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An efficient simulated annealing algorithm for stochastic systems

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ABSTRACT

In the fields of management and administrative science, operations research and industrial engineering, many practical problems can be modeled as discrete stochastic optimization problems where the objective function can be evaluated only through Monte Carlo simulation. In this paper we present a modification of the simulated annealing algorithm (SA) for solving discrete stochastic optimization problems where the acceptance probability depends on whether the objective function values indicate a statistically significant difference at each iteration. Similar to the original SA algorithm, the proposed approach has the hill climbing feature to escape the trap of local optima. However, our method uses constant temperature rather than decreasing temperature, and selects the estimated optimal solution as the state with the best average estimated objective function value obtained from all the previous estimates of the objective function value. We show that the proposed modification converges almost surely to the set of optimal solutions. Computational results and comparisons with other variants are given to demonstrate the performance of the proposed modified SA algorithm.

Keywords: Discrete Parameters; Simulated Annealing; Simulation Optimization.

INTRODUCTION

In this paper we consider a class of optimization problems where the objective function is a stochastic discrete function and can be evaluated only through Monte Carlo simulation. A general problem of discrete stochastic optimization can be defined as

$$\min_{i \in S} \{f(i) := E_{\mathrm{P}}[Y(i,\omega)]\}, \tag{1}$$

where S, the search space, is a large, finite, and discrete set; i is the design parameters; $f(i)$ is the performance measure of interest; and $Y(i,\omega)$ represents the sample performance function based on a sample realization ω (ω can be thought of as representing the randomness in the system (noise), e.g., all the random numbers in a simulation run) where ω is a random vector having probability distribution P, and $E_{\mathrm{P}}[Y(i,\omega)] = \int Y(i,\omega) dP(y)$ is the corresponding

expected value. We assume that the expected value function, $f(i)$, is well defined, i.e., for every $i \in S$, the function $Y(i,\cdot)$ is P-measurable and $E_{\mathrm{P}}\{|Y(i,\omega)|\} < \infty$.

If the expected value $E_p[Y(i,\omega)]$, can be found analytically for all i then (1) represents a deterministic optimization problem which can be solved either analytically or numerically by methods of numerical programming. We are interested in those systems whose $f(i)$ cannot be easily obtained through analytical means and therefore must be estimated from sample paths, e.g., via discrete event simulation. Many real-life systems such as inventory systems, queuing systems, transportation networks, reliability systems and flexible manufacturing systems can be modeled as discrete-event systems. These systems are driven by the occurrence of discrete events. Due to the complex interactions of such discrete events over time, the performance analysis and optimization of these systems can be difficult tasks. At the same time, since such systems are becoming more widespread as a result of modern technological advances, it is important to develop efficient methods for optimizing the parameters of these systems.

Simulated annealing (SA) algorithm was proposed originally by Kirkpatrick *et al.* (1983) for solving complex deterministic optimization problems with discrete space. SA has shown successful application in a wide range of industrial combinatorial optimization problems and this fact has motivated researchers to use SA in simulation optimization. However SA needs to evaluate the objective function values accurately and there have been few studies for the cases where the objective function values can be estimated only through simulation. Haddock and Mittenthal (1992) investigated the feasibility of using a SA algorithm in conjunction with a simulation model to find a combination of input parameter values which maximize the total expected profit of an automated manufacturing system. In their approach, only three input parameters are considered for system evaluation and each evaluation is in fact a simulation run using the selected combination of input parameters. The evaluated value is the corresponding steady-state mean of the simulation run. They treated the point estimate coming from the simulation output as a deterministic value and used it in the simulated annealing algorithm to obtain the optimal solution point. When comparing two solution points, one cannot draw a conclusion and make a decision based on point estimates only, even when steady state behaviors have been reached without running a large number of simulations.

A theoretical analysis of simulated annealing applied to discrete stochastic optimization problems is presented by Gelfand and Mitter (1989). They showed that under suitable conditions of noise, the modified annealing algorithm exhibits the same convergence in probability to the globally optimal states as the original annealing algorithm. Gutjahr and Pflug (1996) generalized the classical convergence result for the SA algorithm to the case where cost function observations are disturbed by random noise. They showed that for a certain

class of noise distributions, the convergence assertion remains valid, provided that the standard deviation of the noise is reduced in the successive steps of cost function evaluation with a speed $O(k^{-\gamma})$, where γ is an arbitrary constant larger than one.

Fox and Heine (1995) also used SA algorithm to solve discrete stochastic optimization problems. Their approach has no restrictive variance assumption. In their approach, each time an estimate of the objective function is needed, they generate a few more observations from the objective function and average them with the previous observations calculated in earlier iterations to obtain the desired estimate of the objective function. They showed that for a properly chosen annealing schedule, their approach is guaranteed to converge in probability to the set of global optimal solutions. Roenko (1990) applied SA algorithm to a stochastic optimization problem. His approach, however, makes it necessary to store all feasible solutions encountered during the execution of the algorithm and to compare them with each newly generated solution. Therefore, this approach does not seem to be realistic for practical applications due to the computational burden involved.

Alrefaei and Andradóttir (1999) presented a modification of the SA algorithm for discrete stochastic optimization problems. Their modification differs from the original SA algorithm in that they use a constant (rather than decreasing) temperature. Alkhamis *et al.* (1999) presented a variant of the SA algorithm for discrete stochastic optimization problems. The basic idea of their modification is to make the comparison between solution point i and solution point j based on whether the objective function value indicates a statistically significant difference at each iteration.

In this paper we extend our work presented in Alkhamis *et al.* (1999) to include two features. First, we use constant temperature rather than decreasing temperature. Second, we select the estimated optimal solution as the state with the best average estimated objective function value obtained from all the previous estimates of the objective function value as in Andradóttir (1999). We present computational results which show that our proposed approach is efficient in finding the optimal solution when used to solve discrete stochastic optimization problems where the objective function values are estimated using simulation; also our computational results demonstrate the efficiency of our approach compared to other variants of SA. The paper is organized as follows. The next section reviews the original SA algorithm and presents our approach to handle objective functions disturbed with noise. The third section presents our new variant of the SA algorithm and proves its convergence to the set of optimal solutions. The fourth section presents computational results and compares the performance of our method with other simulated annealing

algorithms for discrete stochastic optimization problems. Finally, the last section contains some concluding remarks.

SIMULATED ANNEALING FOR STOCHASTIC OBJECTIVE FUNCTIONS

To present the new variant of the simulated annealing algorithm, we need the following definitions and assumptions.

Definition 2.1: For each $i \in S$ there exists a subset $N(i)$ of $S - \{i\}$ which is called the set of neighbors of i, such that each point in $N(i)$ can be reached from i in a single transition.

