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Generalized erdélyi-kober fractional q-integral operator

LEDA GALUÉ

Centro de Investigación de Matemática Aplicada (CIMA). Facultad de Ingeniería. Universidad del Zulia. Email: lgalue@hotmail.com. Apartado de Correo 10482. Maracaibo, Venezuela .

ABSTRACT

In this paper a generalized Erdélyi-Kober fractional q-integral operator is defined, some rules of composition for these operators are presented and it is applied to the basic analogue of the H-function. Various particular cases are obtained and results given by Saxena *et al*. can be derived from our results.

Keywords: Basic analogue of the special functions; basic analogue of the H-function; elementary q-functions; fractional q-integral operator; q-integral; rules of composition.

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INTRODUCTION

Recently, Yadav and Purohit (2004) have investigated the applications of Riemann-Liouville fractional q-integral operator to various basic hypergeometric functions of one variable. On the other hand, Saxena *et al.* (2005) have evaluated Kober fractional basic integral operator of the basic analogue of the H-function defined by Saxena, Modi & Kalla (1983).

In this paper a generalized Erdélyi-Kober fractional q-integral operator is defined, some rules of composition for these operators are presented and it is applied to the basic analogue of the H-function. Various particular cases are obtained and results given earlier by Saxena *et al*. (2005) can be derived from the main resuts.

BASIC ANALOGUE OF THE *H*-FUNCTION

The basic analogue of the H-function is defined by Saxena, Modi & Kalla (1983) in the following form:

$$H_{p,r}^{m,n}\left[x;q\left|\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,r}\end{matrix}\right.\right]=$$

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-B_js}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+A_js}\right)\pi\ x^s}{\prod_{j=m+1}^{r}G\left(q^{1-b_j+B_js}\right)\prod_{j=n+1}^{p}G\left(q^{a_j-A_js}\right)G\left(q^{1-s}\right)\sin(\pi s)}ds \tag{1}$$

where $0\le m\le r,\ 0\le n\le p;\ A_i\ (i=1,...,p)$ and $B_j\,(j=1,...,r)$ are all positive integers; the contour C is a line parallel to $\mathrm{Re}(ws)=0$, with indentations, if necessary, in such a manner that all poles of $G(q^{b_j-B_js})(j=1,...,m)$ are to the right, and those of $G(q^{1-a_j+A_js})(j=1,...,n)$ to the left of C. The integral converges if $\mathrm{Re}(s\log(x)-\log\sin(\pi s))<0$ for large values of $|s|$ on the contour, i.e if $\left|\arg(x)-\frac{w_2}{w_1}\log|x|\right|<\pi$, where $0<|q|<1$, $\log q=-w=-(w_1+iw_2)$, w,w_1,w_2 are definite quantities, w_1,w_2 being real. Further

$$G(q^a)=\left\{\prod_{n=0}^{\infty}(1-q^{a+n})\right\}^{-1}=\frac{1}{(q^a;q)\infty}. \tag{2}$$

If $A_i=B_j=1,\forall i$ and j in Eq. (1), then it reduces to the basic Meijer's G-function, namely

$$H_{p,r}^{m,n}\left[x;q\left|\begin{matrix}(a_j,1)_{1,p}\\(b_j,1)_{1,r}\end{matrix}\right.\right]=G_{p,r}^{m,n}\left[x;q\left|\begin{matrix}a_1,...,a_p\\b_1,...,b_r\end{matrix}\right.\right]=$$

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+s}\right)\pi\ x^s}{\prod_{j=m+1}^{r}G\left(q^{1-b_j+s}\right)\prod_{j=n+1}^{p}G(q^{a_j-s})G(q^{1-s})\sin(\pi s)}ds \tag{3}$$

where $0\le m\le r,0\le n\le p$ and $\mathrm{Re}(s\log(x)-\log\sin(\pi s))<0$ for large values of $|s|$ on the contour C.

Various elementary q-functions and the basic analogue of the special functions are expressed in terms of the basic analogue of G-function, which is a special case of the basic analogue of H-function. Some of them are presented in the Table 1 (Saxena & Kumar 1990, Saxena *et al.* 2005).

