f_2(x)$, $Q(x) = Q_2(x)$

at $x \in (0, 1]$ are given functions, and f_1, f_2, f_3, f_4 are given complex numbers. The conditions (3) and (4) are called the transmission conditions.

Below $W_p^k(a, b)$ is a usual Sobolev space of functions $u(x)$which have generalized derivatives up to the $k-$th order inclusive on (a, b) and the norm

$\|u\|_{W_p^k(a,b)} = \sum_{n=0}^{k} (\int_a^b |u^{(n)}(x)|^p dx)^{1/p}$ is finite.

We shall define the direct sum $W_q^k(-1, 0) \oplus W_q^k(0, 1)$ as

$$W_q^k(-1,0) \oplus W_q^k(0,1) = \left\{ u = \begin{cases} u_1(x), & for \quad x \in \ (-1,0) \\ u_2(x), & for \quad x \in \ (0,1) \end{cases} \middle| u_1 \in W_q^k(-1,0), \right.$$

$$\left. u_2 \in W_q^k(0,1), \|u\| = \|u_1\|_{W_q^k(-1,0)} + \|u_2\|_{W_q^k(0,1)} \right\}$$

Lemma.

Let $a > 0$, $Q_1 \in W_p^1(-1, 0)$, $Q_2 \in W_p^1(0, 1)$ where $1 < p \leq \infty$. Then

$$\int_{-1}^{0} e^{a\lambda(y+1)} Q_1(y)dy = -\frac{1}{a\lambda} Q_1(-1) + O(\frac{1}{|\lambda|^{2-1/p}}),$$

and

$$\int_{0}^{1} e^{-a\lambda(y-1)} Q_2(y)dy = -\frac{1}{a\lambda} Q_2(1) + O(\frac{1}{|\lambda|^{2-1/p}}),$$

for $\frac{\pi}{2} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \varepsilon$**,** $|\lambda| \to \infty$**.**

Proof.

For all $\lambda \in$ C we have by partial integration that

$$\int_{-1}^{0} e^{a\lambda(y+1)} Q_1(y)dy = -\frac{1}{a\lambda} Q_1(-1) + \frac{e^{a\lambda}}{a\lambda} Q_1(0) - \frac{1}{a\lambda} \int_{-1}^{0} e^{a\lambda(y+1)} Q_1'(y)dy. \tag{6}$$

It is obvious that for each $\varepsilon > 0$ there exist $C(\varepsilon) > 0$ such that

$$|e^{a\lambda}| = e^{aRe\lambda} \leq e^{-C(\varepsilon)|\lambda|} \tag{7}$$

for all $\frac{\pi}{2} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \varepsilon$, $|\lambda| \to \infty$.

Consequently

$$\left|\int_{-1}^{0} e^{a\lambda(y+1)}Q'(y)dy\right| \leq C\|Q\|_{W_p^1(-1,0)}\left(\int_{-1}^{0} e^{ap'(y+1)Re\lambda}dy\right)^{1/p'} \leq \frac{C(\varepsilon)}{|\lambda|^{1/p'}}, \quad (8)$$

where$1/p + 1/p' = 1$. Substituting (7) and (8) into (6) we obtain the first asymptotic expression of the Lemma. The other asymptotic expression can be found analogically (see, for example yakubov S. (1998)).

The Green's function of the principal part of the problem (1)-(5)

In this section we construct the Green function for the problem (for the definition of the Green's function see, for example, Naimark (1967)

$$L_0(\lambda, D)u := \lambda^2 u(x) - u''(x) = f(x) \quad (9)$$

$$A_{10}u := u(-1) = f_5 \quad (10)$$

$$A_{20}u := u(-0) = f_2 \quad (11)$$

$$A_{30}u := u(+0) = f_3 \quad (12)$$

$$A_{40}u := u(1) = f_6 \quad . \quad (13)$$

For $\lambda \neq 0$, the fundamental solutions of Eq.(9) are

$$u_1(x) := \begin{cases} e^{\lambda x}, & x \in [-1, 0) \\ 0\ , & x \in (0, 1] \end{cases}, \quad u_2(x) := \begin{cases} e^{-\lambda(x+1)}, & x \in [-1, 0) \\ 0\ , & x \in (0, 1] \end{cases},$$

$$u_3(x) := \begin{cases} 0\ , & x \in [-1, 0) \\ e^{\lambda x}, & x \in (0, 1] \end{cases}, \quad u_4(x) := \begin{cases} 0\ , & x \in [-1, 0) \\ e^{-\lambda(x-1)}, & x \in (0, 1] \end{cases}.$$

So, the general solution of the Eq. (9) we can write in the form

$$u(x) = c_1 u(x) + c_2 u(x) + c_3 u(x) + c_4 u(x) + \int_{-1}^{1} G(x, y, \lambda)f(y)dy \quad (14)$$

where

$$G(x, y, \lambda) := \begin{cases} a_{10}u_1(x) + a_{20}u_2(x), & x \in [-1, y) \\ b_{10}u_1(x) + b_{20}u_{20}(x), & x \in (y, 0) \end{cases}$$

for$y \in [-1, 0)$,

$$G(x, y, \lambda) := \begin{cases} a_{11}u_3(x) + a_{21}u_4(x), & x \in (0, y) \\ b_{11}u_3(x) + b_{21}u_4(x), & x \in (y, 1] \end{cases} \quad (15)$$

for $y \in (0,1]$and numbers c_1, c_2, c_3 and c_4 are any arbitrary constants. By substituting (14) in the conditions (10)-(13), we obtain a system for finding c_i, $i = 1,2,3,4$,

$$c_1 e^{-\lambda} + c_2 = f_5, \quad c_1 + c_2 e^{-\lambda} = f_2, \quad c_3 + c_4 e^{\lambda} = f_3, \quad c_3 e^{\lambda} + c_4 = f_6 \quad ,$$

$$D(\lambda) = \begin{vmatrix} e^{-\lambda} & 1 & 0 & 0 \\ 1 & e^{-\lambda} & 0 & 0 \\ 0 & 0 & 1 & e^{\lambda} \\ 0 & 0 & e^{\lambda} & 1 \end{vmatrix} = e^{2\lambda} + e^{-2\lambda} - 2 = (e^{-\lambda} - e^{\lambda})^2 \neq 0, \ \text{Im } \lambda \neq 0.$$

This system of equations has a unique solution

$$c_1 = \frac{e^{\lambda} f_5 - e^{2\lambda} f_2}{1 - e^{2\lambda}}, \quad c_2 = \frac{e^{\lambda} f_2 - e^{2\lambda} f_5}{1 - e^{2\lambda}}, \quad c_3 = \frac{f_3 - e^{\lambda} f_6}{1 - e^{2\lambda}}, \quad c_4 = \frac{f_6 - e^{\lambda} f_3}{1 - e^{2\lambda}}. \quad (16)$$

