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Numerical study for solving heave and pitch equations of a ship

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ABSTRACT

This paper presents two numerical methods for solving the heave and pitch equations of motion of a ship. For this purpose, the system of differential equations of motion will be transferred to a system of Volterra integral equations and then this system will be solved by using the Trapezoidal method and an Iterative scheme. Finally, by using an example, the analytic solution and the numerical solutions will be compared.

Keywords: Iterative scheme; ship motions; system equations; system of Volterra integral equations; trapezoidal method.

INTRODUCTION

The response of a ship to the sea waves is a complicated phenomenon that is related to the dynamics of the ship and various hydrodynamic forces. The sum of the hydrodynamic forces can be divided into three groups. The first group contains the components, which depend on the condition of motion of the ship without propeller and rudder. The second group contains the forces which act on the rudder. The third group contains the force components, which are caused by the change of circulation around the ship, due to the rudder deflection. However, the motion of the ship in linear manner is considered by many researchers (Lewis 1990). In this paper, we present two numerical methods for solving heave and pitch equations of motion of a ship. In the next section, the mathematical model for heave and pitch motions will be presented. In the third section, the system of differential equations of motion will be transferred to the system of Volterra integral equations and then by using the Trapezoidal method and an Iterative scheme this system will be solved. In the last section , by giving an example, numerical solutions will be compared with the analytic solution.

Nomenclature	
$\vec{F}$	resulting external force acting in the center of gravity
m	mass of the rigid body
$\vec{U}$	instantaneous velocity of the center of gravity
$\vec{M}$	resulting external moment acting about the center of gravity
$\vec{H}$	instantaneous angular momentum about the center of gravity
V	volume of displacement of the body
F_{h3}	hydromechanical force in the z-direction
F_{w3}	exciting wave force in the z-direction
F_{h5}	hydromechanical moment about the z-axis
F_{w5}	exciting wave moment about the z-axis
w_e	circular frequency of encounter

MODELLING HEAVE AND PITCH EQUATIONS OF MOTION

A rigid body's equation of motion of a ship with respect to an Earth-Bound coordinate system follow from Newton's second law. The vector equations for the translation and the rotation about the center of gravity are as follows (Journee & Massie 2001):

$$\vec{F} = \frac{\partial}{\partial t}(m\vec{U}) \text{ and } \vec{M} = \frac{\partial}{\partial t}(\vec{H}). \tag{1}$$

Supposing the system to be linear, the motion of a ship in waves may be considered as a body motion in still water and the forces on the restrained body in waves. The forces of the body are divided into two parts:

a) the forces and hydrodynamical torques related to harmonical motions of the rigid body, and

b) the forces and induced torques of the wave that is produced by the waves that contact the restrain body.

Therefore, the equation of the vertical motion using Newton's second law is as follows:

$$\frac{\partial}{\partial t}(\rho V \cdot \dot{z}) = \rho V \cdot \ddot{z} = F_{h3} + F_{w3}, \tag{2}$$

where, ρ is the density. Similarly the pitch motion fits the following equation:

$$I_{55} \cdot \ddot{\theta} = F_{h5} + F_{w5}, \tag{3}$$

where, I_{55} is the solid mass moment of inertia of the ship.

In linear theory, the radiated hydromechanical forces and moments caused by the coupled motions of the ship in still water can be claimed as the state that are related to the vertical transmissions, speeds and accelerations. Then we may write:

$$\begin{cases} F_{h3} = -[a_{33}\ddot{z} + b_{33}\dot{z} + c_{33}z + a_{35}\ddot{\theta} + b_{35}\dot{\theta} + c_{35}\theta] \\ F_{h5} = -[a_{53}\ddot{z} + b_{53}\dot{z} + c_{53}z + a_{55}\ddot{\theta} + b_{55}\dot{\theta} + c_{55}\theta], \end{cases} \tag{4}$$

where, a_{jk} are the hydrodynamic mass coefficients, b_{jk} are the hydrodynamic damping coefficients and c_{jk} are the restoring spring coefficients. For the sinusoidal motions, the wave forces and moments are as follows:

$$\begin{cases} F_{w3} = |F_3|\cos(w_e t + \varepsilon_3) \\ F_{w5} = |F_5|\cos(w_e t + \varepsilon_5). \end{cases} \tag{5}$$