Table 1. Some Special Cases of the Basic Analogue of G-function

$f(x)$	Eq. No.
$e_q(-x) = G(q)\ G_{0,2}^{1,0}\left[x(1-q);q\left\vert\begin{matrix} -- \\ 0,1 \end{matrix}\right.\right]$	(4)
$\sin_q(x) = \sqrt{\pi}(1-q)^{-1/2}[G(q)]^2\ G_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q\left\vert\begin{matrix} --- \\ \frac{1}{2},0,1 \end{matrix}\right.\right]$	(5)
$\cos_q(x) = \sqrt{\pi}(1-q)^{-1/2}[G(q)]^2\ G_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q\left\vert\begin{matrix} --- \\ 0,\frac{1}{2},1 \end{matrix}\right.\right]$	(6)
$\sinh_q(x) = \frac{\sqrt{\pi}}{i}(1-q)^{-1/2}[G(q)]^2\ G_{0,3}^{1,0}\left[-\frac{x^2(1-q)^2}{4};q\left\vert\begin{matrix} --- \\ \frac{1}{2},0,1 \end{matrix}\right.\right]$	(7)
$\cosh_q(x) = \sqrt{\pi}(1-q)^{-1/2}[G(q)]^2\ G_{0,3}^{1,0}\left[-\frac{x^2(1-q)^2}{4};q\left\vert\begin{matrix} --- \\ 0,\frac{1}{2},1 \end{matrix}\right.\right]$	(8)
$E_q[r;b_j:p;a_j:x] = G_{p,r}^{r,0}\left[x;q\left\vert\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_r \end{matrix}\right.\right]$	(9)
$J_\nu(x;q) = [G(q)]^2\ G_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q\left\vert\begin{matrix} ---- \\ \frac{\nu}{2},\frac{-\nu}{2},1 \end{matrix}\right.\right]$	(10)
$Y_\nu(x;q) = [G(q)]^2\ G_{1,4}^{2,0}\left[\frac{x^2(1-q)^2}{4};q\left\vert\begin{matrix} \frac{-\nu-1}{2} \\ \frac{\nu}{2},\frac{-\nu}{2},\frac{-\nu-1}{2},1 \end{matrix}\right.\right]$	(11)
$K_\nu(x;q) = (1-q)\ G_{0,3}^{2,0}\left[\frac{x^2(1-q)^2}{4};q\left\vert\begin{matrix} ---- \\ \frac{\nu}{2},\frac{-\nu}{2},1 \end{matrix}\right.\right]$	(12)
$H_\nu(x;q) = \left(\frac{1-q}{2}\right)^{1-\alpha}\ G_{1,4}^{3,1}\left[\frac{x^2(1-q)^2}{4};q\left\vert\begin{matrix} \frac{1+\alpha}{2} \\ \frac{\nu}{2},\frac{-\nu}{2},\frac{1+\alpha}{2},1 \end{matrix}\right.\right]$	(13)

Notation:

$E_q[r; b_j : p; a_j : x]$: Mac Robert's E-function.

$J_\nu(x; q)$: q-analogue of the Bessel function $j_\nu(x)$.

$Y_\nu(x; q)$: q-analogue of the Bessel function $Y_\nu(x)$.

$K_\nu(x; q)$: q-analogue of the Bessel function of the third kind $K_\nu(x)$.

$H_\nu(x; q)$: basic analogue of the Struve's function $H_\nu(x)$.

DEFINITION OF THE OPERATOR

We introduce the generalized Erdélyi-Kober fractional q-integral operator as follows:

$$I_q^{\eta,\mu,\beta} f(x) = \frac{\beta x^{-\beta(\eta+\mu)}}{\Gamma_q(\mu)} \int_0^x (x^\beta - t^\beta q)_{\mu-1} t^{\beta(\eta+1)-1} f(t) d_q t \tag{14}$$

where Re(β),Re(μ) > 0, $\eta \in \mathbb{C}$, μ is an arbitrary order of integration.