By using the definition of the Green's function we have (see, for example, Naimark (1967))

$$[a_{10} u_1(y) + a_{20} u_2(y)] - [b_{10} u_1(y) + b_{20} u_2(y)] = 0,$$

$$\left[a_{10} u_1'(y) + a_{20} u_2'(y)\right] - \left[b_{10} u_1'(y) + b_{20} u_2'(y)\right] = 1,$$

$$[a_{11} u_3(y) + a_{21} u_4(y)] - [b_{11} u_3(y) + b_{21} u_4(y)] = 0,$$

$$\left[a_{11} u_3'(y) + a_{21} u_4'(y)\right] - \left[b_{11} u_3'(y) + b_{21} u'_4(y)\right] = 1.$$

Then for $d_1 = b_{10} - a_{10}$, $d_2 = b_{20} - a_{20}$, $d_3 = b_{11} - a_{11}$ and $d_4 = b_{21} - a_{21}$ (17)

we obtain a system for finding d_i, $i = 1,2,3,4$,

$$d_1 u_1(y) + d_2 u_2(y) = 0, \quad d_1 u_1'(y) + d_2 u_2'(y) = -1,$$

$$d_3 u_3(y) + d_4 u_4(y) = 0, \quad d_3 u_3'(y) + d_4 u_4'(y) = -1.$$

Since the determinant of this system

$$\Delta(\lambda) = \begin{vmatrix} e^{\lambda y} & e^{-\lambda(y+1)} & 0 & 0 \\ \lambda e^{\lambda y} & -\lambda e^{-\lambda(y+1)} & 0 & 0 \\ 0 & 0 & e^{\lambda y} & e^{-\lambda(y-1)} \\ 0 & 0 & \lambda e^{\lambda y} & -\lambda e^{-\lambda(y-1)} \end{vmatrix} = 4\lambda^2 \neq 0, \quad \lambda \neq 0$$

we obtain that

$$d_1 = -\frac{e^{-\lambda y}}{2\lambda}, \quad d_2 = \frac{e^{\lambda(y+1)}}{2\lambda}, \quad d_3 = -\frac{e^{-\lambda y}}{2\lambda} \text{ and } d_4 = \frac{e^{\lambda(y-1)}}{2\lambda}. \quad (18)$$

Again by virtue of the definition Green's function we have

$$A_{10}G(.,y,\lambda)=0,\quad A_{20}G(.,y,\lambda)=0$$

$$A_{30}G(.,y,\lambda)=0,\quad A_{40}G(.,y,\lambda)=0.$$

Consequently

$$a_{10}e^{-\lambda}+a_{20}=0,\quad b_{10}+b_{20}e^{-\lambda}=0$$

$$a_{11}+a_{21}e^{\lambda}=0,\quad b_{11}e^{\lambda}+b_{21}=0. \tag{19}$$

From (17) and (19), we obtain the systems for finding b_{ij}, $i=1,2; j=0,1$

$$(b_{10}-d_1)e^{-\lambda}+b_{20}-d_2=0,\quad b_{10}+b_{20}e^{-\lambda}=0$$

and

$$b_{11}-d_3+(b_{21}-d_4)e^{\lambda}=0,\quad b_{11}e^{\lambda}+b_{21}=0.$$

It is easy to verify that

$$b_{10}=\frac{e^{\lambda(y+2)}-e^{-\lambda y}}{2\lambda(1-e^{2\lambda})},\quad b_{20}=\frac{e^{-\lambda(y-1)}-e^{\lambda(y+3)}}{2\lambda(1-e^{2\lambda})},$$

$$b_{11}=\frac{e^{\lambda y}-e^{-\lambda y}}{2\lambda(1-e^{2\lambda})},\quad b_{21}=\frac{e^{-\lambda(y-1)}-e^{\lambda(y+1)}}{2\lambda(1-e^{2\lambda})}, \tag{20}$$

$$a_{10}=\frac{e^{\lambda(y+2)}-e^{-\lambda(y-2)}}{2\lambda(1-e^{2\lambda})}, a_{20}=\frac{e^{-\lambda(y-1)}-e^{\lambda(y+1)}}{2\lambda(1-e^{2\lambda})},$$

$$a_{11}=\frac{e^{\lambda y}-e^{-\lambda(y-2)}}{2\lambda(1-e^{2\lambda})},\quad a_{21}=\frac{e^{-\lambda(y-1)}-e^{\lambda(y-1)}}{2\lambda(1-e^{2\lambda})}. \tag{21}$$

Putting (20) and (21) into (15), we obtain

$$G(x,y,\lambda)=\begin{cases}\dfrac{e^{\lambda(x+y+2)}-e^{-\lambda(x-y)}+e^{-\lambda(x+y)}-e^{-\lambda(y-x-2)}}{2\lambda(1-e^{2\lambda})}, & x\in[-1,y)\\[2ex] \dfrac{e^{\lambda(x+y+2)}-e^{-\lambda(y-x)}+e^{-\lambda(x+y)}-e^{-\lambda(x-y-2)}}{2\lambda(1-e^{2\lambda})}, & x\in(y,0)\end{cases}.$$

for$y\in[-1,0)$,

$$G(x,y,\lambda)=\begin{cases}\dfrac{e^{\lambda(x+y)}-e^{-\lambda(y-x-2)}+e^{-\lambda(x+y-2)}-e^{-\lambda(x-y)}}{2\lambda(1-e^{2\lambda})}, & x\in(0,y)\\ \dfrac{e^{\lambda(x+y)}-e^{-\lambda(x-y-2)}+e^{-\lambda(x+y-2)}-e^{-\lambda(y-x)}}{2\lambda(1-e^{2\lambda})}, & x\in(y,1]\end{cases}. \qquad (22)$$

for$y\in(0,1]$.

Isomorphism and coerciveness with a defect of the principal part of the problem(1)-(5)

Substituting (15) and (18) into (13), we obtain

$$\begin{aligned}u(x)= & \frac{e^{\lambda}f_5-e^{2\lambda}f_2}{1-e^{2\lambda}}e^{\lambda x}+\frac{e^{\lambda}f_2-e^{2\lambda}f_5}{1-e^{2\lambda}}e^{-\lambda(x+1)}\\ & +\int_{-1}^{x}\frac{e^{\lambda(x+y+2)}-e^{-\lambda(x-y)}+e^{-\lambda(x+y)}-e^{-\lambda(y-x-2)}}{2\lambda(1-e^{2\lambda})}f_1(y)dy \qquad (23)\\ & +\int_{x}^{0}\frac{e^{\lambda(x+y+2)}-e^{-\lambda(y-x)}+e^{-\lambda(x+y)}-e^{-\lambda(x-y-2)}}{2\lambda(1-e^{2\lambda})}f_1(y)dy\end{aligned}$$

$Im\lambda\neq0$ for $x\in[-1,0)$,

$$\begin{aligned}u(x)= & \frac{f_3-e^{\lambda}f_6}{1-e^{2\lambda}}e^{\lambda x}+\frac{f_6-e^{\lambda}f_3}{1-e^{2\lambda}}e^{-\lambda(x-1)}\\ & +\int_{0}^{x}\frac{e^{\lambda(x+y)}-e^{-\lambda(y-x-2)}+e^{-\lambda(x+y-2)}-e^{-\lambda(x-y)}}{2\lambda(1-e^{2\lambda})}f_2(y)dy \qquad (24)\\ & +\int_{x}^{1}\frac{e^{\lambda(x+y)}-e^{-\lambda(y-x)}+e^{-\lambda(x+y-2)}-e^{-\lambda(x-y-2)}}{2\lambda(1-e^{2\lambda})}f_2(y)dy\end{aligned}$$

Im $\lambda\neq0$ for $x\in(0,1]$.