The final equations of the coupled heave and pitch motions in regular waves are obtained by using the combination of equations (2), (3), (4), and (5).

$$\begin{cases} (\rho V + a_{33})\ddot{z} + b_{33}\dot{z} + c_{33}z + a_{35}\ddot{\theta} + b_{35}\dot{\theta} + c_{35}\theta = |F_3|\cos(w_e t + \varepsilon_3) \\ a_{53}\ddot{z} + b_{53}\dot{z} + c_{53}z + (I_{55} + a_{55})\ddot{\theta} + b_{55}\dot{\theta} + c_{55}\theta = |F_5|\cos(w_e t + \varepsilon_5). \end{cases} \tag{6}$$

INTEGRAL MODELLING OF SHIP MOTION

In this section, we are going to transfer the coupled differential equations of motion to the system of Volterra integral equations. For this purpose let us transfer the system of Eq. 6. to the following system:

$$\begin{cases} a'_{33}\ddot{z} + b_{33}\dot{z} + c_{33}z + a_{35}\ddot{\theta} + b_{35}\dot{\theta} + c_{35}\theta = F_3\cos w_e t \\ a_{53}\ddot{z} + b_{53}\dot{z} + c_{53}z + a'_{55}\ddot{\theta} + b_{55}\dot{\theta} + c_{55}\theta = F_5\cos w_e t, \end{cases} \tag{7}$$

where, $a'_{33} = a_{33} + \rho V$ and $a'_{55} = I_{55} + a_{55}$. The heave and pitch response to the regular wave excitation are given by $z(t) = z_a \cos w_e t$ and $\theta(t) = \theta_a \cos w_e t$. Then we find (Journee & Massie 2001):

$$\begin{aligned} &z(0) = z_a\ ,\ \theta(0) = \theta_a, \\ &\dot{z}(0) = 0\ ,\ \dot{\theta}(0) = 0. \end{aligned} \tag{8}$$

By twice integrating the system equations (7) in the interval [0,t] and using the initial values (8) we obtain (Kanwal 1971):

$$
(9)\quad \begin{cases} a'_{33}z(t) + f_1(t) + \int_0^t k_1(t,t_1)z(t_1)dt_1 \\ + a_{35}\theta(t) + f_2(t) + \int_0^t k_2(t,t_1)\theta(t_1)dt_1 = E_1(t), \text{ and} \\ a_{53}z(t) + f_3(t) + \int_0^t k_3(t,t_1)z(t_1)dt_1 \\ a'_{55}\theta(t) + f_4(t) + \int_0^t k_4(t,t_1)\theta(t_1)dt_1 = E_2(t), \end{cases}
$$

where

$$
(10)\quad \begin{cases} f_1(t) = -[a'_{33}z_a + b_{33}z_a t], \quad k_1(t,t_1) = [b_{33} + c_{33}(t-t_1)] \\ f_2(t) = -[a_{35}\theta_a + b_{35}\theta_a t], \quad k_2(t,t_1) = [b_{35} + c_{35}(t-t_1)] \\ f_3(t) = -[a_{35}z_a + b_{35}z_a t], \quad k_3(t,t_1) = [b_{53} + c_{53}(t-t_1)] \\ f_4(t) = -[a'_{55}\theta_a + b_{55}\theta_a t], \quad k_4(t,t_1) = [b_{55} + c_{55}(t-t_1)]. \\ E_1(t) = \int_0^t \int_0^{t_1} F_3(t_2)\cos w_e t_2 dt_2 dt_1, \text{ and} \\ E_2(t) = \int_0^t \int_0^{t_1} F_5(t_2)\cos w_e t_2 dt_2 dt_1. \end{cases}
$$

From relations (9) and (10), we obtain a system of Volterra integral equations of the second kind. Following are the numerical Trapezoidal and Iterative methods, and last section shows the accuracy results of the integral equations method for a ship motion.