Making a simple variable change and using the result (Gasper & Rahman 1990, p. 22)

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)\, z}$$

where D_q is the q-derivative operator, we obtain

$$I_q^{\eta,\mu,\beta} f(x) = \frac{\beta}{\Gamma_q(\mu)} \frac{(1-q^{1/\beta}) x^{-\beta(\mu-1)}}{(1-q)} \int_0^1 (x^\beta - x^\beta y q)_{\mu-1} y^\eta f(x y^{1/\beta}) d_q y. \tag{15}$$

Applying the results (Saxena *et al.* 2005, p. 155, Eq. (7), Gasper & Rahman 1990, p. 233, Eq. (I.4))

$$(x-y)_\nu = x^\nu \prod_{k=0}^{\infty} \left[\frac{1 - \left(\frac{y}{x}\right) q^k}{1 - \left(\frac{y}{x}\right) q^{k+\nu}} \right], \quad (a; q)\infty = \prod_{k=0}^{\infty} (1 - a q^k)$$

Eq. (15) reduces to

$$I_q^{\eta,\mu,\beta} f(x) = \frac{\beta}{\Gamma_q(\mu)} \frac{(1-q^{1/\beta})}{(1-q)} \int_0^1 \frac{(yq; q)\infty}{(yq^\mu; q)\infty} y^\eta f(x y^{1/\beta}) d_q y. \tag{16}$$

Jackson (1910) introduced the q-integral (Gasper & Rahman 1990, p. 19, Eq. (1.11.1)) as:

$$\int_0^1 f(t)d_q t = (1-q)\sum_{k=0}^{\infty} q^k f(q^k)$$

and since (Gasper & Rahman 1990, p. 235, Eq. (I.35); p. 6, Eq. (1.2.30))

$$\Gamma_q(x+1) = \frac{(q;q)_\infty}{(q^{x+1};q)_\infty}(1-q)^{-x}, \quad 0<q<1$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

therefore we can write Eq. (16) as:

$$I_q^{\eta,\mu,\beta} f(x) = \beta(1-q^{1/\beta})(1-q)^{\mu-1}\sum_{k=0}^{\infty}\frac{(q^\mu;q)_k}{(q;q)_k}q^{k(\eta+1)}f(xq^{k/\beta}), \qquad (17)$$

where $\mathrm{Re}(\beta)$, $\mathrm{Re}(\mu) > 0$, $\eta \in \mathbb{C}$.

RULES OF COMPOSITION

In this section, we investigate various rules of composition for the generalized Erdélyi-Kober fractional q-integral operators,

$$I_q^{\eta_1,\mu_1,\beta} I_q^{\eta_1+\mu_1,\mu_2,\beta} f(x) = \beta(1-q^{1/\beta})(1-q)^{-1} I_q^{\eta_1,\mu_1+\mu_2,\beta} f(x). \qquad (18)$$

Proof: To prove the result (18), we make use of the relation (17) in the left hand side to obtain

$$I_q^{\eta_1,\mu_1,\beta} I_q^{\eta_1+\mu_1,\mu_2,\beta} f(x) = \beta^2(1-q^{1/\beta})^2(1-q)^{\mu_1+\mu_2-2}$$

$$\cdot \sum_{k_1,k_2=0}^{\infty}\frac{(q^{\mu_1};q)_{k_1}}{(q;q)_{k_1}}\frac{(q^{\mu_2};q)_{k_2}}{(q;q)_{k_2}}q^{k_1(\eta_1+1)}q^{k_2(\eta_1+\mu_1+1)}f(xq^{(k_1+k_2)/\beta})$$

after a change of index, the above expression reduces to

$$I_q^{\eta_1,\mu_1,\beta} I_q^{\eta_1+\mu_1,\mu_2,\beta} f(x) = \beta^2(1-q^{1/\beta})^2(1-q)^{\mu_1+\mu_2-2}$$

$$\cdot \sum_{k=0}^{\infty} q^{k(\eta_1+1)} f(xq^{k/\beta})\sum_{k_2=0}^{k}\frac{(q^{\mu_1};q)_{k-k_2}}{(q;q)_{k-k_2}}\frac{(q^{\mu_2};q)_{k_2}}{(q;q)_{k_2}}q^{\mu_1 k_2}$$

and using the result (Gasper & Rahman 1990, p. 20)

$$\sum_{k=0}^{n} \frac{(a;q)_k (b;q)_{n-k}}{(q;q)_k (q;q)_{n-k}}\ b^k = \frac{(ab;q)_n}{(q;q)_n} \tag{19}$$

we have,

$$I_q^{\eta_1,\mu_1,\beta} I_q^{\eta_1+\mu_1,\mu_2,\beta} f(x) = \beta^2 (1-q^{1/\beta})^2 (1-q)^{\mu_1+\mu_2-2}$$
$$\cdot \sum_{k=0}^{\infty} q^{k(\eta_1+1)} \frac{(q^{\mu_1+\mu_2};q)_k}{(q;q)_k} f(xq^{k/\beta}).$$

Interpreting the above expression with the help of the relation (17), we finally arrive at the result.