Because of (see, for example Yakubov S. And Yakubov Ya. (1999), p 106, Th. in 3.2.4) the operator

$$\tilde{L}_0(\lambda):u\to\tilde{L}_0(\lambda)u:=(L_0(\lambda,D)u,A_{10},A_{20},A_{30},A_{40})$$

from $W_q^2(-1,0,1)$ onto $L_q(-1,0,1)+\mathrm{C}^4$ is an isomorphism and for the solution of the problem (9)-(13) for $\frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon$, $|\lambda|\to\infty$ the estimate

$$\sum_{k=0}^{2}|\lambda|^{2-k}\|u\|_{W_q^k(-1,0,1)}\leq C(\varepsilon)\Big(\|f\|_{L_q(-1,0,1)}+|\lambda|^{2-1/q}(|f_5|+|f_2|+|f_3|+|f_6|\Big),$$

is hold. Consequently

$$|\lambda|^2\|u\|_{L_q(-1,0,1)} \leq C(\varepsilon)\Big(\|f\|_{L_q(-1,0,1)} + |\lambda|^{2-1/q}(|f_5| + |f_2| + |f_3| + |f_6|\Big) \quad ,$$

$$\|u\|_{W_q^2(-1,0,1)} \leq C(\varepsilon)\Big(\|f\|_{L_q(-1,0,1)} + |\lambda|^{2-1/q}(|f_5| + |f_2| + |f_3| + |f_6|\Big).$$

Then for $\frac{\pi}{2} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \varepsilon$, $|\lambda| \to \infty$ we have

$$\left\|\tilde{L}_0(\lambda)^{-1}\right\|_{B(L_q(-1,0,1)+C^4, L_q(-1,0,1)} \leq C(\varepsilon)|\lambda|^{-1/q,} \tag{25}$$

$$\left\|\tilde{L}_0(\lambda)^{-1}\right\|_{B(L_q(-1,0,1)+C^4, W_q^2(-1,0,1)} \leq C(\varepsilon)|\lambda|^{2-1/q.} \tag{26}$$

Coerciveness of the main problem (1)-(5)

Theorem.

Let $Q_1 \in W_p^1(-1,0), Q_2 \in W_p^1(0,1)$, where $1 < p \leq \infty$, $Q_1(-1) \neq 0$, and $Q_2(1) \neq 0$. Then for any $\varepsilon > 0$, there exist $R_\varepsilon > 0$ such that for all complex numbers $|\lambda|$ satisfying $|\lambda| > R_\varepsilon$ and that belong to one of the following two angles

$$\frac{\pi}{2} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \varepsilon \text{ or } |\arg\lambda| < \tfrac{\pi}{2} - \varepsilon$$

the operator $\tilde{L}(\lambda) : u \to \tilde{L}(\lambda)u := (L(\lambda, D), L_1(\lambda)u, L_2u, L_3u, L_4u)$ is an isomorphism from $W_q^2(-1,0,1)$onto $L_q(-1,0,1) + \mathrm{C}^4$, where $q \in (1,\infty)$and these λ hold the following inequalities for a solution to problem (1)-(5)

$$|\lambda|^2\|u\|_{L_q(-1,0,1)} + \|u\|_{W_q^2(-1,0,1)} \leq C(\varepsilon)|\lambda|^{4-1/q}(\|f\|_{L_q(-1,0,1)} + |f_1| + |f_2| + |f_3| + |f_4|). \tag{27}$$

Proof.

Already we know that the solution

$$\begin{aligned} u(x) &= \tilde{L}_0(\lambda)^{-1}(f, f_5, f_2, f_3, f_6) \\ &= \frac{e^{\lambda}f_5 - e^{2\lambda}f_2}{1 - e^{2\lambda}}e^{\lambda x} + \frac{e^{\lambda}f_2 - e^{2\lambda}f_5}{1 - e^{2\lambda}}e^{-\lambda(x+1)} \\ &\quad + \frac{f_3 - e^{\lambda}f_6}{1 - e^{2\lambda}}e^{\lambda x} + \frac{f_6 - e^{\lambda}f_3}{1 - e^{2\lambda}}e^{-\lambda(x-1)} \\ &\quad + \int_{-1}^{x} \frac{e^{\lambda(x+y+2)} - e^{-\lambda(x-y)} + e^{-\lambda(x+y)} - e^{-\lambda(y-x-2)}}{2\lambda(1 - e^{2\lambda})} f_1(y)dy \end{aligned}$$

$$+\int_x^0 \frac{e^{\lambda(x+y+2)} - e^{-\lambda(y-x)} + e^{-\lambda(x+y)} - e^{-\lambda(x-y-2)}}{2\lambda(1-e^{2\lambda})} f_1(y)dy$$

$$+\int_0^x \frac{e^{\lambda(x+y)} - e^{-\lambda(y-x-2)} + e^{-\lambda(x+y-2)} - e^{-\lambda(x-y)}}{2\lambda(1-e^{2\lambda})} f_2(y)dy \quad (28)$$

$$+\int_x^1 \frac{e^{\lambda(x+y)} - e^{-\lambda(y-x)} + e^{-\lambda(x+y-2)} - e^{-\lambda(x-y-2)}}{2\lambda(1-e^{2\lambda})} f_2(y)dy$$

of problem (9)-(13) is given by the formulas (23) and (24).

Let us define the operators $\tilde{Q} : u \to \tilde{Q}u := (Q(x)u(x), 0, 0, 0, 0)$ and $\tilde{L}(\lambda) := \tilde{L}_0(\lambda) + \tilde{Q}$ from $W_q^2(-1, 0, 1)$ into $L_q(-1, 0, 1) + \mathrm{C}^4$. Taking into account that the operator $\tilde{Q}$ from $L_q(-1, 0, 1)$ into $L_q(-1, 0, 1) + \mathrm{C}^4$ is bounded, and applying (25) we have

$$\left\| \tilde{Q}\tilde{L}_0(\lambda)^{-1} \right\|_{B(L_q(-1,0,1)+C^4)} \leq C(\varepsilon)|\lambda|^{-1/q}, \frac{\pi}{2} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \varepsilon, |\lambda| \to \infty. \quad (29)$$

Therefore, from

$$\tilde{L}(\lambda) := \tilde{L}_0(\lambda) + \tilde{Q} = (I + \tilde{Q}\tilde{L}_0(\lambda)^{-1})\tilde{L}_0(\lambda),$$

it follows that for $\frac{\pi}{2} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \varepsilon$, $|\lambda| \to \infty$ there exist the inverse operator $\tilde{L}(\lambda)^{-1}$ for which

$$\tilde{L}(\lambda)^{-1} = \tilde{L}_0(\lambda)^{-1} \sum_{k=0}^{\infty} \left[\tilde{Q}\tilde{L}_0(\lambda)^{-1} \right]^k. \quad (30)$$

Now, we define the function

$$u(x) = \tilde{L}(\lambda)^{-1}(f, f_5, f_2, f_3, f_6). \quad (31)$$

From $\tilde{L}(\lambda)u = (f, f_5, f_2, f_3, f_6)$, we have $(\tilde{L}_0(\lambda) + \tilde{Q})u = (f, f_5, f_2, f_3, f_6)$, namely, $(\lambda^2 u(x) - u''(x) + Q(x)u(x), u(-1), u(-0), u(+0), u(1)) = (f, f_5, f_2, f_3, f_6)$.