TRAPEZOIDAL METHOD

In order to solve the system of Eq. 9. with this method, suppose for any positive constant b, the interval [0, b] is divided into N equal subintervals of length $h = b/N$ such that $0 = s_0 < s_1 < \ldots < s_N = b$, $s_i = ih$, and $i = 0, 1, \ldots, N$.

For i = 1 we obtain:

$$\begin{cases} a'_{33}z_1 + f_1(s_1) + h\sum_{j=0}^{1} k_1(t,s_j)z_j \\ + a_{35}\theta_1 + f_2(s_1) + h\sum_{j=0}^{1} k_2(t,s_j)\theta_j = E_1(s_1), \text{and} \\ a_{53}z_1 + f_3(s_1) + h\sum_{j=0}^{1} k_3(t,s_j)z_j \\ + a'_{55}\theta_1 + f_4(s_1) + h\sum_{j=0}^{1} k_4(t,s_j)\theta_j = E_2(s_1), \end{cases} \tag{11}$$

where $z_i \simeq z(s_i),\ \theta_i \simeq \theta(s_i),\ i = 1, 2, ..., N.$

By simplifying the system (11) we find

$$\begin{cases} A_1 z_1 + B_1\theta_1 = C_1, \text{and} \\ A_2 z_1 + B_2\theta_1 = C_2, \end{cases} \tag{12}$$

where $A_1, A_2, B_1, B_2, C_1, \mathit{and}\ C_2$ are known coefficients. Thus $z_1,\ \theta_1$ may be obtained as solution of the following set of simultaneous equations.

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \begin{pmatrix} z_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{13}$$

The coefficient matrix is nonsingular (Delves & Mohamed 1985). It follows that at the rth stage $z_r\ \mathit{and}\ \theta_r$ are obtained by solving the following system:

$$\begin{cases} a'_{33}z_r + f_1(s_r) + h\sum_{j=0}^{r} k_1(t,s_j)z_j \\ + a_{35}\theta_r + f_2(s_r) + h\sum_{j=0}^{r} k_2(t,s_j)\theta_j = E_1(s_r), \text{and} \\ a_{53}z_r + f_3(s_r) + h\sum_{j=0}^{r} k_3(t,s_j)z_j \\ + a'_{55}\theta_r + f_4(s_r) + h\sum_{j=0}^{r} k_4(t,s_j)\theta_j = E_2(s_r). \end{cases} \tag{14}$$

By simplifying the system (14), we find:

$$\begin{cases} A_{r1} z_r + B_{r1}\theta_r = C_{r1}, \text{ and} \\ A_{r2} z_r + B_{r2}\theta_r = C_{r2}, \end{cases} \tag{15}$$

where $A_{r1}, A_{r2}, B_{r1}, B_{r2}, C_{r1}, and\ C_{r2}$ are known coefficients.

ITERATIVE SCHEME

An Iterative scheme based on the same principle is also available for linear integral equations of the second kind. In this section, we use this method for solving the system of Eq. 9. First we convert the system of Eq. 9. to the system of Eq. 16.

$$\begin{cases} a'_{33}z(t) + a_{35}\theta(t) = \\ E_1(t) - f_1(t) - f_2(t) - \int_0^t k_1(t,t_1)z(t_1)dt_1 - \int_0^t k_2(t,t_1)\theta(t_1)dt_1, \text{ and} \\ a_{53}z(t) + a'_{55}\theta(t) = \\ E_2(t) - f_3(t) - f_4(t) - \int_0^t k_3(t,t_1)z(t_1)dt_1 - \int_0^t k_4(t,t_1)\theta(t_1)dt_1. \end{cases} \tag{16}$$

As a zero-order approximation to the desired functions z_0 and θ_0, $z_0(t) = 0\ and\ \theta_0(t) = 0$ are taken. These are substituted into the right hand side of system of Eq. 16. to give the first-order approximation:

$$\begin{cases} a'_{33}z_1(t) + a_{35}\theta_1(t) = E_1(t) - f_1(t) - f_2(t), \text{ and} \\ a_{53}z_1(t) + a'_{55}\theta_1(t) = E_2(t) - f_3(t) - f_4(t). \end{cases} \tag{17}$$