Analogously, the following rules of composition are obtained:

$$I_q^{\eta_2+\mu_2,\mu_1,\beta} I_q^{\eta_2,\mu_2,\beta} f(x) = \beta(1-q^{1/\beta})(1-q)^{-1} I_q^{\eta_2,\mu_1+\mu_2,\beta} f(x). \tag{20}$$

$$I_q^{\eta_1,\mu_1,\beta} I_q^{\eta_1-\mu_2,\mu_2,\beta} f(x) = \beta(1-q^{1/\beta})(1-q)^{-1} I_q^{\eta_1-\mu_2,\mu_1+\mu_2,\beta} f(x). \tag{21}$$

$$I_q^{\eta_2-\mu_1,\mu_1,\beta} I_q^{\eta_2,\mu_2,\beta} f(x) = \beta(1-q^{1/\beta})(1-q)^{-1} I_q^{\eta_2-\mu_1,\mu_1+\mu_2,\beta} f(x). \tag{22}$$

It is interesting to observe that the composition rule (18) can be extended for n operators in the following form:

$$I_q^{\eta_1,\mu_1,\beta} I_q^{\eta_1+\mu_1,\mu_2,\beta} \ldots I_q^{\eta_1+\mu_1+\ldots+\mu_{n-2},\mu_{n-1},\beta} I_q^{\eta_1+\mu_1+\ldots+\mu_{n-2}+\mu_{n-1},\mu_n,\beta} f(x) =$$
$$\beta^{n-1}(1-q^{1/\beta})^{n-1}(1-q)^{1-n} I_q^{\eta_1,\mu_1+\ldots+\mu_n,\beta} f(x). \tag{23}$$

Proof: To prove (23), we employ the mathematical induction principle with an observation that for $n = 2$ in (23), it yields to Eq. (18) as:

$$I_q^{\eta_1,\mu_1,\beta} I_q^{\eta_1+\mu_1,\mu_2,\beta} f(x) = \beta(1-q^{1/\beta})(1-q)^{-1} I_q^{\eta_1,\mu_1+\mu_2,\beta} f(x). \tag{24}$$

We suppose that Eq. (23) is true for $n = k$, that is

$$I_q^{\eta_1,\mu_1,\beta} I_q^{\eta_1+\mu_1,\mu_2,\beta} \ldots I_q^{\eta_1+\mu_1+\ldots+\mu_{k-2},\mu_{k-1},\beta} I_q^{\eta_1+\mu_1+\ldots+\mu_{k-2}+\mu_{k-1},\mu_k,\beta} f(x) =$$
$$\beta^{k-1}(1-q^{1/\beta})^{k-1}(1-q)^{1-k} I_q^{\eta_1,\mu_1+\ldots+\mu_k,\beta} f(x). \tag{25}$$

On operating both sides of the relation (25) by the operator $I_q^{\eta_1+\mu_1+\ldots+\mu_{k-1}+\mu_k,\mu_{k+1},\beta}(.)$, we obtain

$$I_q^{\eta_1,\mu_1,\beta} I_q^{\eta_1+\mu_1,\mu_2,\beta} \ldots I_q^{\eta_1+\mu_1+\ldots+\mu_{k-1},\mu_k,\beta} I_q^{\eta_1+\mu_1+\ldots+\mu_{k-1}+\mu_k,\mu_{k+1},\beta} f(x) =$$

$$\beta^{k-1}(1-q^{1/\beta})^{k-1}(1-q)^{1-k} I_q^{\eta_1,\mu_1+\ldots+\mu_k,\beta} I_q^{\eta_1+\mu_1+\ldots+\mu_{k-1}+\mu_k,\mu_{k+1},\beta} f(x).$$