Consequently the function (27) will be a solution of problem (1)-(5) if

$$\lambda(f_5 + f_6) + A_{11}\tilde{L}(\lambda)^{-1}(f, f_5, f_2, f_3, f_6) + A_{41}\tilde{L}(\lambda)^{-1}(f, f_5, f_2, f_3, f_6) = f_1 \quad , \quad (32)$$

where $A_{11}u := u'(-1)$ and $A_{41}u := u'(1)$. Because of (24), it follows that

$$\begin{aligned} A_{11}\tilde{L}_0(\lambda)^{-1}(0, f_5, 0, 0, 0) &= \frac{d}{dx}\left[\frac{e^{\lambda}}{1-e^{2\lambda}} e^{\lambda x} - \frac{e^{2\lambda}}{1-e^{2\lambda}} e^{-\lambda(x+1)} \right]\Bigg|_{x=-1} f_5 \\ &= \frac{(\lambda + \lambda e^{2\lambda})f_5}{1-e^{2\lambda}} = \lambda(1+e^{2\lambda})(1+\sum_{k=1}^{\infty} e^{2\lambda k})f_5 \\ &= \lambda f_5 + O(e^{-C(\varepsilon)|\lambda|})f_5 \end{aligned} \quad (33)$$

for $\frac{\pi}{2} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \varepsilon$, $|\lambda| \to \infty$, and similarly

$$\begin{aligned} A_{41}\tilde{L}_0(\lambda)^{-1}(0,0,0,0,f_6) &= \frac{d}{dx}\left[\frac{-e^{\lambda}}{1-e^{2\lambda}}e^{\lambda x}+\frac{1}{1-e^{2\lambda}}e^{-\lambda(x-1)}\right]\Bigg|_{x=1} f_6 \\ &= \frac{(-\lambda-\lambda e^{2\lambda})f_6}{1-e^{2\lambda}} = -\lambda(1+e^{2\lambda})(1+\sum_{k=1}^{\infty}e^{2\lambda k})f_6 \\ &= -\lambda f_6 + O(e^{-C(\varepsilon)|\lambda|})f_6 \end{aligned} \tag{34}$$

for $\frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon$, $|\lambda|\to\infty$.

Expressions (33) and (34) are asymptotic for the first term of the series

$$A_{11}\tilde{L}(\lambda)^{-1}(0,f_5,0,0,0)+A_{41}\tilde{L}(\lambda)^{-1}(0,0,0,0,f_6).$$

To find the second term of this series we shall introduce notations

$$Q(x):=\begin{cases} Q_1(x), & x\in[-1,0) \\ Q_2(x), & x\in(0,1]\end{cases}, Q_j(x,\lambda):=\begin{cases} Q_{1j}(x,\lambda), & x\in[-1,0) \\ Q_{2j}(x,\lambda), & x\in(0,1]\end{cases},\ j=1,2,...$$

$$\tilde{Q}:=\begin{cases}\tilde{Q}_1, & x\in[-1,0) \\ \tilde{Q}_2, & x\in(0,1]\end{cases},$$

and

$$Q_{11}(x,\lambda):=\frac{e^{\lambda}}{1-e^{2\lambda}}e^{\lambda x}-\frac{e^{2\lambda}}{1-e^{2\lambda}}e^{-\lambda(x+1)},\quad x\in[-1,0), \tag{35}$$

$$Q_{21}(x,\lambda):=-\frac{e^{\lambda}}{1-e^{2\lambda}}e^{\lambda x}+\frac{1}{1-e^{2\lambda}}e^{-\lambda(x-1)},\quad x\in(0,1]\quad . \tag{36}$$

Then from (28)

$$\begin{aligned} \sum_{i=1}^{2}\tilde{Q}_i\tilde{L}_0(\lambda)^{-1}(0,f_5,0,0,f_6) &= \tilde{Q}_1\tilde{L}_0(\lambda)^{-1}(0,f_5,0,0,0)+\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}(0,0,0,0,f_6) \\ &= (Q_1(x)Q_{11}(x,\lambda)f_5,0,0,0,0)+(Q_2(x)Q_{21}(x,\lambda)f_6,0,0,0,0) \end{aligned} \tag{37}$$

and from (28), (30), and (37) for the second term of the series

$A_{11}\tilde{L}(\lambda)^{-1}(0,f_5,0,0,0)+A_{41}\tilde{L}(\lambda)^{-1}(0,0,0,0,f_6)$ for $\mathrm{Im}\lambda\neq 0$ we have

$$A_{11}\tilde{L}_0(\lambda)^{-1}(Q_1(x)Q_{11}(x,\lambda)f_5,0,0,0,0)\Big|_{x=-1}+A_{41}\tilde{L}_0(\lambda)^{(-1)}(q_2(x)Q_{21}(x,\lambda)f_6,0,0,0,)\Big|_{x=1}$$

$$= \frac{d}{dx}\left\{\left[\int_{-1}^{x} \frac{e^{\lambda(x+y+2)} - e^{-\lambda(x-y)} + e^{-\lambda(x+y)} - e^{-\lambda(y-x-2)}}{2\lambda(1-e^{2\lambda})} Q_1(y)Q_{11}(y,\lambda)f_5 dy\right.\right.$$

$$\left.\left. + \int_{x}^{0} \frac{e^{\lambda(x+y+2)} - e^{-\lambda(y-x)} + e^{-\lambda(x+y)} - e^{-\lambda(x-y-2)}}{2\lambda(1-e^{2\lambda})} Q_1(y)Q_{11}(y,\lambda)f_5 dy\right]\right|_{x=-1}$$

$$+\left[\int_{0}^{x} \frac{e^{\lambda(x+y)} - e^{-\lambda(y-x-2)} + e^{-\lambda(x+y-2)} - e^{-\lambda(x-y)}}{2\lambda(1-e^{2\lambda})} Q_2(y)Q_{21}(y,\lambda)f_6 dy\right.$$

$$\left.\left.\left. + \int_{x}^{1} \frac{e^{\lambda(x+y)} - e^{-\lambda(y-x)} + e^{-\lambda(x+y-2)} - e^{-\lambda(x-y-2)}}{2\lambda(1-e^{2\lambda})} Q_2(y)Q_{21}(y,\lambda)f_6 dy\right]\right|_{x=1}\right\}$$

$$= \int_{-1}^{0} \frac{e^{-(\lambda-1)} - e^{\lambda(y+1)}}{1-e^{2\lambda}} Q_1(y)Q_{11}(y,\lambda)f_5 dy$$

$$+ \int_{0}^{1} \frac{e^{\lambda(y+1)} - e^{-\lambda(y-1)}}{1-e^{2\lambda}} Q_2(y)Q_21(y,\lambda)f_6 dy. \tag{38}$$