Then, $z_1(t)\ and\ \theta_1(t)$ are obtained from the system of Eq. 17. These functions, when substituted into (16), yield the second approximation. This process is then repeated; the (n + 1)st approximation is obtained by substituting the nth approximation in the right hand side of (16):

$$\begin{cases} a'_{33}z_{n+1}(t) + a_{35}\theta_{n+1}(t) = \\ E_1(t) - f_1(t) - f_2(t) - \int_0^t k_1(t,t_1)z_n(t_1)dt_1 - \int_0^t k_2(t,t_1)\theta_n(t_1)dt_1, \text{ and} \\ a_{53}z_{n+1}(t) + a'_{55}\theta_{n+1}(t) = \\ E_2(t) - f_3(t) - f_4(t) - \int_0^t k_3(t,t_1)z_n(t_1)dt_1 - \int_0^t k_4(t,t_1)\theta_n(t_1)dt_1. \end{cases} \tag{18}$$

If $z_n(t)$ and $\theta_n(t)$ tend uniformly to the limits as $n \to \infty$, then these limits are the required solutions, $z(t) = \lim_{n\to\infty} z_n(t)$,and \ $\theta(t) = \lim_{n\to\infty} \theta_n(t)$, (Kanwal 1971).

NUMERICAL RESULTS

Suppose the hydrodynamic coefficients for heave and pitch motions of a ship are as follows:

$$a_{ij} = b_{ij} = c_{ij} = \begin{cases} 1 & i \neq j \\ -1 & i = j \end{cases} \tag{19}$$

and

$$\begin{aligned} & m = 1000(kg), \\ & I_{xx} = I_{yy} = I_{zz} = 2000(kg.m^3), \\ & z_a = \theta_a = 2(m), \\ & F_3 = 125\ \pi^2(N),\ F_5 = 250\ \pi^2(N),\ \text{and} \\ & w_e = \pi/4(rad),\ T_e = 8(s),\ t = 2(s). \end{aligned} \tag{20}$$

By the above conditions the equations of motion are as follows:

$$\begin{cases} -1001\ddot{z} - \dot{z} - z + \ddot{\theta} + \dot{\theta} + \theta = 125\pi^2 \cos(\frac{\pi}{4} t) \\ \ddot{z} + \dot{z} + z - 2001\ddot{\theta} - \dot{\theta} - \theta = 250\pi^2 \cos(\frac{\pi}{4} t), \end{cases} \tag{21}$$

with exact solution $z(t) = 2\cos(\frac{\pi}{4} t)\ and\ \theta(t) = 2\cos(\frac{\pi}{4} t)$.

We solve this system by an Iterative scheme (Shisfar & Garshasbi 2005). As a zero approximation let $z_0(t) = 0,\ \theta_0(t) = 0$, then by using a program written in Matlab, Version 6.5 we find $z_1(t) = 2\cos(\frac{\pi}{4} t)\ and\ \theta_1(t) = 2\cos(\frac{\pi}{4} t)$.

Now we solve this system by the Trapizoidal method. Let $h = 0.001$, we find $s_i = 0.001i$. The numerical results are shown in Table1.

Table 1: Numerical solution by the Trapezoidal method

t	z(t)	$\theta(t)$
0.0	2	2
0.001	1.9999993831	1.9999993831
0.002	1.9999975325	1.9999975325
0.003	1.9999944483	1.9999944483
⋮	⋮	⋮
0.01	1.999938315	1.999938315
0.02	1.999753264	1.999753264
⋮	⋮	⋮
0.1	1.9938346674	1.9938346674
0.2	1.9753766811	1.9753766811
⋮	⋮	⋮
1	1.4142135623	1.4142135623

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دراسة عددية لحل معادلات هيف وبتش لسفينة

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خلاصة

تستعرض هذه الورقة طريقتين عدديتين لحل معادلات هيف وبيتش لحركة سفينة. ولهذا الغرض، فإن نظام المعادلات التفاضلية سوف يتم تحويله إلى نظام معادلات فولتيرا التكميلية ومن ثم يتم حلها باستخدام طريقة المعين المنحرف وخطة التكرار. وأخيراً وباستخدام مثال، ستتم المقارنة بين الحل العددي والتحليلي.