Definition 2.2: A function $G : S \times S \rightarrow [0,1]$ is said to be a generating probability function for S and N if:

$$(1)\ \ G_{ij} > 0 \Leftrightarrow j \in N(i)\ \textit{and}$$

$$(2)\ \sum_{j \in S} G_{ij} = 1\ \textit{for all}\ i,j \in S.$$

G_{ij} is the probability of generating solution point j as a candidate for the next solution point when the system is in solution point i. We will consider G_{ij} such that the probability is distributed uniformly over $N(i)$. Given $i \in S$, a candidate solution is selected among $N(i)$ such that the probability of selecting a neighbour $j \in N(i)$ is equal to G_{ij} , where

$$G_{ij} = \begin{cases} \frac{1}{|N(i)|} & \textit{for}\ j \in N(i) \\ 0 & \textit{otherwise} \end{cases} .$$

Assumption 2.1: For any pair $(i, j) \in S \times S$, j is reachable from i , i.e. there exists a finite sequence, $\{n_m\}_{m=0}^{\ell}$ for some ℓ, such that $i_{n_0} = i$, $i_{n_\ell} = j$ and $i_{n_{m+1}} \in N(i_{n_m}), m = 0, 1, 2, ..., \ell - 1$.

Assumption 2.2: The feasible region S is a finite set containing at least two solution points and the set of all global optimal solutions $S^*$is a proper subset of S.

In SA, one needs a sequence of positive real numbers $\{T_k, k = 0, 1, ...\}$ satisfying $T_k \geq 0$, $T_{k+1} < T_k \forall k, \textit{and}\ lim_{k \rightarrow \infty} T_k = 0. T_k$ is called the temperature at the kth iteration and the sequence $\{T_k, k = 0, 1, ...\}$ is called the cooling schedule. The SA algorithm can be described as a sequence of Markov chains with the state space being the domain of the objective function to be optimized and X_k denotes the state of the system visited by the SA algorithm at the kth step.

SA algorithm needs to evaluate the objective function value $f(i)$ accurately. In this paper we consider situations where $f(i)$ can only be evaluated via Monte carlo simulation. Since the input processes driving the simulation are random,

the output from the simulation is also random. The runs of the simulation do not directly yield the desired measures of system performance but they only give estimates of the performance measures. Since the estimators are themselves random variables, they are subject to sampling error. Accordingly, SA algorithm must be modified to handle the stochastic nature of the simulation output.

Now we describe our modification for the simulated annealing algorithm. On the kth iteration with current configuration (state) i and a candidate configuration j, we generate L_k independent observations of the difference $D_{ji} = Y_j - Y_i$ between the objective function value Y_j of solution point j and the objective function value Y_i of solution point i. Let:

$$\bar{D}_{ji} = \bar{Y}_j(L_k) - \bar{Y}_i(L_k) = \frac{1}{L_k}\sum_{\ell=1}^{L_k} D_{ji}^{\ell} \text{ and } \hat{\sigma}_k = \frac{1}{\sqrt{L_k}}\sqrt{\frac{1}{L_k - 1}\sum_{\ell=1}^{L_k}(D_{ji}^{\ell} - \bar{D}_{ji})^2}$$

respectively, denote the sample mean and sample standard error of the mean based on the observed sample of D_{ji}. Let t_k denote a selected upper critical value of student's t-distribution with $L_k - 1$ degrees of freedom. In the proposed modification of SA algorithm, the accepting probability has the form $\min\left\{1, exp\left[\frac{-[\bar{Y}_j(L_k) - \bar{Y}_i(L_k) - t_{L_k}\hat{\sigma}_{L_k}]}{T}\right]\right\}$. The transition matrix for the kth step is given by:

$$\tilde{P}_{ij}(k) = P\{X_{k+1} = j | X_k = i\}$$

$$= \begin{cases} G_{ij}P\left\{U_k \leq \exp\left[\frac{-[\bar{Y}_j(L_k) - \bar{Y}_i(L_k) - t_{L_k}\hat{\sigma}_{L_k}]^+}{T}\right]\right\} & j \in N(i) \\ 1 - \sum_{\ell \in N(i)} P_{i\ell}(k) & j = i \end{cases}$$

(*where for all* $a \in R, a^+ = a$ *if* $a > 0$, *and* $a^+ = 0$ *otherwise*),

where U_k is a uniform random variable defined on the interval [0,1]. Note that if $j \in N(i)$, then:

$$P\left\{U_k \leq \exp\left[\frac{-[\bar{Y}_j(L_k) - \bar{Y}_i(L_k) - t_{L_k}\hat{\sigma}_{L_k}]^+}{T}\right]\right\}$$

$$= E\left\{\exp\left[\frac{-[\bar{Y}_j(L_k) - \bar{Y}_i(L_k) - t_{L_k}\hat{\sigma}_{L_k}]^+}{T}\right]\right\}$$

In our modification, the acceptance probabilities are larger than the ones in the regular SA algorithm, since we consider whether the difference between estimated objective values is statistically significant. In the fourth section, computational experience with three test cases demonstrates the advantage of our modification.

In the next section, we present two variants of the simulated annealing algorithm that use the above modified Metropolis criterion. Both variants use constant temperature but they differ in the way of estimating the optimal solution. The first variant, presented in Ahmed (2005), uses the state that is visited most often by the algorithm (divided by a normalizer) as the estimated optimal solution. The second variant selects the estimated optimal solution as the state with the best average estimated objective function value obtained from all the previous estimates of the objective function value.

VARIANTS OF THE MODIFIED SIMULATED ANNEALING ALGORITHM

First, we review a variant of SA algorithm as presented in Ahmed (2005) so as to further motivate our approach. In this variant (denoted as Algorithm 1), a constant temperature was used and the state that had been visited most often by the algorithm (divided by a normalizer) was selected as an estimate of the optimal solution. Let, for all $i \in S$ and $k \geq 0$, $V_k(i)$ be the number of times that the Markov chain $\{X_k, k=0,1,...\}$has visited state i in the first k iterations, and X_k^* is the state that the search process has visited most often after k iterations. Letting$\tilde{V}_k(i) = \dfrac{V_k(i)}{|N(i)|}$, then $X_k^* = \{i : \tilde{V}_k(i) \geq \underset{j \in S}{Max}\, \tilde{V}_k(j)\}$. It is clear that the value of X_k^*, say j, changes only when the search process visits point i and $\tilde{V}_k(i) > \tilde{V}_k(j)$. Before we state the algorithm, we need the following assumption.

Assumption 3.1 Let $\{L_k\}$ be a sequence of positive integers such that $\lim_{k \to \infty} L_k = \infty$.

The steps of Algorithm 1 are as follows:

Algorithm 1

Parameters: N, T, $\{L_k\}$.