On applying the relation (24) to the right hand side of the above expression, we finally obtain:

$$I_q^{\eta_1,\mu_1,\beta} I_q^{\eta_1+\mu_1,\mu_2,\beta} \ldots I_q^{\eta_1+\mu_1+\ldots+\mu_{k-1},\mu_k,\beta} I_q^{\eta_1+\mu_1+\ldots+\mu_{k-1}+\mu_k,\mu_{k+1},\beta} f(x) =$$

$$\beta^{(k+1)-1}(1-q^{1/\beta})^{(k+1)-1}(1-q)^{1-(k+1)} I_q^{\eta_1,\mu_1+\ldots+\mu_{k+1},\beta} f(x)$$

which is true for $n = k+1, (k = 1, 2, \ldots, n-1)$. This completes the proof of the result (23).

Similarly, we can generalize the index composition rules given by equations (20)-(22) to the following results:

$$I_q^{\eta_n+\mu_n+\ldots+\mu_2,\mu_1,\beta} I_q^{\eta_n+\mu_n+\ldots+\mu_3,\mu_2,\beta} \ldots I_q^{\eta_n+\mu_n,\mu_{n-1},\beta} I_q^{\eta_n,\mu_n,\beta} f(x) =$$

$$\beta^{n-1}(1-q^{1/\beta})^{n-1}(1-q)^{1-n} I_q^{\eta_n,\mu_1+\ldots+\mu_n,\beta} f(x). \tag{26}$$

$$I_q^{\eta_1,\mu_1,\beta} I_q^{\eta_1-\mu_2,\mu_2,\beta} \ldots I_q^{\eta_1-\mu_2-\ldots-\mu_{n-2}-\mu_{n-1},\mu_{n-1},\beta} I_q^{\eta_1-\mu_2-\ldots-\mu_{n-1}-\mu_n,\mu_n,\beta} f(x) =$$

$$\beta^{n-1}(1-q^{1/\beta})^{n-1}(1-q)^{1-n} I_q^{\eta_1-\mu_2-\ldots-\mu_{n-1}-\mu_n,\mu_1+\ldots+\mu_n,\beta} f(x). \tag{27}$$

$$I_q^{\eta_n-\mu_{n-1}-\ldots-\mu_1,\mu_1,\beta} I_q^{\eta_n-\mu_{n-1}-\ldots-\mu_2,\mu_2,\beta} \ldots I_q^{\eta_n-\mu_{n-1},\mu_{n-1},\beta} I_q^{\eta_n,\mu_n,\beta} f(x) =$$

$$\beta^{n-1}(1-q^{1/\beta})^{n-1}(1-q)^{1-n} I_q^{\eta_n-\mu_{n-1}-\ldots-\mu_1,\mu_1+\ldots+\mu_n,\beta} f(x). \tag{28}$$

APPLICATIONS

This section envisages the applications of the generalized Erdélyi-Kober fractional q-integral operator to the basic analogue of the H-function.

We operate the operator (14) to the definition (1) with argument as x^λ , it yields after some simplifications:

$$I_q^{\eta,\mu,\beta} H_{p,r}^{m,n}\left[x^{\lambda};q\left|\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,r}\end{matrix}\right.\right]=\frac{\beta(1-q^{1/\beta})(1-q)^{\mu-1}}{2\pi i}$$

$$\cdot\int_C\frac{\prod_{j=1}^{m}G(q^{bj-B_js})\prod_{j=1}^{n}G(q^{1-a_j+A_js})\pi x^{\lambda s}}{\prod_{j=m+1}^{r}G(q^{1-b_j+B_js})\prod_{j=n+1}^{p}G(q^{a_j-A_js})G(q^{1-s})sin(\pi s)}$$

$$\cdot\left\{\sum_{k=0}^{\infty}\frac{(q^{\mu};q)_k}{(q;q)_k}q^{k(\eta+1)}q^{\lambda ks/\beta}\right\}ds$$

where we have interchanged the order of integral and sum on the basis of the absolute convergence.