Therefore, because of (35) and (36), for Im $\lambda \neq 0$ it follows that

$$A_{11}\tilde{L}_0(\lambda)^{-1}\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}(0,f_5,0,0,0) + A_{41}\tilde{L}_0(\lambda)^{-1}\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}(0,0,0,0,f_6)$$

$$= \frac{1}{(1-e^{2\lambda})^2}\int_{-1}^{0} (e^{-\lambda(y-1)} - e^{\lambda(y+1)})^2 Q_1(y)f_5 dy$$

$$- \frac{1}{(1-e^{2\lambda})^2}\int_{0}^{1} (e^{-\lambda(y-1)} - e^{\lambda(y+1)})^2 Q_2(y)f_6 dy. \tag{39}$$

Applying the above Lemma and using the inequalities

$$\left|\int_{-1}^{0} (-2e^{-\lambda(y-1)}e^{\lambda(y+1)} + e^{2\lambda(y+1)})Q_1(y)dy\right| \leq Ce^{Re\lambda}\|Q_1\|_{C[-1,0]} \leq Ce^{-C(\varepsilon)|\lambda|},$$

and

$$\left|\int_{0}^{1} (-2e^{-\lambda(y-1)}e^{\lambda(y+1)} + e^{2\lambda(y+1)})Q_1(y)dy\right| \leq Ce^{Re\lambda}\|Q_2\|_{C[0,1]} \leq Ce^{-C(\varepsilon)|\lambda|},$$

$$\frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon,\quad |\lambda|\to\infty,$$

we have

$$\int_{-1}^{0}\left(e^{-\lambda(y-1)}-e^{\lambda(y+1)}\right)^{2}Q_1(y)dy$$

$$=\frac{1}{2\lambda}Q_1(-1)+O\left(\frac{1}{|\lambda|^{2-1/p}}\right),\ \text{for}\ \frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon,|\lambda|\to\infty, \tag{40}$$

and

$$\int_{0}^{1}\left(e^{-\lambda(y-1)}-e^{\lambda(y+1)}\right)^{2}Q_2(y)dy$$

$$=-\frac{1}{2\lambda}Q_2(1)+O\left(\frac{1}{|\lambda|^{2-1/p}}\right),\ \text{for}\ \frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon,|\lambda|\to\infty \tag{41}$$

Putting (40) and (41) into (39) we get

$$A_{11}\tilde{L}_0(\lambda)^{-1}\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}(0,f_5,0,0,0)+A_{41}\tilde{L}_0(\lambda)^{-1}\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}(0,0,0,0,f_6)$$

$$=\frac{1}{2\lambda}(Q_1(-1)f_5+Q_2(1)f_6)+O\left(\frac{1}{|\lambda|^{2-1/p}}\right)(f_5+f_6), \tag{42}$$

$$\frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon,\quad |\lambda|\to\infty.$$

Denote

$$\begin{aligned}Q_{12}(x,\lambda):=&\int_{-1}^{x}\frac{e^{\lambda(x+z+2)}-e^{-\lambda(x-z)}+e^{-\lambda(x+z)}-e^{-\lambda(z-x-2)}}{2\lambda(1-e^{2\lambda})}Q_1(z)Q_{11}(z,\lambda)dz\\&+\int_{x}^{0}\frac{e^{\lambda(x+z+2)}-e^{-\lambda(z-x)}+e^{-\lambda(x+z)}-e^{-\lambda(x-z-2)}}{2\lambda(1-e^{2\lambda})}Q_1(z)Q_{11}(z,\lambda)dz,\end{aligned} \tag{43}$$

and

$$\begin{aligned}Q_{22}(x,\lambda):=&\int_{0}^{x}\frac{e^{\lambda(x+z)}-e^{-\lambda(z-x-2)}+e^{-\lambda(x+z-2)}-e^{-\lambda(x-z)}}{2\lambda(1-e^{2\lambda})}Q_2(z)Q_{21}(z,\lambda)dz\\&+\int_{x}^{1}\frac{e^{\lambda(x+z)}-e^{-\lambda(x-z-2)}+e^{-\lambda(x+z-2)}-e^{-\lambda(z-x)}}{2\lambda(1-e^{2\lambda})}Q_2(z)Q_{21}(z,\lambda)dz.\end{aligned} \tag{44}$$

Then from (28) and (37) for Im $\lambda \neq 0$ it follows that

$$\left[\tilde{Q}\tilde{L}_0(\lambda)^{-1}\right]^2(0,f_5,0,0,f_6)$$
$$= \left[\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}\right]^2(0,f_5,0,0,0) + \left[\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}\right]^2(0,0,0,0,f_6)$$

$$= \tilde{Q}_1\tilde{L}(\lambda)^{-1}(Q_1(x)Q_{11}(x,\lambda)f_5,0,0,0,0) + \tilde{Q}_2\tilde{L}(\lambda)^{-1}(Q_2(x)Q_{21}(x,\lambda)f_6,0,0,0,0). \quad (45)$$

Appling the same technique as in (37) and using (28), (30) and (45) for the third term of the series we have

$$A_{11}\tilde{L}_0(\lambda)^{-1}(0,f_5,0,0,0) + A_{41}\tilde{L}_0(\lambda)^{-1}(0,0,0,0,f_6) \text{ for Im } \lambda \neq 0$$

$$A_{11}\tilde{L}_0(\lambda)^{-1}\left[\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}\right]^2(0,f_5,0,0,0) + A_{41}\tilde{L}_0(\lambda)^{-1}\left[\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}\right]^2(0,0,0,0,f_6)$$

$$= A_{11}\tilde{L}_0(\lambda)^{-1}(Q_1(x)Q_{12}(x,\lambda)f_5,0,0,0,0) + A_{41}\tilde{L}_0(\lambda)^{-1}(Q_2(x)Q_{22}(x,\lambda)f_6,0,0,0,0)$$

$$= \int_{-1}^{0}\frac{e^{-\lambda(y-1)} - e^{\lambda(y+1)}}{1-e^{2\lambda}}Q_1(y)Q_{12}(y,\lambda)f_5dy$$

$$+ \int_{0}^{1}\frac{e^{\lambda(y+1)} - e^{-\lambda(y-1)}}{1-e^{2\lambda}}Q_2(y)Q_{22}(y,\lambda)f_6dy. \quad (46)$$

Because of (35), (36) and (43), (44), expression (46) consist of sum of expression of the form

$$\frac{-C_1}{\lambda(1-e^{2\lambda})^3}\int_{-1}^{0}e^{-\lambda(y-1)}Q_1(y)\left(\int_{-1}^{y}e^{-\lambda(z-y)}Q_1(z)e^{-\lambda(z-1)}dz\right)f_5dy$$

$$+\frac{C_2}{\lambda(1-e^{2\lambda})^3}\int_{0}^{1}e^{-\lambda(y-1)}Q_2(y)\left(\int_{0}^{y}e^{-\lambda(z-y)}Q_2(z)e^{-\lambda(z-1)}dz\right)f_6dy.$$