Step 1. Select a starting point $i_0 \in S$. Let $V_0(i_0) = 1$ and $V_0(j) = 0$ for all $j \in S$, $j \neq i_0$. Let $k = 0$ and $X_k^* = i_0$. Go to step 2.

Step 2. Given $X_k = i$, choose a candidate $Z_k \in N(i)$ with probability distribution

$$P[Z_k = j | X_k = i] = G_{ij},\ j \in N(i).$$

Step 3. Given $Z_k = j$, generate two L_k independent observations $Y_i^1, Y_i^2, ..., Y_i^{L_k}$ and $Y_j^1, Y_j^2, ..., Y_j^{L_k}$. Evaluate $\bar{Y}_i(L_k)$, $\bar{Y}_j(L_k)$ and $\hat{\sigma}_{L_k}$.

Step 4. Given $Z_k = j$, generate $U_k \sim U[0, 1]$, (U_k is uniformly distributed) and set

$$X_{k+1} = \begin{cases} Z_k & \text{if } U_k \leq exp\left[\frac{-[\bar{Y}_j(L_k) - \bar{Y}_i(L_k) - t_{L_k}\hat{\sigma}_{L_k}]^+}{T}\right] \\ X_k & \text{otherwise} \end{cases}$$

Step 5: Set $k = k+1$, $V_k(X_k) = V_{k-1}(X_k) + 1$ and $V_k(j) = V_{k-1}(j)$, for all $j \in S$ and $j \neq X_k$. If $\frac{V_k(X_k)}{|N(X_k)|} > \frac{V_k(X_{k-1}^*)}{|N(X_{k-1}^*)|}$, then let $X_k^* = X_k$; otherwise let $X_k^* = X_{k-1}^*$. Update L_k. Go to step 2.

The stochastic random process $\{X_k\}$ produced by Algorithm 1 is a discrete-time inhomogeneous Markov chain defined over states S, and its state transition probability is given by:

$$\tilde{P}_{ij}(k) = \begin{cases} G_{ij} E\left\{\exp\left[\frac{-[\bar{Y}_j(L_k) - \bar{Y}_i(L_k) - t_{L_k}\hat{\sigma}_{L_k}]^+}{T}\right]\right\} & j \in N(i) \\ 1 - \sum_{\ell \in N(i)} P_{i\ell}(k) & j = i \end{cases} \tag{2}$$

Let P_{ij} be the transition probability matrix defined by:

$$P_{ij} = \begin{cases} G_{ij} \exp\left[\frac{-[f(j) - f(i)]^+}{T}\right] & j \in N(i) \\ 1 - \sum_{\ell \in N(i)} P_{i\ell} & j = i \end{cases} . \tag{3}$$

Convergence of Algorithm 1 can be found in Ahmed (2005).

Now we present another variant of the simulated annealing algorithm (which we will refer to as Algorithm 2). This variant follows the same procedure as Algorithm 1, but uses a different approach for estimating the optimal solution. Instead of using the most visited point to estimate the optimal solution, the new approach selects the state with the best average estimated objective function value obtained from all the previous estimates of the objective function values to be the estimated optimal solution. This approach requires maintaining two

variables for each point $i \in S$. Let $\overline{\overline{L}}_k(i)$ be the number of estimates of $f(i)$ generated in the first k iterations and let $\overline{\overline{Y}}_k(i)$ be the sum of all estimates of $f(i)$ that have been generated in the first k iterations. The estimate of the optimal solution would then be a point $i^* \in \arg\min_{i \in S} \frac{\overline{\overline{Y}}_k(i)}{\overline{\overline{L}}_k(i)}$. Before we state the steps of Algorithm 2 we need the following assumption.

Assumption 3.2 Let $\{L_k\}$ be a sequence of positive integers such that $\lim_{k \to \infty} L_k = L \leq \infty$.

The steps for the modified SA algorithm are as follows:

Algorithm 2.

Parameters: N, T, $\{L_k\}$.

Step 1. Select a starting point $X_0 \in S$. Let $V_0(X_0) = 1$ and $V_0(i) = 0$ for all $i \in S$, $X \neq X_0$.

Let $\overline{\overline{Y}}_0(i) = 0$ and $\overline{\overline{L}}_0(i) = 0$ for all $i \in S$. Let $k = 0$ and $X_k^* = X_0$. Go to step 1.

Step 2. Given $X_k = i$, choose a candidate $Z_k \in N(i)$with probability distribution

$$P[Z_k = j | X_k = i] = G_{ij},\ j \in N(i).$$

Step 3. Given $Z_k = j$, generate two L_k independent observations $Y_i^1, Y_i^2, ..., Y_i^{L_k}$ and $Y_j^1, Y_j^2, ..., Y_j^{L_k}$. Evaluate $\bar{Y}_i(L_k)$, $\bar{Y}_j(L_k)$ and $\hat{\sigma}_{L_k}$. Let $\overline{\overline{L}}_{k+1}(\ell) = \overline{\overline{L}}_k(\ell) + L_k$ and $\overline{\overline{Y}}_{k+1}(\ell) = \overline{\overline{Y}}_k(\ell) + L_k$ for $\ell = i, j$.

Step 4. Given $Z_k = j$, generate $U_k \sim U[0, 1]$, and set

$$X_{k+1} = \begin{cases} Z_k & \text{if } U_k \leq exp\left[\frac{-[\bar{Y}_j(L_k) - \bar{Y}_i(L_k) - t_{L_k}\hat{\sigma}_{L_k}]^+}{T}\right] \\ X_k & \text{otherwise} \end{cases}$$

Step 5. Set $k = k + 1$ and select $X_k^* \in \arg\min_{i \in S} \frac{\overline{\overline{Y}}_k(i)}{\overline{\overline{L}}_k(i)}$. Go to step 2.

The stochastic process $\{X_k, k = 0, 1, 2, ...\}$ generated by Algorithm 2 is a time inhomogeneous Markov chain with transition matrix given by (2).

Convergence of the modified SA algorithm

In this section we discuss the convergence of Algorithm 2 and show that our approach is guaranteed to converge almost surely to the set of global optimal

solutions.

Let P'_{ij} be the transition probability matrix defined by

$$P'_{ij} = \begin{cases} G_{ij} \quad E\left[\exp\left[\frac{-[\bar{Y}_j - \bar{Y}_i - t\hat{\sigma}]^+}{T}\right]\right] j \in N(i) \\ 1 - \sum_{\ell \in N(i)} P'_{i\ell} \qquad\qquad\qquad j = i \end{cases} \tag{4}$$

where $\bar{Y}_j = \frac{1}{L}\sum_{\ell=1}^{L} Y_j^{\ell}$, $\bar{Y}_i = \frac{1}{L}\sum_{\ell=1}^{L} Y_i^{\ell}$ and $\hat{\sigma} = \frac{1}{\sqrt{L}}\sqrt{\frac{1}{L-1}\sum_{\ell=1}^{L}(D_{ji}^{\ell} - \bar{D}_{ji})^2}$.