In view of the result (Gasper & Rahman 1990, p. 7, Eq. (1.3.2))

$${}_1\phi_0(a;-;q,z)=\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty},|z|<1,\quad |q|<1,$$

and from Eq. (2), we can write

$$\sum_{k=0}^{\infty}\frac{(q^{\mu};q)_k}{(q;q)_k}q^{k(\eta+1)}q^{\lambda ks/\beta}=\frac{(q^{\mu+\eta+1+\lambda s/\beta};q)_\infty}{(q^{\eta+1+\lambda s/\beta};q)_\infty}=\frac{G(q^{\eta+1+\lambda s/\beta})}{G(q^{\mu+\eta+1+\lambda s/\beta})}.$$

Therefore,

$$I_q^{\eta,\mu,\beta} H_{p,r}^{m,n}\left[x^{\lambda};q\left|\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,r}\end{matrix}\right.\right]=\frac{\beta(1-q^{1/\beta})(1-q)^{\mu-1}}{2\pi i}$$

$$\cdot\int_C\frac{\prod_{j=1}^{m}G(q^{bj-B_js})\prod_{j=1}^{n}G(q^{1-a_j+A_js})G(q^{\eta+1+\lambda_s/\beta})\ \pi\ x^{\lambda s}\ ds}{\prod_{j=m+1}^{r}G(q^{1-b_j+B_js})\prod_{j=n+1}^{p}G(q^{a_j-A_js})G(q^{1-s})G(q^{\mu+\eta+1+\lambda_s/\beta})\ sin(\pi s)}$$

on interpretation of the right hand side of the above expression in light of the definition (1), we finally obtain the following results:

$$I_q^{\eta,\mu,\beta} H_{p,r}^{m,n}\left[x^{\lambda}; q \,\middle|\, \begin{matrix} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,r} \end{matrix}\right] = \beta(1-q^{1/\beta})(1-q)^{\mu-1}$$

$$\cdot\; H_{p+1,r+1}^{m,n+1}\left[x^{\lambda}; q \,\middle|\, \begin{matrix} (-\eta, \frac{\lambda}{\beta}), (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,r}, (-\mu-\eta, \frac{\lambda}{\beta}) \end{matrix}\right] \text{ if } \lambda > 0, \tag{29}$$

or

$$I_q^{\eta,\mu,\beta} H_{p,r}^{m,n}\left[x^{\lambda}; q \,\middle|\, \begin{matrix} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,r} \end{matrix}\right] = \beta(1-q^{1/\beta})(1-q)^{\mu-1}$$

$$\cdot\; H_{p+1,r+1}^{m+1,n}\left[x^{\lambda}; q \,\middle|\, \begin{matrix} (a_j, A_j)_{1,p}, (\mu+\eta+1, -\frac{\lambda}{\beta}) \\ (\eta+1, -\frac{\lambda}{\beta}), (b_j, B_j)_{1,r} \end{matrix}\right] \text{ if } \lambda < 0. \tag{30}$$

From Eq. (3) and Eq. (29), we obtain

$$I_q^{\eta,\mu,\beta} G_{p,r}^{m,n}\left[x^{\lambda}; q \,\middle|\, \begin{matrix} a_1, \ldots.., a_p \\ b_1, \ldots.., b_r \end{matrix}\right] = \beta(1-q^{1/\beta})(1-q)^{\mu-1}$$

$$\cdot\; H_{p+1,r+1}^{m,n+1}\left[x^{\lambda}; q \,\middle|\, \begin{matrix} (-\eta, \frac{\lambda}{\beta}), (a_j, 1)_{1,p} \\ (b_j, 1)_{1,r}, (-\mu-\eta, \frac{\lambda}{\beta}) \end{matrix}\right] \text{ if } \lambda > 0, \tag{31}$$

an analogous result can be deduced for $\lambda < 0$ using Eq. (3) and Eq. (30).

Interestingly, by making a suitable change to the complex parameters a_j $(j = 1, 2, ..., p)$, b_j $(j = 1, 2, ..., r)$, and the argument x in conjunction with definitions given in Table 1, we obtain the following results given in Table 2.