These expressions can be estimated as

$$\left|\frac{-C_1}{\lambda(1-e^{2\lambda})^3}\int_{-1}^{0}e^{-\lambda(y-1)}Q_1(y)\left(\int_{-1}^{y}e^{-\lambda(z-y)}Q_1(z)e^{-\lambda(z-1)}dz\right)f_5dy\right|$$

$$\leq \frac{C_1(\varepsilon)\|Q_1\|^2_{C[-1,0]}}{|\lambda|}\int_{-1}^{0}e^{-Re\lambda(y-1)}dy|f_5| \leq \frac{C_1(\varepsilon)\|Q_1\|^2_{C[-1,0]}}{|\lambda|^2}|f_5|$$

and similarly

$$\left|\frac{C_2}{\lambda(1-e^{2\lambda})^3}\int_0^1 e^{-\lambda(y-1)}Q_2(y)\left(\int_0^y e^{-\lambda(z-y)}Q_2(z)e^{-\lambda(z-1)}dz\right)f_6dy\right|$$

$$\le \frac{C_2(\varepsilon)\|Q_2\|^2_{C[0,1]}}{|\lambda|}\int_0^1 e^{-Re\lambda(y-1)}dy|f_6|$$

$$\le \frac{C_2(\varepsilon)\|Q_2\|^2_{C[0,1]}}{|\lambda|^2}|f_6| \text{ for } \frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon, |\lambda|\to\infty.$$

Consequently, for $\frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon$, $|\lambda|\to\infty$

$$A_{11}\tilde{L}_0(\lambda)^{-1}\left[\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}\right]^2(0,f_5,0,0,0)+A_{41}\tilde{L}_0(\lambda)^{-1}\left[\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}\right]^2(0,0,0,0,f_6)$$

$$\le \frac{C_1(\varepsilon)\|Q_1\|^2_{C[-1,0]}}{|\lambda|^2}|f_5|+\frac{C_2(\varepsilon)\|Q_2\|^2_{C[0,1]}}{|\lambda|^2}|f_6|$$

$$\le \frac{C_1(\varepsilon)}{|\lambda|^2}|f_5|+\frac{C_2(\varepsilon)}{|\lambda|^2}|f_6|\le\frac{C(\varepsilon)}{|\lambda|^2}(|f_5|+|f_6|) \tag{47}$$

Let

$$\begin{aligned} Q_{13}(x,\lambda) := & \int_{-1}^x \frac{e^{\lambda(x+z+2)}-e^{-\lambda(x-z)}+e^{-\lambda(x+z)}-e^{-\lambda(z-x-2)}}{2\lambda(1-e^{2\lambda})}Q_1(z)Q_{12}(z,\lambda)dz \\ & +\int_x^0 \frac{e^{\lambda(x+z+2)}-e^{-\lambda(z-x)}+e^{-\lambda(x+z)}-e^{-\lambda(x-z-2)}}{2\lambda(1-e^{2\lambda})}Q_1(z)Q_{12}(z,\lambda)dz, \end{aligned} \tag{48}$$

and

$$\begin{aligned} Q_{23}(x,\lambda) := & \int_{0}^x \frac{e^{\lambda(x+z)}-e^{-\lambda(z-x-2)}+e^{-\lambda(x+z-2)}-e^{-\lambda(x-z)}}{2\lambda(1-e^{2\lambda})}Q_2(z)Q_{22}(z,\lambda)dz \\ & +\int_x^1 \frac{e^{\lambda(x+z)}-e^{-\lambda(x-z-2)}+e^{-\lambda(x+z-2)}-e^{-\lambda(z-x)}}{2\lambda(1-e^{2\lambda})}Q_2(z)Q_{22}(z,\lambda)dz. \end{aligned} \tag{49}$$

Then from (28) and (45) for Im $\lambda\neq 0$, it follows that

$$\begin{aligned} & \left[\tilde{Q}\tilde{L}_0(\lambda)^{-1}\right]^3(0,f_5,0,0,f_6) \\ & = \left[\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}\right]^3(0,f_5,0,0,0)+\left[\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}\right]^3(0,0,0,0,f_6) \\ & = \tilde{Q}_1\tilde{L}(\lambda)^{-1}(Q_1(x)Q_{12}(x,\lambda)f_5,0,0,0,0)+\tilde{Q}_2\tilde{L}(\lambda)^{-1}(Q_2(x)Q_{22}(x,\lambda)f_6,0,0,0,0) \\ & = (Q_1(x)Q_{13}(x,\lambda)f_5+(Q_2(x)Q_{23}(x,\lambda)f_6 \end{aligned} \tag{50}$$

Again by applying the same technique as in (38) and using (28), (30) and (50) for the fourth term of the series we have

$$A_{11}\tilde{L}_0(\lambda)^{-1}(0,f_5,0,0,0)+A_{41}\tilde{L}_0(\lambda)^{-1}(0,0,0,0,f_6)\text{for Im }\lambda\neq 0$$

$$A_{11}\tilde{L}_0(\lambda)^{-1}\left[\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}\right]^3(0,f_5,0,0,0)+A_{41}\tilde{L}_0(\lambda)^{-1}\left[\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}\right]^3(0,0,0,0,f_6)$$

$$=A_{11}\tilde{L}_0(\lambda)^{-1}(Q_1(x)Q_{13}(x,\lambda)f_5,0,0,0,0)+A_{41}\tilde{L}_0(\lambda)^{-1}(Q_2(x)Q_{23}(x,\lambda)f_6,0,0,0,0) \quad (51)$$

$$=\int_{-1}^{0}\frac{e^{-\lambda(y-1)}-e^{\lambda(y+1)}}{1-e^{2\lambda}}Q_1(y)Q_{13}(y,\lambda)f_5dy$$

$$+\int_{0}^{1}\frac{e^{\lambda(y+1)}-e^{-\lambda(y-1)}}{1-e^{2\lambda}}Q_2(y)Q_{23}(y,\lambda)f_6dy.$$

Because of (43), (44) and (48), (49), expression (51) consist of sum of expressions of the form

$$\frac{-C_1}{\lambda(1-e^{2\lambda})^4}\int_{-1}^{0}e^{-\lambda(y-1)}Q_1(y)\left(\int_{-1}^{y}e^{-\lambda(z-y)}Q_1(z)(\int_{-1}^{z}e^{-\lambda(v-z)}Q_1(v)dv)dz\right)f_5dy$$

$$+\frac{C_2}{\lambda(1-e^{2\lambda})^3}\int_{0}^{1}e^{-\lambda(y-1)}Q_2(y)\left(\int_{0}^{y}e^{-\lambda(z-y)}Q_2(z)(\int_{0}^{z}e^{-\lambda(v-z)}Q_2(v)dv)dz\right)f_6dy.$$