To prove the convergence of Algorithm 2, we need the following lemma proved by Andradóttir (1995).

Lemma 3.1 Suppose that $\{X_k\}$is a non-homogeneous Markov chain with a finite state space S and transition matrix $\tilde{P}_{ij}(k) \to P_{ij}$ as $k \to \infty$ for all $i, j \in S$, and P is an irreducible and aperiodic Markov chain. Let $g : S \to R$ be a real valued function on the state space S. Then $\lim \frac{1}{m}\sum_{k=1}^{m} g(X_k) = \sum_{i \in S} \pi_i g(i)$ almost surely *as* $m \to \infty$, where $\{\pi_j,\ j \in S\}$ is the equilibrium distribution of P.

Corollary 3.1 Let $g_j : S \to R$ be the indicator function of the set {j}, then:

$g_j(X_k) = 1_{\{X_k = j\}} = \begin{cases} 1 & \textit{if } X_k = j, \\ 0 & \textit{otherwise}. \end{cases}$. Applying Lemma 1, we obtain:

$P(\lim_{M \to \infty} \frac{1}{M}\sum_{k=1}^{M} 1_{\{X_k = j\}} = \pi_j) = 1$. In other words, if we observe the process $\{X_k\}$, the average number of visits to state j during the first M iterations converges to π_j for large M. Note that $V_M(j) = \sum_{k=1}^{M} 1_{\{X_k = j\}}$ which implies that $\frac{V_M(j)}{M}$ asymptotically equals π_j.

theorem 3.1. Suppose that assumptions (2.1), (2.2), and (3.2) are satisfied**.** Then the sequence $\{X_k^*\}$ generated by Algorithm 2 converges almost surely to S^*; i.e.,

$P\{\lim_{k \to \infty} 1_{\{X_k^* = S^*\}} = 1\} = 1$, where S^* is the set that contains the optimal solutions.

Proof: We need to prove the theorem for two cases, (i) for $L = \infty$ and (ii) for $L < \infty$. For case 1, the Markov chain $\{X_k\}$generated by Algorithm 2 coincides with the Markov chain generated by Algorithm 1. The proof for case 1 requires that $\tilde{P}_{ij}(k) \to P_{ij}$ as $k \to \infty$, for all $i, j \in S$ where $\tilde{P}_{ij}(k)$ and P_{ij} are the transition probabilities matrices given in equations (2) and (3) respectively, and P_{ij} is irreducible, aperiodic and has a stationary distribution, π, with $\pi_j > 0$ for all

$j \in S$. Now, if $i \neq j$, then:

$$\lim_{k\to\infty} \tilde{P}_{ij}(k) = G_{ij} \lim_{k\to\infty} E\left[exp\left[\frac{-[\bar{Y}_j(L_k) - \bar{Y}_i(L_k) - t_{L(k)}\hat{\sigma}_{L(k)}]^+}{T}\right]\right].$$

Since $\left|exp\left[\frac{-[\bar{Y}_j(L_k) - \bar{Y}_i(L_k) - t_{L(k)}\hat{\sigma}_{L(k)}]^+}{T}\right]\right| \leq 1$, the bounded convergence theorem gives:

$$\begin{aligned}
\lim_{k\to\infty} \tilde{P}_{ij}(k) &= G_{ij} \lim_{k\to\infty} E\left[exp\left[\frac{-[\bar{Y}_j(L_k) - \bar{Y}_i(L_k) - t_{L(k)}\hat{\sigma}_{L(k)}]^+}{T}\right]\right] \\
&= G_{ij} E\left[\lim_{k\to\infty} exp\left[\frac{-[\bar{Y}_j(L_k) - \bar{Y}_i(L_k) - t_{L(k)}\hat{\sigma}_{L(k)}]^+}{T}\right]\right] \\
&= G_{ij} E\left[exp\left[\frac{-[f(j) - f(i)]^+}{T}\right]\right] \\
&= P_{ij}
\end{aligned}$$

where the third equality follows from the strong law of large numbers and the assumption that $L_k \to \infty$ as $k \to \infty$. Note that $\frac{1}{L_k - 1}\sum_{\ell=1}^{L_k} (D_{ji}^{\ell} - \bar{D}_{ji})^2 \xrightarrow{p} \sigma_j^2 + \sigma_i^2$ as $L_k \to \infty$; therefore, $\hat{\sigma}_{L(k)} \xrightarrow{p} 0$ as $L_k \to \infty$.

The proof that P_{ij} is irreducible follows directly from Assumption 2.1 and Equation (3). To prove that P_{ij} is aperiodic, we need to show that $P_{ii} > 0$, since periodicity is a class property. Let $i^* \in S^*$, whereS^* is the set that contains the optimal configurations, and $i \in N(i^*)$ with $f(i^*) < f(i)$. Then, from the definition of P, $P_{i^*i^*} > 0$ and therefore, P is aperiodic. Now since S is finite from Assumption 2.2, and P_{ij} is irreducible and aperiodic, then P has a stationary distribution, π, with $\pi_j > 0$ for all $j \in S$. By Lemma 1, $\frac{V_k(j)}{k} \to \pi_j > 0$ almost surely as $k \to \infty$, for all $j \in S$, where $V_k(j)$ is the number of visits the Markov chain $\{X_k\}$makes to state j in the first k iterations for all $j \in S$. This implies that $V_k(j) \to \infty$ almost surely as $k \to \infty$, for all $j \in S$. Clearly $\overline{\overline{L}}_k(j) \geq V_k(j) - 1$ for all $j \in S$ where $\overline{\overline{L}}_k(j)$ is the number of times an estimate of $f(j)$ has been obtained in the first k iterations for all $j \in S$. Therefore by the strong law of large

numbers, we have that $\dfrac{\overline{\overline{Y}}_k(j)}{\overline{\overline{L}}_k(j)} \to f(j)$ almost surely as $k \to \infty$, for all $j \in S$. This proves the theorem for the case $L = \infty$.