Table 2. The generalized Erdélyi-Kober fractional q-integral operator of some elementary q-functions

$f(x)$	$I_q^{\eta,\mu,\beta} f(x) = \frac{\beta x^{-\beta(\eta+\mu)}}{\Gamma_q(\mu)} \int_0^x (x^\beta - t^\beta q)_{\mu-1} t^{\beta(\eta+1)-1} f(t)\, d_q t,\ Re(\beta),\ Re(\mu) > 0,\ \eta \in C$	Eq. No.
$e_q(-x)$	$\beta(1-q^{1/\beta})(1-q)^{\mu-1}\, G(q)\, H_{1,3}^{1,1}\left[x(1-q);\, q \middle\vert \begin{matrix} (-\eta, \frac{1}{\beta}) \\ (0, 1), (1, 1), (-\mu-\eta, \frac{1}{\beta}) \end{matrix} \right]$	(32)
$\sin_q(x)$	$\beta\sqrt{\pi}(1-q^{1/\beta})(1-q)^{\mu-3/2}\, [G(q)]^2\, H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};\, q \middle\vert \begin{matrix} (-\eta, \frac{2}{\beta}) \\ (\frac{1}{2}, 1), (0, 1), (1, 1), (-\mu-\eta, \frac{2}{\beta}) \end{matrix} \right]$	(33)
$\cos_q(x)$	$\beta\sqrt{\pi}(1-q^{1/\beta})(1-q)^{\mu-3/2}\, [G(q)]^2\, H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};\, q \middle\vert \begin{matrix} (-\eta, \frac{2}{\beta}) \\ (0, 1), (\frac{1}{2}, 1), (1, 1), (-\mu-\eta, \frac{2}{\beta}) \end{matrix} \right]$	(34)
$\sinh_q(x)$	$\beta\frac{\sqrt{\pi}}{i}(1-q^{1/\beta})(1-q)^{\mu-3/2}\, [G(q)]^2\, H_{1,4}^{1,1}\left[-\frac{x^2(1-q)^2}{4};\, q \middle\vert \begin{matrix} (-\eta, \frac{2}{\beta}) \\ (\frac{1}{2}, 1), (0, 1), (1, 1), (-\mu-\eta, \frac{2}{\beta}) \end{matrix} \right]$	(35)
$\cosh_q(x)$	$\beta\sqrt{\pi}(1-q^{1/\beta})(1-q)^{\mu-3/2}\, [G(q)]^2\, H_{1,4}^{1,1}\left[-\frac{x^2(1-q)^2}{4};\, q \middle\vert \begin{matrix} (-\eta, \frac{2}{\beta}) \\ (0, 1), (\frac{1}{2}, 1), (1, 1), (-\mu-\eta, \frac{2}{\beta}) \end{matrix} \right]$	(36)

Table 2. Continuation...

$E_q\left[r;\, b_j : p; a_j : x\right]$	$\beta(1-q^{1/\beta})(1-q)^{\mu-1}\ H^{r,1}_{p+1,r+1}\left[x;\, q\left\vert \begin{matrix} (-\eta,\frac{1}{\beta}),(a_j,\,1)_{1,p} \\ (b_j,\,1)_{1,r},(-\mu-\eta,\frac{1}{\beta}) \end{matrix}\right.\right]$	(37)
$J_\nu(x;\, q)$	$\beta(1-q^{1/\beta})(1-q)^{\mu-1}\,[G(q)]^2\ H^{1,1}_{1,4}\left[\frac{x^2(1-q)^2}{4};\, q\left\vert \begin{matrix} (-\eta,\frac{2}{\beta}) \\ (\frac{\nu}{2},\,1),\,(\frac{-\nu}{2},\,1),(1,\,1),(-\mu-\eta,\frac{2}{\beta}) \end{matrix}\right.\right]$	(38)
$Y_\nu(x;\, q)$	$\beta(1-q^{1/\beta})(1-q)^{\mu-1}\,[G(q)]^2\ H^{2,1}_{2,5}\left[\frac{x^2(1-q)^2}{4};\, q\left\vert \begin{matrix} (-\eta,\frac{2}{\beta}),\,(\frac{-\nu-1}{2},\,1) \\ (\frac{\nu}{2},\,1),\,(\frac{-\nu}{2},\,1),\,(\frac{-\nu-1}{2},\,1),(1,\,1),(-\mu-\eta,\frac{2}{\beta}) \end{matrix}\right.\right]$	(39)
$K_\nu(x;\, q)$	$\beta(1-q^{1/\beta})(1-q)^{\mu}\ H^{2,1}_{1,4}\left[\frac{x^2(1-q)^2}{4};\, q\left\vert \begin{matrix} (-\eta,\frac{2}{\beta}) \\ (\frac{\nu}{2},\,1),\,(\frac{-\nu}{2},\,1),(1,\,1),(-\mu-\eta,\frac{2}{\beta}) \end{matrix}\right.\right]$	(40)
$H_\nu(x;\, q)$	$2^{\alpha-1}\beta(1-q^{1/\beta})(1-q)^{\mu-\alpha}\ H^{3,2}_{2,5}\left[\frac{x^2(1-q)^2}{4};\, q\left\vert \begin{matrix} (-\eta,\frac{2}{\beta}),\,(\frac{1+\alpha}{2},\,1) \\ (\frac{\nu}{2},\,1),\,(\frac{-\nu}{2},\,1),\,(\frac{1+\alpha}{2},\,1),(1,\,1),(-\mu-\eta,\frac{2}{\beta}) \end{matrix}\right.\right]$	(41)