All these expression can be estimated as

$$\left|\frac{-C_1}{\lambda(1-e^{2\lambda})^4}\int_{-1}^{0}e^{-\lambda(y-1)}Q_1(y)\left(\int_{-1}^{y}e^{-\lambda(z-y)}Q_1(z)(\int_{-1}^{z}e^{-\lambda(v-z)}Q_1(v)dv)dz\right)f_5dy\right|$$

$$+\left|\frac{C_2}{\lambda(1-e^{2\lambda})^3}\int_{0}^{1}e^{-\lambda(y-1)}Q_2(y)\left(\int_{0}^{y}e^{-\lambda(z-y)}Q_2(z)(\int_{0}^{z}e^{-\lambda(v-z)}Q_2(v)dv)dz\right)f_6dy\right|$$

$$\leq\frac{C_1(\varepsilon)\|Q_1\|^3_{C[-1,0]}}{|\lambda|}\int_{-1}^{0}e^{-Re\lambda(y-1)}dy|f_5|+\frac{C_2(\varepsilon)\|Q_2\|^3_{C[0,1]}}{|\lambda|}\int_{0}^{1}e^{-Re\lambda(y-1)}dy|f_6|$$

$$\leq\frac{C_1(\varepsilon)\|Q_1\|^3_{C[-1,0]}}{|\lambda|^2}|f_5|+\frac{C_2(\varepsilon)\|Q_2\|^3_{C[0,1]}}{|\lambda|^2}|f_6|$$

$$\leq\frac{C_1(\varepsilon)}{|\lambda|^2}|f_5|+\frac{C_2(\varepsilon)}{|\lambda|^2}|f_6|\leq\frac{C(\varepsilon)}{|\lambda|^2}(|f_5|+|f_6|)$$

Consequently, for $\frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon$, $|\lambda|\to\infty$

$$\left|A_{11}\tilde{L}_0(\lambda)^{-1}\left[\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}\right]^3(0,f_5,0,0,0)+A_{41}\tilde{L}_0(\lambda)^{-1}\left[\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}\right]^3(0,0,0,0,f_6)\right|$$

$$\leq\frac{C(\varepsilon)}{|\lambda|^2}(|f_5|+|f_6|). \quad (52)$$

Taking into account (43), (44), (45) and (48), (49), (50) we see that the estimate (47) (estimate (52) holds for the next terms of the series

$$A_{11}\tilde{L}_0(\lambda)^{-1}(0,f_5,0,0,0)+A_{41}\tilde{L}_0(\lambda)^{-1}(0,0,0,0,f_6),\frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon,|\lambda|\to\infty$$

$$\left|A_{11}\tilde{L}_0(\lambda)^{-1}\left[\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}\right]^k(0,f_5,0,0,0)+A_{41}\tilde{L}_0(\lambda)^{-1}\left[\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}\right]^k(0,0,0,0,f_6)\right|$$

$$\leq\frac{C(\varepsilon)}{|\lambda|^2}(|f_5|+|f_6|),\quad k=2,3,\ldots\quad . \tag{53}$$

By virtue of (30) and (32) it can be shown easily that Eq. (32) can be written as in the form

$$\lambda(f_5+f_6)+A_{11}\tilde{L}_0(\lambda)^{-1}(f,f_5,f_2,f_3,f_6)+A_{41}\tilde{L}_0(\lambda)^{-1}(f,f_5,f_2,f_3,f_6)$$

$$+A_{11}\tilde{L}_0(\lambda)^{-1}\sum_{k=2}^{n-1}\left[\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}\right]^k(0,f_5,0,0,0)$$

$$+A_{41}\tilde{L}_0(\lambda)^{-1}\sum_{k=2}^{n-1}\left[\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}\right]^k(0,0,0,0,f_6)$$

$$+A_{11}\tilde{L}_0(\lambda)^{-1}\sum_{k=n}^{\infty}\left[\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}\right]^k(0,f_5,0,0,0)$$

$$+A_{41}\tilde{L}_0(\lambda)^{-1}\sum_{k=n}^{\infty}\left[\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}\right]^k(0,0,0,0,f_6)$$

$$=f_1-A_{11}\tilde{L}_0(\lambda)^{-1}(f,f_5,f_2,f_3,f_6)-A_{41}\tilde{L}_0(\lambda)^{-1}(f,f_5,f_2,f_3,f_6). \tag{54}$$

Because of (25), (26), (29), for $\frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon$, $|\lambda|\to\infty$ we obtain

$$\sum_{m=1}^{2}\left\|A_{m^2 1}\tilde{L}_0(\lambda)^{-1}\sum_{k=n}^{\infty}\left[\tilde{Q}\tilde{L}_0(\lambda)^{-1}\right]^k_{B(L_q(-1,0,1)+C^4,C)}\right\|$$

$$\leq\left\|A_{11}\tilde{L}_0(\lambda)^{-1}\sum_{k=n}^{\infty}\left[\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}\right]^k_{B(L_q(-1,0)+C^4,C)}\right\|+\left\|A_{41}\tilde{L}_0(\lambda)^{-1}\sum_{k=n}^{\infty}\left[\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}\right]^k_{B(L_q(0,1)+C^4,C)}\right\|$$

$$\leq\|A_{11}\|_{B(C^1[-1,0],C)}\left\|\tilde{L}_0(\lambda)^{-1}\right\|_{B(L_q(-1,0)+C^4,C^1[-1,0])}\sum_{k=n}^{\infty}\left\|\tilde{Q}_1\tilde{L}_0(\lambda)^{-1}\right\|^k_{B(L_q(-1,0)+C^4)}$$

$$+\|A_{41}\|_{B(C^1[0,1],C)}\left\|\tilde{L}_0(\lambda)^{-1}\right\|_{B(L_q(0,1)+C^4,C^1[0,1])}\sum_{k=n}^{\infty}\left\|\tilde{Q}_2\tilde{L}_0(\lambda)^{-1}\right\|^k_{B(L_q(0,1)+C^4)}$$

$$\leq (C_1(\varepsilon)+C_2(\varepsilon))|\lambda|^{2-1/q}\frac{(|\lambda|^{-1/q})^n}{1-|\lambda|^{-1/q}} \leq C(\lambda)|\lambda|^{2-(n+1)/q.} \tag{55}$$

Taking n large that $4/(n+1) < 1/q < 1$. Then $2-(n+1)/q < -2$ and, because of (33), (42), (53), and (55) from Equation (54), it follows that

$$\begin{aligned}&\frac{1}{2\lambda}(Q_1(-1)f_5+Q_2(1)f_6)+O\left(\frac{1}{|\lambda|^{2-1/p}}\right)(f_5+f_6)\\&=f_1-A_{11}\tilde{L}_0(\lambda)^{-1}(f,f_5,0,0,0)-A_{41}\tilde{L}_0(\lambda)^{-1}(f,0,0,0,f_6)\\&\frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon,\quad |\lambda|\to\infty.\end{aligned} \tag{56}$$

By virtue of (25), (26), (29), (30) for $\frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon$, $|\lambda|\to\infty$ we have

$$\left\|\tilde{L}(\lambda)^{-1}\right\|_{B(L_q(-1,0,1)+\mathrm{C}^4,L_q(-1,0,1))}\leq C_1|\lambda|^{-1/q}, \tag{57}$$

and

$$\left\|\tilde{L}(\lambda)^{-1}\right\|_{B(L_q(-1,0,1)+\mathrm{C}^4,W_q^2(-1,0,1))}\leq C_2|\lambda|^{2-1/q}, \tag{58}$$