For case 2, we follow the same steps as in Theorem 5 of Alrefaei and Andradottir (1999). It is easy to show that $\tilde{P}_{ij}(k) \to P'_{ij}$ as $k \to \infty$, for all $i,j \in S$ where $\tilde{P}_{ij}(k)$ *and* P'_{ij} are the transition probabilities matrices given in Equations (2) and (4) respectively. To show that P'_{ij} is irreducible we need to show that $E\left[\exp\left[\dfrac{-[\bar{Y}_j - \bar{Y}_i - t\hat{\sigma}]^+}{T}\right]\right]$ is positive. Now, by Jensen's inequality

$$\begin{aligned} E\left[\exp\left[\frac{-[\bar{Y}_j - \bar{Y}_i - t\hat{\sigma}]^+}{T}\right]\right] &\geq \exp\left[\frac{-1}{T} E\left[\bar{Y}_j - \bar{Y}_i - t\hat{\sigma}\right]^+\right] \\ &\geq \exp\left[\frac{-1}{T} E\left[\left|\bar{Y}_j - \bar{Y}_i - t\hat{\sigma}\right|\right]\right] \\ &\geq \exp\left[\frac{-1}{T} E\left[\left|\bar{Y}_j\right| - \left|\bar{Y}_i\right| - \left|t\hat{\sigma}\right|\right]\right] > 0. \end{aligned}$$

The second inequality is true since $E[X] = E[X^+] + E[X^-]$ and both expectations on the right hand side are non-negative, therefore, $E\left[\left|\bar{Y}_j - \bar{Y}_i - t\hat{\sigma}\right|\right] \geq E\left[\bar{Y}_j - \bar{Y}_i - t\hat{\sigma}\right]^+$. The last inequality holds since $E[|Y(j)|] \leq E[|Y(j,\omega)|] < \infty \; \forall j \in S$. Therefore, by Assumption 2.1, P'_{ij} is irreducible. Therefore, since S is finite, P'_{ij} has a stationary distribution, ν, with $\nu_j > 0$ for all $j \in S$.

To prove that P'_{ij} is aperiodic, we need to show that there exists $i \in S$ such that $P'_{ii} > 0$. Suppose that $P'_{ii} = 0$ for all $i \in S$. By Assumption 2.1, Equation (4), and the fact that $\exp\left[-\left[\bar{Y}_j - \bar{Y}_i - t\hat{\sigma}\right]^+ / T\right] \leq 1, \forall\, i,j \in S$, we have:

$$\begin{aligned} &P'_{ii} = 0 \text{ for all } i \in S \\ &\Leftrightarrow E\left[\exp\left[\frac{-[\bar{Y}_j - \bar{Y}_i - t\hat{\sigma}]^+}{T}\right]\right] = 1 \quad \forall\, i \in S,\ j \in N(i) \\ &\Leftrightarrow \exp\left[\frac{-[\bar{Y}_j - \bar{Y}_i - t\hat{\sigma}]^+}{T}\right] = 1 \text{ a.s.}, \forall\, i \in S,\ j \in N(i) \\ &\Leftrightarrow \bar{Y}_j - \bar{Y}_i - t\hat{\sigma} = 0 \ a.s., \forall\, i \in S,\ j \in N(i) \\ &\Leftrightarrow \bar{Y}_j - \bar{Y}_i = 0 \ a.s., \forall\, i,\ j \in S \\ &\Leftrightarrow f(i) = f(j), \forall\, i,\ j \in S. \end{aligned}$$

Since this contradicts Assumption 2.2, we have shown that P'_{ij} is aperiodic.

Since $\tilde{P}_{ij}(k) \to P'_{ij}$ as $k \to \infty$ where P'_{ij} is irreducible and aperiodic and P'_{ij} has

a stationary distribution, ν, with $\nu_j > 0$ for all $j \in S$, by Lemma 1, $\frac{V_k(j)}{k} \to \nu_j > 0$ almost surely as $k \to \infty$, for all $j \in S$, where $V_k(j)$ is the number of visits the Markov chain $\{X_k\}$makes to state j in the first k iterations for all $j \in S$. This implies that $V_k(j) \to \infty$ almost surely as $k \to \infty$, for all $j \in S$. The rest of the proof follows the same steps for the case $L = \infty$. This completes the proof.

COMPUTATIONAL RESULTS

Inventory and queuing models are widely used in the design and analysis of many industrial systems. In this section, we present examples of three different discrete simulation optimization problems. In the first example, we implement the modified SA algorithm to solve an inventory example with 101 feasible solutions, similar to the one used in Andradóttir (1999). In the second example we solve four instances of an (s, S) inventory problem. In the third example, we consider the optimization of M/M/1 queuing system in transient analysis and compare our results with the one presented in Alrefaei and Andradóttir (1999).

Inventory example

Suppose that the demand for a certain product has a Poisson distribution with parameter λ. Consider the problem of determining the order size n that maximizes the probability that the demand D equals n, subject to the constraint that at most N units can be ordered. This is an optimization problem of the form (1) with $S = \{0, 1, \ldots, N\}$. It is clear that the objective function, f, of this optimization problem is given by $f(i) = -\lambda^i \frac{e^{-\lambda}}{i!}$ for all $i \in S$, so the solution to this optimization problem can be determined analytically. We apply Algorithms 1 and 2 to solve this optimization problem with $T = 0.01$, $\lambda = 10$, $N = 100$, and $L_k = 10 + \left\lfloor \frac{k}{300} \right\rfloor$ for Algorithm 1. We use two neighborhood structures. The first neighborhood structure is given by $N1(i) = \{j \in S : |j - i| \leq 5\}$, while the second neighborhood structure is given by $N2(i) = \{j \in S : |j - i| \leq 10\}$. For this problem setting, we have two global maxima at $i = 9$ and at $i = 10$ with values equal to 0.125. Figures 1 and 2 compare the performance of Algorithms 1 and 2 in solving this optimization problem with the choice of parameters described above using neighborhood structure $N1$ and $N2$, respectively. The x-axis shows the iteration number, while the y-axis shows the average estimated optimal objective value at the estimated optimal solution over 100 replications. Figure 1 shows that Algorithm 2 converges faster than Algorithm 1. More specifically, Algorithm 2 converges very close to the optimal value (i.e. 0.125)

after about 2000 iterations, whereas Algorithm 1 converges after about 6500 iterations. In Figure 2, Algorithm 2 converges to the optimal value after about 950 iterations, whereas Algorithm 1 reaches the optimal value after performing 2500 iterations. Figures 1 and 2 indicate that Algorithm 2 outperforms Algorithm 1 for this setting of the inventory problem.