PARTICULAR CASES

(i) If we take $\beta = 1$ in Eq. (14) we obtain

$$I_q^{\eta,\mu,1}f(x) = I_q^{\eta,\mu}f(x) = \frac{x^{-(\eta+\mu)}}{\Gamma_q(\mu)}\int_0^x (x-tq)_{\mu-1}t^{\eta}f(t)d_qt, \tag{42}$$

where $\mathrm{Re}(\mu) > 0, \eta \in \mathbb{C}$, and $I_q^{\eta,\mu}(.)$ is the Kober fractional q-integral operator defined by Agarwal (1969).

(ii) If we let $q \to 1^-$ in the definition (16), we get

$$\lim_{q\to 1^-} I_q^{\eta,\mu,\beta}f(x) = \lim_{q\to 1^-}\left\{\frac{\beta}{\Gamma_q(\mu)}\frac{(1-q^{1/\beta})}{(1-q)}\int_0^1 \frac{(yq;q)_\infty}{(yq^{\mu};q)_\infty}y^{\eta}f(xy^{1/\beta})d_qy\right\}$$

on using the results (Gasper & Rahman 1990, p. 3, Eq. (1.2.13); p. 16, Eq. (1.10.3); p. 9, Eq. (1.3.19)), namely

$$\lim_{q\to 1^-}\frac{1-q^a}{1-q} = a, \qquad \lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x)$$

$$\lim_{q\to 1^-}\frac{(q^az;q)_\infty}{(z;q)_\infty} = (1-z)^{-a}, \quad |z| < 1, \qquad a \text{ being real}$$

and ,

$$\lim_{q\to 1^-} d_qz = dz$$

we obtain,

$$\lim_{q\to 1^-} I_q^{\eta,\mu,\beta}f(x) = \frac{1}{\Gamma(\mu)}\int_0^1 (1-y)^{\mu-1}y^{\eta}f(xy^{1/\beta})dy = I_\beta^{\eta,\mu}f(x) \tag{43}$$

where μ, $\mathrm{Re}(\beta) > 0$, $\eta \in \mathbb{C}$ and $I_\beta^{\eta,\mu}(.)$ is a generalization of the Erdélyi-Kober operators (Kalla & Galué 1993, Sneddon 1975).

(iii) Further, if we put $\beta = 1$ in the Table 2, we obtain the results given by Saxena *et al*. (2005).

(iv) Finally, for $\beta = 1$ in Eq. (18), Eq. (20)-(23) and Eq. (26)-(28), we have rules of composition for the Kober fractional q-integral operator given by Eq. (42).

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تعميم جزئية إرديلي - كوبر مشغل تكاملي q-

ليدا جاليو

مركز الرياضيات التطبيقية - كلية الهندسة - فنزويلا

خلاصة

في هذه الورقة فإن إرديلي - كوبر الجزئي المشغل التكاملي q- قد تم تعريفها. كما تم عرض بعض قواعد تركيب هذه المشغلات وتم تطبيقها للنظير الأساسي لوظيفة H-.

تم الحصول على بعض الحالات الخاصة، كما أن النتائج التي توصل إليها الباحث ساكينا وآخرون من الممكن إشتقاقها من نتائج هذا البحث.