Applying the Theorem 3.10.4 of Besow (1978) and (55)-(56), for $\frac{\pi}{2}+\varepsilon<\arg\lambda<\frac{3\pi}{2}-\varepsilon$, $|\lambda|\to\infty$ we obtain

$$|\lambda|\left|\frac{d}{dx}\tilde{L}(\lambda)^{-1}(f,f_1,f_2,f_3,f_4)|_{x=-1}+\frac{d}{dx}\tilde{L}(\lambda)^{-1}(f,f_1,f_2,f_3,f_4)|_{x=1}\right|$$

$$\leq|\lambda|\left|\frac{d}{dx}\tilde{L}(\lambda)^{-1}(f,f_1,f_2,f_3,f_4)|_{z=-1}\right|+|\lambda|\left|\frac{d}{dx}\tilde{L}(\lambda)^{-1}(f,f_1,f_2,f_3,f_4)|_{z=1}\right|$$

$$\leq C_1\left(|\lambda|^{2+1/q}\left\|\tilde{L}(\lambda)^{-1}(f,f_1,f_2,f_3,f_4)\right\|_{L_q(-1,0)}+|\lambda|^{1/q}\left\|\tilde{L}(\lambda)^{-1}(f,f_1,f_2,f_3,f_4)\right\|_{W_q^2(-1,0)}\right)$$

$$+C_2\left(|\lambda|^{2+1/q}\left\|\tilde{L}(\lambda)^{-1}(f,f_1,f_2,f_3,f_4)\right\|_{L_q(0,1)}+|\lambda|^{1/q}\left\|\tilde{L}(\lambda)^{-1}(f,f_1,f_2,f_3,f_4)\right\|_{W_q^2(0,1)}\right)$$

$$\leq C_1|\lambda|^2\|(f,f_1,f_2,f_3,f_4)\|_{L_q(-1,0)+C^4}+C_2|\lambda|^2\|(f,f_1,f_2,f_3,f_4)\|_{L_q(0,1)+C^4}$$

$$\leq C|\lambda|^2\|(f,f_1,f_2,f_3,f_4)\|_{L_q(-1,0,1)+C^4}.$$

Consequently,

$$\left\| A_{11}\tilde{L}_0(\lambda)^{-1} + A_{41}\tilde{L}_0(\lambda)^{-1} \right\|_{B(L_q(-1,0,1)+C^4,C)} \le C|\lambda|, \frac{\pi}{2} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \varepsilon, |\lambda| \to \infty.$$

By using (56) we see that equation (56) has a solution and

$$|f_5 + f_6| \le |f_5| + |f_6| \le C_1(\varepsilon)|\lambda|^2(\|f\|_{L_q(-1,0)} + |f_1| + |f_2| + |f_3| + |f_4|)$$

$$+C_2(\varepsilon)|\lambda|^2(\|f\|_{L_q(0,1)} + |f_1| + |f_2| + |f_3| + |f_4|)$$

$$\le C(\varepsilon)|\lambda|^2(\|f\|_{L_q(-1,0,1)} + |f_1| + |f_2| + |f_3| + |f_4|). \tag{59}$$

Putting (59) in to (31) and taking in view (57) and (58) we get

$$\|u\|_{L_q(-1,0,1)} = \|u\|_{L_q(-1,0)} + \|u\|_{L_q(0,1)}$$

$$\le \left\| \tilde{L}(\lambda)^{-1}(f, f_5, f_2, f_3, f_4) \right\|_{L_q(-1,0)} + \left\| \tilde{L}(\lambda)^{-1}(f, f_6, f_2, f_3, f_4) \right\|_{L_q(0,1)}$$

$$\le C(\varepsilon)|\lambda|^{-1/q}(\|f\|_{L_q(-1,0)} + |f_5| + |f_2| + |f_3| + |f_4|)$$

$$+C(\varepsilon)|\lambda|^{-1/q}(\|f\|_{L_q(0,1)} + |f_6| + |f_2| + |f_3| + |f_4|)$$

$$\le C(\varepsilon)|\lambda|^{2-1/q}(\|f\|_{L_q(-1,0,1)} + |f_1| + |f_2| + |f_3| + |f_4|)$$

and similarly

$$\|u\|_{W_q^2(-1,0,1)} = \|u\|_{W_q^2(-1,0)} + \|u\|_{W_q^2(0,1)}$$

$$\le \left\| \tilde{L}(\lambda)^{-1}(f, f_5, f_2, f_3, f_4) \right\|_{W_q^2(-1,0)} + \left\| \tilde{L}(\lambda)^{-1}(f, f_6, f_2, f_3, f_4) \right\|_{W_q^2(0,1)}$$

$$\le C(\varepsilon)|\lambda|^{2-1/q}(\|f\|_{L_q(-1,0)} + |f_5| + |f_2| + |f_3| + |f_4|)$$

$$+C(\varepsilon)|\lambda|^{2-1/q}(\|f\|_{L_q(0,1)} + |f_6| + |f_2| + |f_3| + |f_4|)$$

$$\le C(\varepsilon)|\lambda|^{4-1/q}(\|f\|_{L_q(-1,0,1)} + |f_1| + |f_2| + |f_3| + |f_4|).$$

The proof of (27) is complete.

CONCLUSION

There is a substantial literature on Birkhof-regular boundary value problems, but not one systematically investigates the Birkhof-irregular situation. It is well-known that many mathematical physics problems deal with Birkhof-irregular problems. Particularly, in recent years, highly important result this field have been obtained for the continuous case. The literature on such results is voluminous and we refer to (see, Krawitskii (1968), Regge (1963), Tikhonov (1968), Yakubov S. (1994,1998, 2000), Yakubav Ya. (1998), Yakubov S. and Yakubav Ya (1999)) and corresponding references cited therein. But very little is known about discontinuities (see, Akdogan, Demirci and Mukhtarov (2007), Kandemir, Mukhtarov and Yakubov Ya. (accepted 2009), Kandemir and Yakubov Ya. (accepted 2009), Mukhtarov (1994), Mukhtarov and Demir (1999), Mukhtarov, Kandemir and Kuruoğlu (2002), Mukhtarov and Kandemir (2002), Mukhtarov and Yakubov S. (2002)).

In this paper we extended some results which deal with Birkhof-irregular Regge problems, to the discontinuous case.

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طريقة لحل مشاكل قيمة الحدود غير المنتظمة مع شروط التوصيل

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خلاصة

في هذه الورقة تم اعتبار مشكلة قيمة الحدود مع شروط التوصيل. إن مشكلة قيمة الحدود التي تم دراستها في هذا البحث تشتمل على قيمة عنصر eigen والذي يعتبر درجة ثانية في المعادلة ومن الدرجة الأولى في إحدى شروط الحد. لقد تم اتخاذ طرق مختلفة لمعرفة مشكلة قيمة الحدود هذه والتي بنيت على مشكلة Regge .