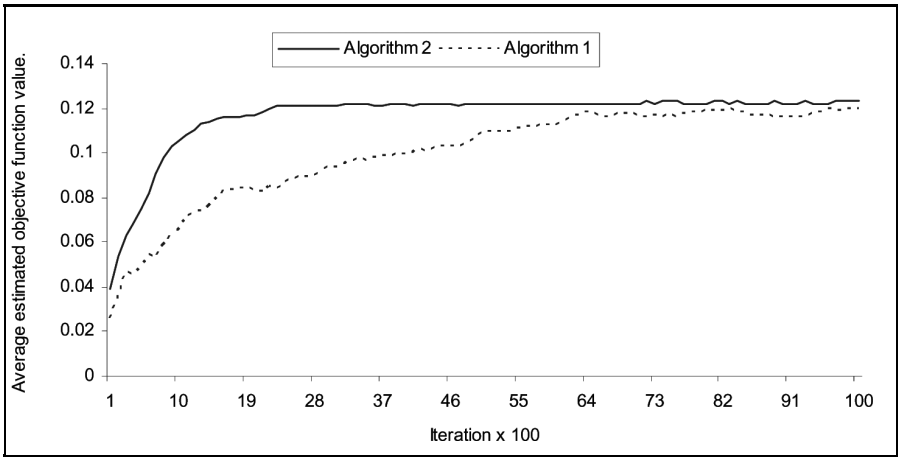

Figure 1: Performance of Algorithm 1 and Algorithm 2 for the inventory example using neighborhood structure N1

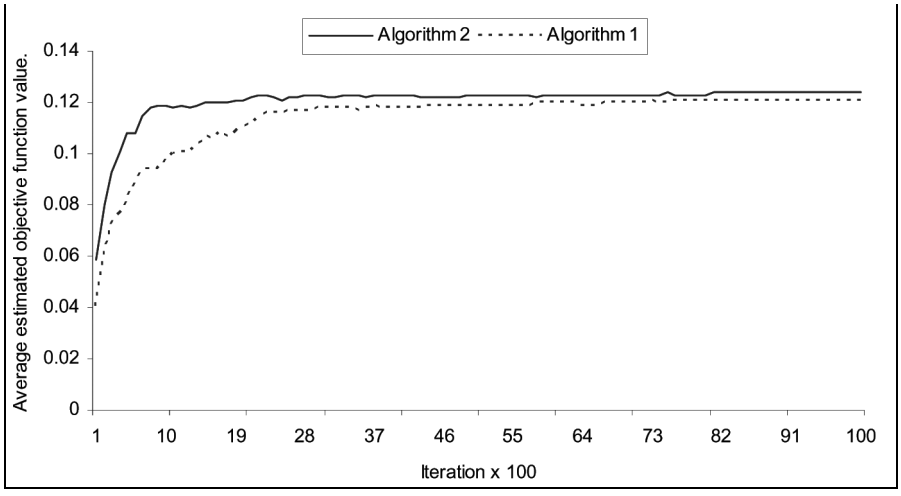

Figure 2: Performance of Algorithm 1 and Algorithm 2 for the inventory example using neighborhood structure N2

(s, S) inventory problem

Consider the standard infinite horizon , single item periodic review inventory

model with zero lead times (Fu & Hu 1997). In this model, the demands in the successive periods are mutually independent, non-negative and identically distributed random variables. At the beginning of each period, an order may be placed for any positive quantity of stock. When the demand during a period exceeds the inventory on hand, then the excess demand is backlogged until it is subsequently filled by a delivery.

The following costs are involved. There is a fixed set-up cost for each order, a linear purchase cost, a holding and shortage cost functions. Under the assumption of linear costs, a policy of the (s,S) type is optimal. An (s,S) ordering policy specifies that an order be placed when the level of inventory is found to be below s units, and that the amount of the order be the difference between S and the inventory position. The usual means of finding the optimal values is either through dynamic programming or through stationary analysis. The dynamic programming method is a recursive means of finding the optimal values. The stationary analysis can only be applied to restricted cases and usually involves numerical methods that do not allow for easy optimization.

Since determining the actual optimal values of (s,S) can be computationally quite complex, one obvious way to analyze them is via simulation. While evaluating alternative systems through simulation is fairly routine, optimization through simulation is a challenging problem.

Let X_n = the inventory position in period n,
W_n = the inventory level in period n,
D_n = the demand in period n, and
$F(.)$ = the distribution function of D_n.

The performance measure of interest is the long-run average cost function per period, i.e., the limit of the n period average cost per period, which is defined as follows:

$$f(s,S) = \lim_{n\to\infty} \frac{\sum_{s,S} g_n(s,S)}{n}$$

where $g_n(s,S)$ is the one-period cost function defined as

$$g_n(s,S) = 1_{\{X_n<s\}}(q + c(S - X_n)) + hW_n^+ + pW_n^-,$$

where h = holding costs/ period/ unit of inventory,
p = shortage costs/period/ unit of inventory,
q = set-up cost for placing an order,
c = per-unit ordering cost, and

$X^+ = \max(0, X)$ and $X^- = \max(0, -X)$, and

$$1_{\{X_n<s\}} = \begin{cases} 1 & \text{if } X_n < s \\ 0 & \text{otherwise} \end{cases}.$$

In this example, we present our computational results for the above (s, S) problem and compare the performance of our algorithm with two variants of SA algorithms. The first variant is the method proposed by Gelfand and Mitter (1989) and similarly by Gutjahr and Pflug (1996) (which we will refer to as GM-GP). The second variant is the one proposed by Alkhamis and Ahmed (1999) (which we will refer to as the AL-AH approach). GM-GP and AL-AH variants use a decreasing annealing schedule $\{T_k\}$, whereas our approach uses an annealing schedule with constant temperature T. Also, GM-GP and AL-AH variants use the state that is visited by the algorithm in iteration k as the estimated optimal solution in that iteration**;** our approach selects the state with the best average estimated objective function value obtained from all the previous estimates of the objective function values to be the estimated optimal solution.

In this example, we choose our test cases to have exponentially distributed demands, so that we can easily compare the simulation optimization results to the following analytical results:

$$s_{opt} = -\frac{1}{\lambda}\ln\left(\frac{h+\sqrt{2qh\lambda}}{h+p}\right), S_{opt} = s_{opt} + \sqrt{\frac{2q}{\lambda h}},$$

$$f^*(s,S) = \frac{c}{\lambda} + \frac{q + h\left(s - E(D) + \lambda(S-s)\left(\frac{s+S}{2}\right)\right) + (h+p)E[D]e^{-\lambda}}{1+\lambda(S-s)},$$

where $\lambda = \dfrac{1}{E(D)}$. Four instances will be considered for this problem. Table 1 presents the data for the four instances.

Table 1: Test cases for (s, S) inventory problem.

						Analytical solution		
Case #	**E(D)**	***p***	***q***	***c***	***h***	***s***	***S***	$f^*(s, S)$
1	30	10	30	1	1	45	88	118
2	35	10	30	1	1	55	100	135
3	40	10	30	1	1	64	113	153
4	45	10	30	1	1	73	125	170

Figures 3-6 compare the performance of Algorithm 2 with that of GM-GP and AL-AH algorithms when they are applied to solve the four instances of the

(s, S) optimization problem presented above with neighborhood structure given by $N(i) = \{j \in S : |j - i| \leq 5\}$ and the following choices of the parameters T, the annealing sequence $\{T_k\}$ (for the GM-GP and AL-AH algorithms), and the sequence $\{L_k\}$: $T = 5$, $T_k = 15/\ln(10 + k)$, $L_k = 10$ for Algorithm 2, $L_k = 10 + \lfloor k/200 \rfloor$ for the GM-GP and AL-AH algorithms. In Figures 3-6 the x-axis shows the iteration number, while the y-axis shows the average estimated optimal objective value at the estimated optimal solution over 50 replications. For all instances, Algorithm 2 outperforms GM-GP and AL-AH algorithms by converging faster to the optimal solution.

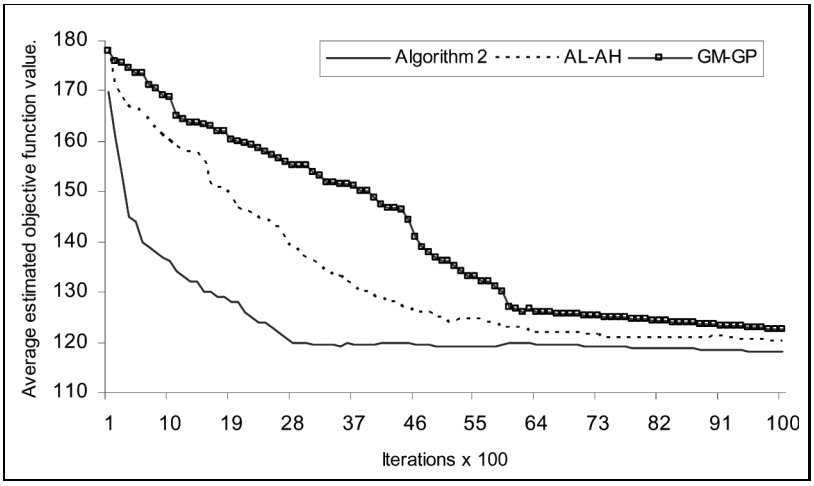

Figure 3: Performance of Algorithm 2, AL-AH and GM-GP for the (s, S) problem test case 1.

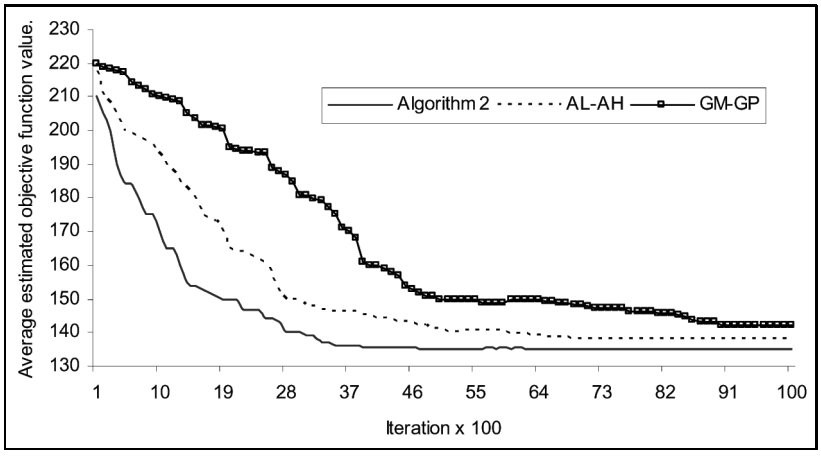

Figure 4: Performance of Algorithm 2, AL-AH and GM-GP for the (s, S) problem test case 2.

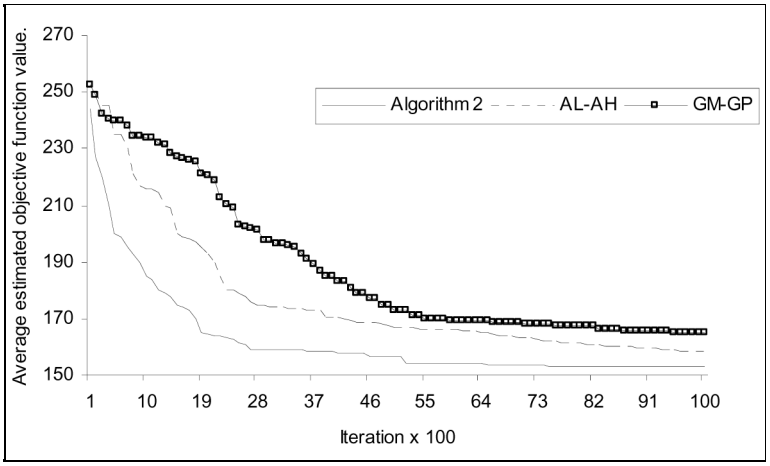

Figure 5: Performance of Algorithm 2, AL-AH and GM-GP for the (s, S) problem test case 3.

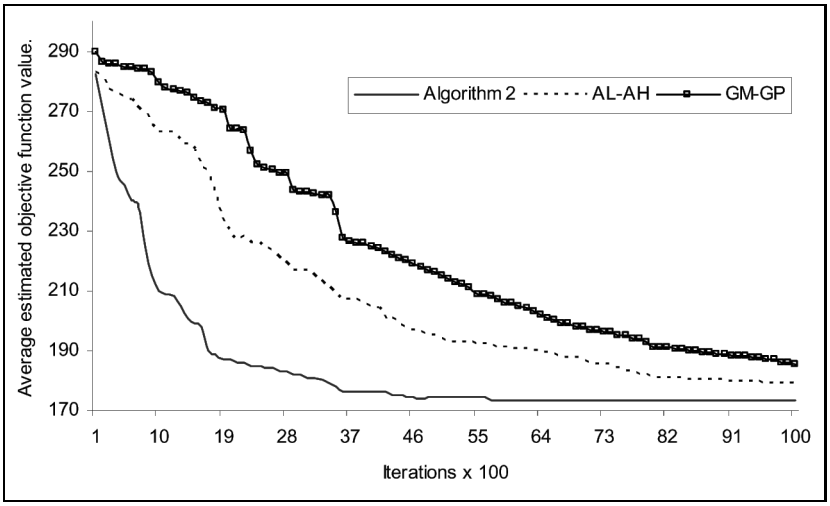

Figure 6: Performance of Algorithm 2, AL-AH and GM-GP for the (s, S) problem test case 4.

M/M/1 queuing problem

This problem involves the optimization of the M/M/1 queuing system similar to the one presented in Alrefaei and Andradóttir (1999). Consider an M/M/1 queuing system with arrival rate λ and service rate μ. Let W_i be the system time of job i, B_i the production time of job i, and A_i the inter-arrival time between jobs i - 1 and i, where $A_0 = 0$ and $W_0 = 0$. To generate the system waiting time of job c, W_c , one can generate B_i, A_i, for $i = 1, \ldots, c$ and then use the following well known recursive formula: $W_i = \max\{B_i , W_{i-1} + B_i - A_i\}$

to obtain W_i for $i = 1, \ldots, c$. We are interested in solving the following optimization problem:

$$\min_{x \in S} f(x) = E\{W(x)\}$$

where $S = [1,..., 50]$ and $W(x)$ is the system time per job in an M/M/1 queuing system with fixed arrival rate $\lambda = 1$ and service rate $\mu(x) > 1$ for all $x \in S$, and where the values of $\mu(x)$ for all $x \in S$ are given in Table 2.

Table 2: The values of the service rates $\mu(x)$ for all $x \in S$.

μ (1),...,μ (10)	1.65	1.6	1.5	1.6	1.7	1.75	1.65	1.6	1.55	1.5
μ (11),...,μ (20)	1.47	1.45	1.5	1.55	1.6	1.65	1.6	1.55	1.5	1.47
μ (21),...,μ (30)	1.45	1.5	1.55	1.6	1.65	1.7	1.75	2.0	1.7	1.6
μ (31),...,μ (40)	1.55	1.5	1.47	1.5	1.6	1.65	1.7	1.75	1.65	1.6
μ (41),...,μ (50)	1.55	1.5	1.47	1.5	1.6	1.65	1.7	1.6	1.5	1.45

We are interested in minimizing the expected average system time per job for the first 100 jobs, $f_i(x) = E\left[\sum_{i=1}^{100} W_i(x)/100\right]$ where x∈S. We apply Algorithm 2 to solve this problem with two different neighborhood structures. The first neighborhood structure is given by:

$$N1(i) = \begin{cases} \{2\} & \textit{if } i = 1, \\ \{49\} & \textit{if } i = 50, \\ \{i-1,\ i+1\} & \textit{otherwise}. \end{cases}$$

The second neighborhood structure is given by

$$N2(i) = \{j \in S : |j - i| \leq 3\}.$$

Note that, in the transient setting, the estimated optimal objective function value is 0.9790 which occurs at $x = 28$. We select the initial state x_0, of Algorithm 2 randomly (uniformly) over S and we run 100 replications to estimate the average performance of Algorithm 2.

Now we compare the performance of Algorithm 2 with the variant presented in Alrefaei and Andradóttir (1999) which we denote by AL-AN. Table 3 compares the performance of Algorithm 2 with that of AL-AN to solve the transient version of the optimization problem using $T = 0.01$ and $L_k = 20$ with two neighborhood structures N1 and N2. Table 3 presents the number of

convergent paths these algorithms have done so far out of a total of 100 replications (i.e., the number of replications in which the estimated optimal solution equals the (true) global optimal solution). For both neighborhood settings, Algorithm 2 converges more rapidly to the optimal global solution than Algorithm 3. Using neighborhood N1, 50 replications have converged after 1000 iterations when Algorithm 2 is used, compared to 36 replications for AL-AN. For the same temperature setting and using N2, 75 replications have converged after 1000 iterations when Algorithm 2 is used, compared to 52 replications for AL-AN. Table 4 shows the same information for a different temperature setting, $T = 1$. Using neighborhood N1, 86 replications have converged after 1000 iterations when Algorithm 2 is used, compared to 70 replications for AL-AN. Using N2 with $T = 1$, both algorithms perform almost equably. Tables 3 and 4 indicate the efficiency of our modified SA variant when applied to solve the transient version of the above queuing optimization problem using the parameter values given previously.

Table 3: A comparison of the performance of Algorithm 2 and AL-AN algorithm, for the M/M/1 problem with different neighborhood structures and T = .01

	Number of convergent paths.			
	Neighborhood N1 , T = .01.		**Neighborhood N2 , T = .01**	
Iteration	**Algorithm 2**	**AL-AN**	**Algorithm 2**	**AL-AN**
10	17	14	20	21
50	26	26	31	28
100	26	26	37	32
200	30	27	41	36
300	35	29	47	37
400	37	29	51	40
500	44	30	53	42
600	45	31	59	47
700	46	34	61	48
800	48	34	67	48
900	49	34	68	50
1000	50	36	75	52

Table 4: A comparison of the performance of Algorithm 2 and AL-AN algorithm, for the M/M/1 problem with different neighborhood structures and T = 1

	Number of convergent paths.			
	Neighborhood N1 , T = 1		Neighborhood N2 , T = 1	
Iteration	Algorithm 2	AL-AN	Algorithm 2	AL-AN
10	13	9	19	19
50	24	20	47	42
100	36	24	62	64
200	43	31	78	78
300	51	46	87	89
400	56	50	92	94
500	64	56	97	98
600	70	56	99	99
700	74	64	100	100
800	76	65	100	100
900	81	67	100	100
1000	86	70	100	100

CONCLUSION

In this paper, we proposed a new variant of the simulated annealing algorithm for solving discrete stochastic optimization problems where the objective function is stochastic and can be evaluated only through Monte Carlo simulation. This variant is important when either the objective function cannot be computed exactly or such an evaluation is computationally expensive. In the proposed variant, the Metropolis criterion depends on whether the objective function values indicate a statistically significant difference at each iteration. The differences between objective function values are considered to be statistically significant based on confidence intervals associated with these values. Unlike the original SA, our method uses a constant temperature and uses a convergence criterion that selects the state with the best average estimated objective function value obtained from all the previous estimates of the objective function values to be the estimated optimal solution. The efficiency of the proposed algorithm has been verified for four test cases. An experimental evaluation for larger instances is a topic of possible future research.

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خوارزمية المحاكاة البطيئة للنظم العشوائية

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خلاصة

معظم المشاكل العملية في مجال العلوم الإدارية وبحوث العمليات والهندسة الصناعية يمكن نمذجتها كمسألة عشوائية متقطعة لإيجاد حلها الأمثل بحيث لا يمكن إيجاد دالة الهدف إلا عن طريق المحاكاة.

في هذا البحث نقدم تطويراً لخوارزمية المحاكاة البطيئة لإيجاد الحل الأمثل للنظم العشوائية المتقطعة حيث يكون احتمال قبول الحل يعتمد على مدى الفرق الإحصائي عند كل خطوة. إن الخوارزمية المقترحة لها نفس خاصية خوارزمية المحاكاة البطيئة الأصلية، لكن طريقتنا تستعمل درجة حرارة ثابتة وليست متناقصة وأيضاً تختار الحالة التي لها أفضل متوسط تقديري لدالة الهدف من كل التقديرات السابقة لدالة الهدف لتكون هي الحل التقديري الأمثل.

وقد بينا أن الإضافات المقترحة سوف تؤدي بالتأكيد إلى الحل الأمثل. وقد أعطينا نتائج حسابية ومقارنات مع طرق أخرى وذلك لتوضيح مدى فاعلية خوارزمية المحاكاة البطيئة المقترحة.